Notre Dame Journal of Formal Logic
Volume XVIII, Number 4, October 1977
NDJFAM

A SEMANTICAL ANALYSIS OF THE CALCULI $\mathbf{C}_n$

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1 *Introduction* $\mathbf{C}_1$ is a propositional calculus which can serve as a basis for inconsistent, but non trivial deductive theories (see [1] and [2]). The axiomatic basis of $\mathbf{C}_1$ is as follows:

1. Primitive symbols of $\mathbf{C}_1$: 1.1. $\supset$ (implication), & (conjunction), $\vee$ (disjunction), and $\neg$ (negation); 1.2. propositional variables: $p, q, r, \ldots, p', q', r', \ldots$; 1.3. Parentheses.

The notion of formula and the symbol of equivalence ($\equiv$) are defined in the standard way. Roman capitals will be used as syntactical variables for formulas. A° is an abbreviation of $\neg(A \,\&\, \neg A)$.

Definition 1 $\neg^* A =_{df} \neg A \,\&\, A^\circ$.

2. Postulates (axiom schemata and deduction rule) of $\mathbf{C}_1$:

(1) $A \supset (B \supset A)$,
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$,
(3) $\dfrac{A \quad A \supset B}{B}$,
(4) $A \,\&\, B \supset A$,
(5) $A \,\&\, B \supset B$,
(6) $A \supset (B \supset A \,\&\, B)$,
(7) $A \supset A \vee B$,
(8) $B \supset A \vee B$,
(9) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$,
(10) $A \vee \neg A$,
(11) $\neg\neg A \supset A$,
(12) $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$,
(13) $A^\circ \,\&\, B^\circ \supset (A \,\&\, B)^\circ$,
(14) $A^\circ \,\&\, B^\circ \supset (A \vee B)^\circ$,
(15) $A^\circ \,\&\, B^\circ \supset (A \supset B)^\circ$.

(Formal) proof, deduction and the symbol $\vdash$ are introduced as in Kleene's book [4].

Received October 12, 1976

In $\mathbf{C}_1$ the symbols $\supset$, &, v, and $\equiv$ have all properties of the classical positive logic. For instance, in $\mathbf{C}_1$ Peirce's law, $((A \supset B) \supset A) \supset A$, is valid. $\neg^*$ has all properties of classical negation; for example, we have in $\mathbf{C}_1$:

$$\vdash (A \supset B) \supset ((A \supset \neg^* B) \supset \neg^* A); \vdash \neg^* A \supset (A \supset B); \vdash A \vee \neg^* A.$$

In this paper we present a *two-valued semantics* for $\mathbf{C}_1$, which constitutes a generalization of the common semantics of the classical propositional calculus. It seems to us that the proposed semantics for $\mathbf{C}_1$ agrees with some views of the young Łukasiewicz, as he presented them in his work [5], section **18**, item b'. In this paper Łukasiewicz discusses, among other things, the possibility of denying the principle of contradiction.

In the sequel the symbols $\Rightarrow$ and $\Leftrightarrow$ will be employed as metalinguistic abbreviations for implication and equivalence.

2 *A two-valued semantics for* $\mathbf{C}_1$ Let $\mathcal{F}$ denote the set of formulas of $\mathbf{C}_1$. Γ and Δ will designate any subsets of $\mathcal{F}$. $\overline{\Gamma}$ will be an abbreviation of the set $\{A \in \mathcal{F}: \Gamma \vdash A\}$.

Definition 2 The set Γ of formulas is said to be trivial if $\overline{\Gamma} = \mathcal{F}$; otherwise, Γ is called nontrivial.

Definition 3 Γ is said to be consistent if there is at least one formula A such that $A, \neg A \in \overline{\Gamma}$; otherwise, Γ is called consistent.

Definition 4 Γ is a maximal nontrivial set if it is not trivial and, for all A, if $A \notin \Gamma$, then $\Gamma \cup \{A\}$ is trivial.

Theorem 1 *If* Γ *is a maximal nontrivial set of formulas, then:*

1. $\Gamma \vdash A \Leftrightarrow A \in \Gamma$; 2. $A \in \Gamma \Rightarrow \neg^* A \notin \Gamma$; $\neg^* A \in \Gamma \Rightarrow A \notin \Gamma$; 3. $A \in \Gamma$ or $\neg^* A \in \Gamma$; 4. $\vdash A \Rightarrow A \in \Gamma$; 5. $A, A^\circ \in \Gamma \Rightarrow \neg A \notin \Gamma$; $\neg A, A^\circ \in \Gamma \Rightarrow A \notin \Gamma$; 6. $A, A \supset B \in \Gamma \Rightarrow B \in \Gamma$; 7. $A^\circ \in \Gamma \Rightarrow A \notin \Gamma$ or $\neg A \notin \Gamma$; 8. $A^\circ \in \Gamma \Rightarrow (\neg A)^\circ \in \Gamma$.

Proof: We shall prove only three of the above properties:

1. $\Gamma \vdash A \Leftrightarrow A \in \Gamma$:

Let us suppose that $\Gamma \vdash A$, but that $A \notin \Gamma$. Then, since Γ is maximal, $\Gamma \cup \{A\} \vdash A \,\&\, \neg^* A$. Hence, $\Gamma \vdash A \supset (A \,\&\, \neg^* A)$, and $\Gamma \vdash \neg^* A$. But, taking into account that $\Gamma \vdash A$, it follows that $\Gamma \vdash A \,\&\, \neg^* A$; therefore, Γ would be trivial, which is absurd.

3. $A \in \Gamma$ or $\neg^* A \in \Gamma$:

If $A \notin \Gamma$ and $\neg^* A \notin \Gamma$, then $\Gamma \cup \{A\} \vdash B \,\&\, \neg^* B$ and $\Gamma \cup \{\neg^* A\} \vdash B \,\&\, \neg^* B$. Consequently, $\Gamma \cup \{A \vee \neg^* A\} \vdash B \,\&\, \neg^* B$, and Γ would be trivial.

7. $A^\circ \in \Gamma \Rightarrow A \notin \Gamma$ or $\neg A \notin \Gamma$:

Let us admit that $A^\circ \in \Gamma$ and that $A, \neg A \in \Gamma$. Then, $\Gamma \vdash A \,\&\, \neg A$; but, since $\Gamma \vdash A^\circ$, it results that $\Gamma \vdash A \,\&\, \neg^* A$, and Γ would again be trivial.

Definition 5 A valuation of $\mathbf{C}_1$ is a function v: $\mathcal{F} \to \{0, 1\}$ such that:

1. $v(A) = 0 \Rightarrow v(\neg A) = 1$; 2. $v(\neg\neg A) = 1 \Rightarrow v(A) = 1$; 3. $v(B^\circ) = v(A \supset B) = v(A \supset \neg B) = 1 \Rightarrow v(A) = 0$; 4. $v(A \supset B) = 1 \Leftrightarrow v(A) = 0$ or $v(B) = 1$; 5. $v(A \,\&\, B) = 1 \Leftrightarrow v(A) = v(B) = 1$; 6. $v(A \vee B) = 1 \Leftrightarrow v(A) = 1$ or $v(B) = 1$; 7. $v(A^\circ) = v(B^\circ) = 1 \Rightarrow v((A \vee B)^\circ) = v((A \,\&\, B)^\circ) = v((A \supset B)^\circ) = 1$.

Theorem 2 *If v is a valuation of $\mathbf{C}_1$, then v has the following properties:*

$v(A) = 1 \Leftrightarrow v(\neg^* A) = 0$; $v(A) = 0 \Leftrightarrow v(\neg^* A) = 1$; $v(A) = 0 \Leftrightarrow v(A) = 0$ and $v(\neg A) = 1$; $v(A) = 1 \Leftrightarrow v(A) = 1$ or $v(\neg A) = 0$.

Proof: Immediate.

Definition 6 A valuation v is called singular if there exists at least one formula A such that $v(A) = v(\neg A) = 1$. Otherwise, v is said to be normal.

Definition 7 A formula A is valid if, for every valuation v, $v(A) = 1$.

Definition 8 A valuation v is a model of a set Γ of formulas, if $v(A) = 1$ for any element A of Γ.

Definition 9 If, for any model v of Γ, we have $v(A) = 1$, we say that A is a semantical consequence of Γ, and we write $\Gamma \vDash A$. In particular, $\vDash A$ is an abbreviation of $\emptyset \vDash A$, and this means that A is valid.

Theorem 3 $\Gamma \vdash A \Rightarrow \Gamma \vDash A$.

Proof: By induction on the length of a deduction of A from Γ.

Corollary $\vdash A \Rightarrow \vDash A$.

Lemma 1 *Every nontrivial set of formulas is contained in a maximal nontrivial set.*

Proof: By an obvious adaptation of the proof of the corresponding classical theorem.

Corollary *There are maximal nontrivial inconsistent sets.*

Proof: It is easy to see that $\{p, \neg p\}$ is an inconsistent but nontrivial set. Hence, by the preceding theorem, it is contained in a maximal nontrivial set, which is inconsistent.

Lemma 2 *Every maximal nontrivial set Γ of formulas has a model.*

Proof: We define the function v: $\mathcal{F} \rightarrow \{0, 1\}$ as follows: for every formula A, if $A \in \Gamma$, we put $v(A) = 1$; otherwise, $v(A) = 0$. Then, we prove that v satisfies all conditions of the definition of valuation.

Corollary 1 *Any nontrivial set of formulas has a model.*

Corollary 2 *There are singular valuations (and, of course, also normal valuations).*

Proof: $\{p, \neg p\}$ is inconsistent but nontrivial. Therefore, this set is contained in a maximal nontrivial set, which has a model v; but obviously v is singular.

Theorem 4 $\Gamma \vDash A \Rightarrow \Gamma \vdash A$.

Proof: If $\Gamma \vDash A$, then, for every valuation v, which is a model of Γ, we have that $v(A) = 1$. Hence, by Theorem 2, there is no valuation v, which is a model of Γ and such that $v(\neg^* A) = 1$. Then, $\Gamma \cup \{\neg^* A\}$ does not have a model. But, by Lemmas 1 and 2, any nontrivial set has a model. Consequently, $\Gamma \cup \{\neg^* A\}$ is trivial, and $\Gamma \cup \{\neg^* A\} \vdash \neg^*\neg^* A$. Since $\Gamma \cup \{A\} \vdash \neg^*\neg^* A$, and $\neg^*$ behaves like the classical negation, we have $\Gamma \cup \{\neg^* A \vee A\} \vdash \neg^*\neg^* A$ and $\Gamma \vdash \neg^*\neg^* A$. Finally, it follows that $\Gamma \vdash A$.

Corollary 1 $\vDash A \Rightarrow \vdash A$.

Corollary 2 $\Gamma \vDash A \Leftrightarrow \Gamma \vdash A$.

Theorem 5 *There are inconsistent (but nontrivial) sets of formulas which have models.*

Remarks: 1. The first (or the second) property of Theorem 2 implies conditions 1 and 3 of the definition of valuation (Definition 5); 2. The value of a valuation v for an arbitrary formula is not in general determined by the values of v for the propositional variables.

Definition 10 Let Δ be the set $\{A^\circ \in \mathcal{F}: \vdash A\}$. Γ is said to be strongly nontrivial, if $\Gamma \cup \Delta$ is not trivial. Let now Δ denote the set $\{A^\circ \in \mathcal{F}: A$ is not a propositional variable$\}$; Γ is said to be strictly nontrivial, if $\Gamma \cup \Delta$ is not trivial.

Theorem 6 *There exist sets of formulas which are strongly nontrivial and sets of formulas which are strictly nontrivial.*

Proof: We shall prove only the first part of the theorem, i.e., that there are strongly nontrivial sets of formulas. In effect, if Δ is the set $\{A^\circ \in \mathcal{F}: \vdash A\}$, then Δ is consistent. But this implies that Δ is also nontrivial. Hence, Δ is contained in a maximal nontrivial set Δ'. Let Δ'' be the set $\Delta' - \Delta$. Δ'' is evidently a strongly nontrivial set of formulas.

3 *The decidability of* $\mathbf{C}_1$ By means of the above semantics for $\mathbf{C}_1$, we can obtain as a byproduct a decision method for that calculus. This will be the objective of the present section of this paper.

Definition 11 (of quasi-matrix) For each formula of $\mathbf{C}_1$ we can construct tables according to the instructions below, which we shall call *quasi-matrices*.

In order to construct a quasi-matrix for a formula A, the procedure is as follows:

1. Make a list of all the propositional variables which occur in A, and arrange them in a line.
2. Under the list of the propositional variables, place in successive lines all the possible combinations of 0's and 1's which can be attributed to these variables.
3. Then make a list of all the negations of propositional variables and

calculate their value, in each line, as follows: if a variable was given value 0, the negation gets value 1. If a variable was given value 1, bifurcate the line in which that occurred, writing in the first part value 0 for the negation, and, in the second part, value 1 for the negation. Every time there is a bifurcation, the values are the same for the two lines in the part on the left of it.

4. Make a list and calculate, for each line, the value of each subformula of A and, if it is a proper subformula, of its negation, whose proper subformulas and their negations had already been listed and calculated, as follows:

(i) When no negations are involved proceed as in a truth-table for the classical propositional calculus;

(ii) If any of the formulas under consideration is a negation and so of the form $\neg A'$, write value 1 under it, on the lines in which A' has value 0. On the lines in which A' has value 1, proceed as follows:

(1°) If A' is of the form $\neg B$, check if the value of B is equal to the value of $\neg B$. If that is the case, bifurcate the line, writing the value 0 in the first part and, in the second, the value 1. If the value of B is different from the value of $\neg B$, simply write the value 0.

(2°) If A' is of the form $B \S C$, where § is $\supset$, $\vee$, or &, there are two cases to be considered:

a. A' is of the form $D \& \neg D$, or of the form $\neg D \& D$. In this case, write the value 0 for the formula $\neg A'$.

b. A' is not of the form $D \& \neg D$, or of the form $\neg D \& D$. In this case, check if the value of B is equal to the value of $\neg B$, or if the value of C is equal to the value of $\neg C$. If this is true, bifurcate the line, writing the value 0 in the first part and, in the second, the value 1. If, on the contrary, the value of B is different from the value of $\neg B$, and the value of C is different from the value of $\neg C$, simply write the value 0.

Lemma 3 $v: \mathcal{J} \to \{0, 1\}$ *is a valuation if and only if:*

1. $v(\neg A) = 0 \Rightarrow v(A) = 1$,
2. $v(\neg\neg A) = 1 \Rightarrow v(A) = 1$,
3. $v(B^\circ) = v(A \supset B) = v(A \supset \neg B) = 1 \Rightarrow v(A) = 0$,
4. $v(A \supset B) = 1 \Leftrightarrow v(A) = 0$ or $v(B) = 1$,
5. $v(A \& B) = 1 \Leftrightarrow v(A) = v(B) = 1$,
6. $v(A \vee B) = 1 \Leftrightarrow v(A) = 1$ or $v(B) = 1$,
7. $v((A \& B)^\circ) = 0 \Rightarrow v(A^\circ) = 0$ or $v(B^\circ) = 0$,

7′. $v((A \& B)^\circ) = 0 \Rightarrow v(A^\circ) = 0$ or $v(B^\circ) = 0$,

7″. $v((A \vee B)^\circ) = 0 \Rightarrow v(A^\circ) = 0$ or $v(B^\circ) = 0$.

Lemma 4 $v(A^\circ) = 0 \Leftrightarrow v(A) = v(\neg A) = 1$.

Proof: (a) $v(A^\circ) = 0 \Rightarrow v(A \& \neg A) = 1 \Rightarrow v(A) = v(\neg A) = 1$. (b) Suppose that $v(A) = v(\neg A) = 1$; if $v(A^\circ) = 1$, then $v(A) = v(\neg A) = v(A^\circ) = 1$, that is, $v(A) = v(\neg^* A) = 1$, and v would not be a valuation. Hence $v(A^\circ) = 0$. Therefore, $v(A) = v(\neg A) = 1 \Rightarrow v(A^\circ) = 0$.

Lemma 5 *v: $\mathcal{F} \to \{0, 1\}$ is a valuation if and only if the conditions* 1-6 *of* Lemma 3 *are present and:*

7i. $v((A \supset B)^\circ) = 0 \Rightarrow v(A) = v(\neg A) = 1$ or $v(B) = v(\neg B) = 1$,
7ii. $v((A \,\&\, B)^\circ) = 0 \Rightarrow v(A) = v(\neg A) = 1$ or $v(B) = v(\neg B) = 1$,
7iii. $v((A \vee B)^\circ) = 0 \Rightarrow v(A) = v(\neg A) = 1$ or $v(B) = v(\neg B) = 1$.

Definition 12 Let v be a valuation and let F be a formula. Then, v_F is the restriction of v to the set of subformulas of F and negations of proper subformulas of F.

Lemma 6 *For every valuation v and for every formula F, $v(F) = v_F(F)$.*

Definition 13 Let v be a valuation and Γ be a set of formulas. Then, v_Γ is the restriction of v to the set Γ.

Definition 14 We say that a line of a quasi-matrix *corresponds* to v_Γ, if $v_\Gamma(A)$ is the value corresponding to A in that line, for every $A \in \Gamma$, where Γ is the set of formulas of the table.

Lemma 7 *Given a quasi-matrix $\mathfrak{Q}$, for every valuation v there is a line of $\mathfrak{Q}$ which corresponds to v_Γ, where Γ is the set of formulas of $\mathfrak{Q}$.*

Proof: By induction on the number of columns of $\mathfrak{Q}$.

Definition 15 Let $\mathfrak{Q}$ be a quasi-matrix for a formula A and let Γ be the set of subformulas and negations of proper subformulas of A. Let k be a line of that quasi-matrix and $k(F)$ be the value *atributed* to F in k. Then, $\Delta(\Gamma, k)$ is a set of formulas such that, for every formula F,

I. If $F \in \Gamma$, then $F \in \Delta(\Gamma, k)$ iff $k(F) = 0$.
II. If $F \notin \Gamma$, then $F \in \Delta(\Gamma, k)$ iff:

a. F is atomic; or
b. $F = \neg F_1$ and $F_1 \notin \Delta(\Gamma, k)$; or
c. $F = F_1 \,\&\, F_2$ and $F_1 \in \Delta(\Gamma, k)$ or $F_2 \in \Delta(\Gamma, k)$; or
d. $F = F_1 \vee F_2$ and $F_1 \in \Delta(\Gamma, k)$ and $F_2 \in \Delta(\Gamma, k)$; or
e. $F = F_1 \supset F_2$ and $F_1 \notin \Delta(\Gamma, k)$ and $F_2 \in \Delta(\Gamma, k)$.

Some properties of the sets $\Delta(\Gamma, k)$:

1. $\neg A \in \Delta(\Gamma, k) \Rightarrow A \notin \Delta(\Gamma, k)$,
2. $A \in \Delta(\Gamma, k) \Rightarrow \neg\neg A \in \Delta(\Gamma, k)$,
3. $\neg^* A \in \Delta(\Gamma, k) \Leftrightarrow A \notin \Delta(\Gamma, k)$,
4. $A \supset B \notin \Delta(\Gamma, k) \Leftrightarrow A \in \Delta(\Gamma, k)$ or $B \notin \Delta(\Gamma, k)$,
5. $A \in \Delta(\Gamma, k)$ or $B \in \Delta(\Gamma, k) \Leftrightarrow A \,\&\, B \in \Delta(\Gamma, k)$,
6. $A \notin \Delta(\Gamma, k)$ or $B \notin \Delta(\Gamma, k) \Leftrightarrow A \vee B \notin \Delta(\Gamma, k)$,
7. $(A \,\S\, B)^\circ \in \Delta(\Gamma, k) \Rightarrow A \notin \Delta(\Gamma, k)$ and $\neg A \notin \Delta(\Gamma, k)$, or $B \notin \Delta(\Gamma, k)$ and $\neg B \notin \Delta(\Gamma, k)$ (where § is &, $\vee$, or $\supset$).

Lemma 8 (A. Loparić) *For every line k of a quasi-matrix $\mathfrak{Q}$, there is a valuation v, such that v_Γ corresponds to k, where Γ is the set of formulas of $\mathfrak{Q}$.*

Proof: Let f be the function $\mathcal{F} \to \{0, 1\}$ such that, for every $A \in \mathcal{F}, f(A) = 0$, if $A \in \Delta(\Gamma, k)$, and $f(A) = 1$ if $A \notin \Delta(\Gamma, k)$. Then, by the properties 1-7 of the sets $\Delta(\Gamma, k)$, f is a valuation v. Since v_Γ and k "coincide", there is a valuation v, such that v_Γ corresponds to k.

Theorem 7 (M. Fidel) $\mathbf{C}_1$ *is decidable.*

Proof: Consequence of Lemmas 6, 7, and 8: the formula A is a theorem of $\mathbf{C}_1$ if and only if in any quasi-matrix for A the last column contains only 1's; in effect, in this case we have for any valuation v: $v(A) = v_A(A) = 1$.

Examples:

1. $\neg(A \vee B) \supset \neg A \,\&\, \neg B$ is not valid in $\mathbf{C}_1$:

<table>
<tr><th>A</th><th>B</th><th>$\neg A$</th><th>$\neg B$</th><th>$A \vee B$</th><th>$\neg(A \vee B)$</th><th>$\neg A \,\&\, \neg B$</th><th>$\neg(A \vee B) \supset \neg A \,\&\, \neg B$</th></tr>
<tr><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">1</td><td>0</td><td>1</td><td>1</td></tr>
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<tr><td rowspan="7">1</td><td rowspan="7">1</td><td rowspan="3">0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
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<tr><td rowspan="4">1</td><td rowspan="2">0</td><td rowspan="2">1</td><td>0</td><td>0</td><td>1</td></tr>
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2. $\neg(A \,\&\, B) \supset \neg A \vee \neg B$ is valid in $\mathbf{C}_1$:

<table>
<tr><th>A</th><th>B</th><th>$\neg A$</th><th>$\neg B$</th><th>$A \,\&\, B$</th><th>$\neg(A \,\&\, B)$</th><th>$\neg A \vee \neg B$</th><th>$\neg(A \,\&\, B) \supset \neg A \vee \neg B$</th></tr>
<tr><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
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4 *The calculi* $\mathbf{C}_n$, $1 \leqslant n < \omega$ The calculi $\mathbf{C}_n$, $1 \leqslant n < \omega$, have the same language as that of $\mathbf{C}_1$. It is convenient to abbreviate $A^{\circ\circ\cdots\circ}$, where the symbol $^\circ$ appears n times, $n \geqslant 1$, by A^n, and $A^1 \,\&\, A^2 \,\&\, \ldots \,\&\, A^n$ by $A^{(n)}$.

The postulates of $\mathbf{C}_n$, $1 \leqslant n < \omega$, are those of $\mathbf{C}_1$, excepting the postulates (12)-(15), which are replaced by the following:

(12′) $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$,
(13′) $A^{(n)} \,\&\, B^{(n)} \supset (A \,\&\, B)^{(n)}$,
(14′) $A^{(n)} \,\&\, B^{(n)} \supset (A \vee B)^{(n)}$,
(15′) $A^{(n)} \,\&\, B^{(n)} \supset (A \supset B)^{(n)}$.

The extension of the semantics of $\mathbf{C}_1$ to the systems $\mathbf{C}_n$, $1 \leqslant n < \omega$, is immediate. All definitions and theorems are the same, only with evident modifications as regards the strong negation (for example, $\neg^* A$ becomes $\neg^{(n)} A$, which is an abbreviation of $\neg A \,\&\, A^{(n)}$), and the symbol $^\circ$ (for example, A° becomes $A^{(n)}$).

In the construction of the quasi-matrices, the only change worth mentioning is that when A is $D^{n-1} \,\&\, \neg D^{n-1}$ or $\neg D^{n-1} \,\&\, D^{n-1}$, we write value 0 for the formula $\neg A$. In fact, we must have $v(\neg A) = 0$, since, in the opposite case, we would have, by virtue of clause 7 of the definition of valuation, that $v(D \,\&\, \neg^{(n)} D) = 1$ and v would not be a valuation. This clause is exactly the one that characterizes the quasi-matrices of the system $\mathbf{C}_n$, $1 \leqslant n < \omega$.

Example The schema $(A^{n-1} \,\&\, \neg A^{n-1})^{(n)}$ is valid in $\mathbf{C}_n$, but not in $\mathbf{C}_m$, $m > n$. We shall show, using the quasi-matrices, that the schema $(A \,\&\, \neg A)^\circ$ is valid in $\mathbf{C}_1$ but not in $\mathbf{C}_2$.

In $\mathbf{C}_1$:

A	$\neg A$	$A \,\&\, \neg A$	$\neg(A \,\&\, \neg A)$	$(A \,\&\, \neg A) \,\&\, \neg(A \,\&\, A)$	$\neg((A \,\&\, \neg A) \,\&\, \neg(A \,\&\, \neg A))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1

In $\mathbf{C}_2$:

A	$\neg A$	$A \,\&\, \neg A$	$\neg(A \& \neg A)$	$(A \,\&\neg A) \,\&\, \neg(A \,\&\, \neg A)$	$\neg((A \,\&\, \neg A) \,\&\, \neg(A \,\&\, \neg A))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1
			1	1	0
					1

5 *Modal calculi based on* $\mathbf{C}_1$ We can build modal calculi from calculi $\mathbf{C}_n$, $1 \leqslant n < \omega$, exactly in the same way as systems of modal logic are built from the classical propositional calculus. We shall deal only with the construction of modal calculi associated with $\mathbf{C}_1$, namely, $\mathbf{C}_1$T, $\mathbf{C}_1 S_4$, $\mathbf{C}_1$B, and $\mathbf{C}_1 S_5$, which are the calculi corresponding to the systems T of Feys-von Wright, S4 of Lewis, the Browerian and S5 of Lewis.

The primitive symbols of the modal calculi we are going to present are exactly the same as the ones of $\mathbf{C}_1$ to which we add the monadic operator of necessity ($\Box$). The symbol $\Diamond$ is defined as follows:

$$\Diamond A =_{df} \neg^* \Box \neg^* A$$

The other modal operators are similarly defined. Formula is defined as usual. The postulates (axiom schemata and deduction rules) of $\mathbf{C}_1$T are the same as those of $\mathbf{C}_1$ plus the following:

$$\Box A \supset A$$
$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$
$$A^\circ \supset (\Box A)^\circ$$
$$A/\Box A$$

The postulates of $\mathbf{C}_1S_4$ are the same as those of $\mathbf{C}_1$T and:

$$\Box A \supset \Box\Box A$$

The postulates of $\mathbf{C}_1$B are the same as those of $\mathbf{C}_1$T and:

$$A \supset \Box\Diamond A$$

The postulates of $\mathbf{C}_1S_5$ are the same as those of $\mathbf{C}_1$T and:

$$\Diamond A \supset \Box\Diamond A$$

From the semantic point of view, we can adapt, in a convenient way, our method of section 2, using the technique of Kripke's models (see Hughes & Cresswell [3], chapter 4), hence getting semantics which will be appropriate to the calculi described above.

The relation between the systems $\mathbf{C}_1$T, $\mathbf{C}_1S_4$, $\mathbf{C}_1$B, and $\mathbf{C}_1S_5$ is the same as the one between their corresponding systems T, S_4, B and S_5. As to the modal calculi associated with $\mathbf{C}_n$, $1 < n < \omega$, they are totaly analogous to the calculi we obtain starting from $\mathbf{C}_1$.

Notice that the modal systems based on $\mathbf{C}_n$, $1 \leqslant n < \omega$, have the fundamental properties of these latter systems. In fact, if we use the semantic methods indicated, it would not be difficult to show, for example, that the schema $A \,\&\, \urcorner A \supset B$ is not valid in $\mathbf{C}_1$T. We also note that one of the paradoxes of strict implication, that is, the schema $\Box\urcorner A \supset \Box(A \supset B)$, is not valid in $\mathbf{C}_1$T, although it is valid in T. This shows that the new systems have interesting properties. Nevertheless, we shall not go into the matter in detail, since we have the intention of developing it in future works.

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