Notre Dame Journal of Formal Logic
Volume XVIII, Number 4, October 1977
NDJFAM

CRITICAL POINTS OF NORMAL FUNCTIONS. I.

JOHN L. HICKMAN

In this paper* we are interested in obtaining some idea of the conditions that must be imposed upon a class of ordinals in order that it be the class of critical points of some normal function. Exact criteria appear to be not so easy to obtain, and in the main this paper is devoted to determining necessary conditions and to exhibiting some examples that may (hopefully) shed some light on the general problem. The paper concludes with a few results on the "algebraic" character of **CR**, the collection of all classes of critical points of normal functions. We have consistently sinned against "ordinary" set theories by speaking about collections of proper classes, functions with proper classes as domains, etc., as if such objects exist within the theory. We feel that the intuitive clarity thereby gained, together with the fact that all of our results could be formalized within an appropriate extension of such a theory, provides sufficient justification for this attitude. When constructing our proofs, we have had **ZFC** in mind, but any of the usual set theories with Choice would do as well.

Generally, lower-case Greek letters denote arbitrary ordinals, the finite ordinals being denoted by "m", "n", . . ., "t". Ordinals are assumed to be defined in such a way that each is the set of all smaller ones. ON is the class of all ordinals, ω is the first transfinite ordinal, and ω_α is the $(\alpha + 1)$th initial ordinal. The aleph corresponding to ω_α is denoted by "$\aleph_\alpha$". Cardinality (of a set) is denoted by "$|\,|$". A function will always be taken to be a map $f: X \to Y$, with $X, Y \subseteq \mathrm{ON}$. Functions are denoted by "f", "g", "h". Frequently we shall make assumptions concerning the extent of the domain, $\mathrm{dom}(f)$, and range, $\mathrm{ran}(f)$ of a function f. These assumptions will always be made explicit in definitions, but shall remain tacit elsewhere if the context makes them clear.

Definition 1 $\mathrm{LIM} = \{\alpha; \sim\exists\beta(\alpha = \beta + 1)\}$; $\mathrm{LIM}^* = \mathrm{LIM} - \{0\}$.

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*This paper was written while the author held a Research Fellowship at the Australian National University.

Received January 14, 1975

SUC $=$ ON $-$ LIM.
INT $= (\{\alpha;\ \forall\beta < \alpha(|\beta| < |\alpha|)\}) - \omega, = \{\omega_\alpha;\ \alpha \in \mathrm{ON}\}$.

Definition 2 The cofinality function, *cf*.
$\mathrm{dom}(\mathrm{cf}) = \mathrm{ON} - \{0\}$.
$\mathrm{cf}(\alpha) = \min\{\beta\colon \exists f(f\colon \beta \to \alpha\ \&\ \alpha = \bigcup(f''\beta))\},\ \alpha \in \mathrm{dom}(\mathrm{cf})$.

Definition 3 REG $= \mathrm{cf}''\mathrm{LIM}^*$; SIN $=$ INT-REG.

We assume as known the following results:

(i) $\mathrm{cf}^2 = \mathrm{cf}$.
(ii) REG $\subseteq$ INT.
(iii) $\forall\alpha(\omega_{\alpha+1} \in \mathrm{REG})$.

(i) is easy enough to prove; and the proofs of (ii), (iii) are given respectively in [2], p. 281 and [2], p. 285. Choice is required for the proof of (iii).

Definition 4 Let f be a function.

(i) f is nondecreasing if $\forall\alpha, \beta(\alpha \leqslant \beta \Rightarrow f(\alpha) \leqslant f(\beta))$. f is increasing if $\forall\alpha, \beta(\alpha < \beta \Rightarrow f(\alpha) < f(\beta))$.
(ii) For $\alpha \in \mathrm{LIM}$, f is continuous at α if for every increasing sequence $(\tau_\xi)_{\xi<\lambda}$, $\lambda \in \mathrm{LIM}$, such that $\lim_{\xi<\lambda}\tau_\xi = \alpha$, we have $\sup\{f(\tau_\xi);\ \xi < \lambda\} = f(\alpha)$, $\alpha + 1 \subseteq \mathrm{dom}(f)$. f is continuous if f is continuous at α for all $\alpha \in \mathrm{LIM} \cap \mathrm{dom}(f)$.
(iii) f is normal if f is increasing and continuous.
(iv) f is semi-normal if f is nondecreasing, continuous, and if $\forall\alpha(\alpha \leqslant f(\alpha))$.

It is easy to show that:

(i) Every normal function is semi-normal.
(ii) Not every semi-normal function is normal.
(iii) Not every nondecreasing, continuous function is semi-normal.

Definition 5 Let f be a function.
$\mathbf{CR}_f = \{\alpha\colon f(\alpha) = \alpha\}$.
α is called a "critical point" of f is $\alpha \in \mathbf{CR}_f$.

Our main interest in this paper lies with the classes $\mathbf{CR}_f$ for normal f; since, however, it is sometimes easier to work with semi-normal functions rather than normal functions, it is natural to ask whether we could obtain the same results by looking at the classes $\mathbf{CR}_f$ for semi-normal f—in other words, whether for each normal f there is a semi-normal g such that $\mathbf{CR}_f = \mathbf{CR}_g$. As we shall see later on in this paper, unfortunately this is not always the case.

Definition 6 Let f be a semi-normal function with $\mathrm{ran}(f) \subseteq \mathrm{dom}(f)$. The function C_f:

$$\mathrm{dom}(\mathrm{C}_f) = \mathrm{dom}(f), \text{ and } \mathrm{C}_f(\alpha) = \lim_{n<\omega} f^n(\alpha),\ \alpha \in \mathrm{dom}(\mathrm{C}_f).$$

Lemma *Let $\{\tau_{ij}\}_{i\in I, j\in J}$ be a doubly indexed set of ordinals. Then:*

$$\sup\{\sup\{\tau_{ij};\ i \in I\};\ j \in J\} = \sup\{\sup\{\tau_{ij};\ j \in J\};\ i \in I\}.$$

Proof: Routine calculation of sups.

Theorem 1 *Let f be semi-normal. Then:*

(i) C_f *is semi-normal.*
(ii) C_f *is normal if and only if f is the identity function ι.*

Proof: (i) Since f is semi-normal, we have $f^n(\alpha) \leqslant f^{n+1}(\alpha)$ for all α and n, and thus $\alpha \leqslant \mathrm{C}_f(\alpha)$. Also, for $\alpha \leqslant \beta$, we have $f^n(\alpha) \leqslant f^n(\beta)$ for all n, and so $\mathrm{C}_f(\alpha) \leqslant \mathrm{C}_f(\beta)$. It remains to check continuity. Take $\lambda \in \mathrm{LIM}$, let (τ_ξ) be an increasing λ-sequence, and put $\alpha = \lim_{\xi<\lambda}\tau_\xi$. The double sequence $(f^n(\tau_\xi))_{n<\omega,\,\xi<\lambda}$ is such that for each $m < \omega$ the λ-sequence $(f^m(\tau_\xi))_{\xi<\lambda}$ is nondecreasing, as is the ω-sequence $(f^n(\tau_\zeta))_{n<\omega}$ for each $\zeta < \lambda$. Therefore, our lemma tells us that

$$\lim_{\xi<\lambda}\mathrm{C}_f(\tau_\xi) = \lim_{\xi<\lambda}\lim_{n<\omega} f^n(\tau_\xi) = \lim_{n<\omega}\lim_{\xi<\lambda} f^n(\tau_\xi) = \lim_{n<\omega} f^n(\alpha) = \mathrm{C}_f(\alpha).$$

(ii) Suppose that $f = \iota$. Then obviously $\mathrm{C}_f = \iota$, and so C_f is normal. Now assume that $f(\beta) \neq \beta$ for some β; since f is semi-normal, we must have $f(\beta) > \beta$. Thus we obtain $\mathrm{C}_f(\beta) > \beta$. However, we also have

$$f(\mathrm{C}_f(\beta)) = f(\lim_{n<\omega} f^n(\beta)) = \lim_{n<\omega} f^{n+1}(\beta) = \mathrm{C}_f(\beta),$$

whence it follows that $\mathrm{C}_f(\mathrm{C}_f(\beta)) = \mathrm{C}_f(\beta)$. Therefore, C_f is not increasing, and so not normal.

Theorem 2 *Let f be semi-normal. Then:*

(i) $\mathrm{C}_f''\mathrm{ON} = \mathbf{CR}_f = \mathbf{CR}_{\mathrm{C}_f}$.
(ii) $\forall\alpha(\alpha \notin \mathbf{CR}_f \Rightarrow \mathrm{C}_f(\alpha) = \min\{\beta \in \mathbf{CR}_f;\ \beta > \alpha\})$.

Proof: The proof of this result can be found (substantially) in most texts; see, e.g., [2], p. 237.

(i) We have already seen in the proof of Theorem 1 (ii) that $f(\mathrm{C}_f(\alpha)) = \mathrm{C}_f(\alpha)$ for all α, i.e., $\mathrm{C}_f''\mathrm{ON} \subseteq \mathbf{CR}_f$. But if $\alpha \in \mathbf{CR}_f$, then clearly $f^n(\alpha) = \alpha$ for all n, and so $\mathrm{C}_f(\alpha) = \alpha$. This shows that $\mathrm{C}_f''\mathrm{ON} = \mathbf{CR}_f \subseteq \mathbf{CR}_{\mathrm{C}_f}$, while the one remaining inclusion follows from the fact that if $\alpha \in \mathbf{CR}_{\mathrm{C}_f}$, then $\alpha \leqslant f(\alpha) \leqslant \mathrm{C}_f(\alpha) = \alpha$.

(ii) Take $\alpha \notin \mathbf{CR}_f$; then $\mathrm{C}_f(\alpha) \geqslant f(\alpha) > \alpha$, and so from (i) we see that $\{\beta \in \mathbf{CR}_f;\ \beta > \alpha\} \neq \emptyset$, and hence $\gamma = \min\{\beta \in \mathbf{CR}_f;\ \beta > \alpha\}$ is well-defined. From (i) again we have $\gamma \leqslant \mathrm{C}_f(\alpha)$; on the other hand, since it is clear that $f^n(\beta) \leqslant \gamma$ for all $\beta < \gamma$ and all n, we must have $\mathrm{C}_f(\alpha) \leqslant \gamma$.

Theorem 2 is the basic result in this field, and shows that for any semi-normal f (with $\mathrm{dom}(f) = \mathrm{ON}$), $\mathbf{CR}_f$ is a proper class. If we drop condition (c) and require merely that f be nondecreasing and continuous, then we can show that $\mathbf{CR}_f \neq \emptyset$, but not much else. In fact, given any nonzero cardinal Γ (finite or infinite), it is not difficult to construct a nondecreasing continuous function f with $|\mathbf{CR}_f| = \Gamma$.

Despite the fact that some of the following results require only semi-normality, we shall in future confine our attention to normal functions. Obviously for any normal f we have $\mathsf{C}_f''(\mathsf{ON} - \mathbf{CR}_f) \subseteq \mathsf{LIM}$; equally obviously, some normal functions have nonlimit critical points. The following simple result presents itself immediately.

Theorem 3 *Put* $\omega^* = \omega - \{0\}$, *and let* f *be normal. We have* $\mathsf{LIM} \subseteq \mathbf{CR}_f$ *if and only if for each* $\alpha \in \mathsf{LIM}^*$ *there is an increasing function* $f_\alpha: \omega^* \to \omega^*$ *such that* $f(\alpha + n) = \alpha + f_\alpha(n)$, $n \in \omega^*$.

Proof: Trivial.

Using this result, we see that there are normal functions f for which $\mathbf{CR}_f = \mathsf{LIM}$; we need only take these maps f_α in such a way that $f_\alpha(1) \neq 1$. On the other hand, a simple continuity argument shows that there is no normal f with $\mathbf{CR}_f = \mathsf{SUC}$.

Since we have $\mathsf{INT} \subset \mathsf{LIM}$, it is clear that there are normal functions f for which $\mathsf{INT} \subset \mathbf{CR}_f$. We now give an example of a normal function d for which $\mathsf{INT} \subset \mathbf{CR}_\mathsf{d}$ but $\mathsf{LIM} \not\subseteq \mathbf{CR}_\mathsf{d}$, and require a preliminary result that appears to be a slight improvement on a theorem given by Kuratowski and Mostowski [2], Theorem 9, p. 281.

Theorem 4 *Let* e *be the (normal) function defined by* $\mathsf{e}(\alpha) = \omega^\alpha$.

Then $\mathsf{INT} - \{\omega\} \subset \mathbf{CR}_\mathsf{e}$.

Proof: Take $\rho = \omega_\alpha$, $\alpha > 0$. We shall show that $\mathsf{e}(\gamma) < \rho$ for all $\gamma < \rho$. Given this, by normality we shall have $\mathsf{e}(\rho) = \lim_{\xi<\rho}\mathsf{e}(\xi) = \rho$. Thus, let $\gamma < \rho$ be given, and define the set Z by

$$Z = \{f: \gamma \to \omega: |\gamma - f^{-1}(\{0\})| < \aleph_0\}.$$

Clearly Z can be partitioned into subsets Z_n, $n < \omega$, where

$$Z_n = \{f \in Z: |\gamma - f^{-1}(\{0\})| = n\}.$$

Now if we put $|\gamma| = \aleph_\beta$, then an easy calculation gives

$$|Z_n| = (\aleph_0)^n(\aleph_\beta)^n = \aleph_\beta, \text{ whence } |Z| = \aleph_0\aleph_\beta = \aleph_\beta.$$

But of course antilexicographical ordering of Z yields the ordinal ω^γ; thus $|\omega^\gamma| = \aleph_\beta$. Since $\gamma < \rho = \omega_\alpha$, we must have $\beta < \alpha$, and so we conclude that $\mathsf{e}(\gamma) = \omega^\gamma < \rho$, as desired. Finally, to show that $\mathsf{INT} - \{\omega\} \neq \mathbf{CR}_\mathsf{e}$, we merely observe that $\omega < \varepsilon_0 = \mathsf{C}_\mathsf{e}(0) < \omega_1$.

We now consider the normal function d defined by $\mathsf{d}(\alpha) = 2^\alpha$. We have $\mathsf{d}(\omega) = \omega$, while for $\rho \in \mathsf{INT} - \{\omega\}$ we have $\rho \leqslant \mathsf{d}(\rho) \leqslant \omega^\rho = \rho$. Thus $\mathsf{INT} \subseteq \mathbf{CR}_\mathsf{d}$: $\mathsf{INT} \neq \mathbf{CR}_\mathsf{d}$, since $\mathsf{C}_\mathsf{d}(\omega + 1) = \varepsilon_0$; and $\mathsf{LIM} \not\subseteq \mathbf{CR}_\mathsf{d}$, since $\mathsf{d}(\omega 2) = \omega^2 > \omega 2$. Now since $\mathsf{REG} \subset \mathsf{INT} \subset \mathbf{CR}_\mathsf{d}$, we see that d has critical points of every possible infinite cofinality. Our aim is to show that this is not an isolated phenomenon, but is in fact a property possessed by every normal function.

Lemma *Take* $\lambda \in \mathsf{LIM}^*$, *let* (τ_ξ) *be an increasing* λ-*sequence, and put* $\alpha = \lim_{\xi<\lambda}\tau_\xi$. *Then* $\mathsf{cf}(\alpha) = \mathsf{cf}(\lambda)$.

Proof: Trivial. Put $X = \{\tau_\xi: \xi < \lambda\}$. Then X is a cofinal subset of α, with order-type λ.

Corollary (Bukovský, [1]) *Take* $\rho \in \mathrm{REG}$, *let* (τ_ξ) *be an increasing* ρ-*sequence, and put* $\alpha = \lim_{\xi<\rho}\tau_\xi$. *Then* $\mathrm{cf}(\alpha) = \rho$.

Theorem 5 $\forall f$ (*f normal* $\Rightarrow \mathrm{cf}''(\mathrm{LIM}^* \cap \mathbf{CR}_f) = \mathrm{REG}$).

Proof: (a) Assume that there is some α_0 such that $\mathrm{ON} - \alpha_0 \subseteq \mathbf{CR}_f$. Now it is a simple matter to show, via Bukovský's result, that for any α, $\mathrm{cf}''(\mathrm{LIM}^* - \alpha) = \mathrm{REG}$. Hence our conclusion is immediate in this case.

(b) Assume that for each α there is $\beta > \alpha$ with $\beta \notin \mathbf{CR}_f$. We define an increasing ON-sequence of elements of $\mathbf{CR}_f$ as follows.

(1) $\tau_0 = \mathrm{C}_f(0)$;
(2) For $\alpha \in \mathrm{LIM}^*$, put $\tau_\alpha = \lim_{\xi<\alpha}\tau_\xi$;
(3) $\tau_{\beta+1} = \mathrm{C}_f(\gamma)$, where $\gamma = \min\{\delta > \tau_\beta;\ \delta \notin \mathbf{CR}_f\}$.

Our assumption shows that each τ_ξ is well-defined, and Bukovský's result tells us that $\mathrm{cf}(\tau_\rho) = \rho$ for each $\rho \in \mathrm{REG}$.

Just as there is no normal f with $\mathbf{CR}_f = \mathrm{SUC}$, so there is no normal f with $\mathbf{CR}_f = \mathrm{REG}$; a simple induction argument shows that if $\mathrm{REG} \subseteq \mathbf{CR}_f$ then $\mathrm{INT} \subseteq \mathbf{CR}_f$, for any normal f. The normal function A defined by $A(\alpha) = \omega_\alpha$ shows that there are normal functions f for which $\mathrm{INT} \supseteq \mathbf{CR}_f$, but perhaps it is not quite so obvious that there is no normal f for which $\mathrm{INT} = \mathbf{CR}_f$. This is a consequence of our next result.

Theorem 6 $\forall f, \alpha$ (*f normal* & $\omega_{\alpha+1} \in \mathbf{CR}_f \Rightarrow |\mathbf{CR}_f \cap (\omega_{\alpha+1} - \omega_\alpha)| = \aleph_{\alpha+1}$).

Proof: Put $X = \mathbf{CR}_f \cap (\omega_{\alpha+1} - \omega_\alpha)$; we show first that $X \neq \emptyset$. Take $\beta \in \omega_{\alpha+1} - \omega_\alpha$; clearly we may assume that $f(\beta) > \beta$, when $\beta < \mathrm{C}_f(\beta) \leqslant \omega_{\alpha+1}$. But $\mathrm{cf}(\mathrm{C}_f(\beta)) = \omega < \omega_{\alpha+1} = \mathrm{cf}(\omega_{\alpha+1})$. Hence $\mathrm{C}_f(\beta) < \omega_{\alpha+1}$, and so $\mathrm{C}_f(\beta) \in X$. We now show that $|X| = \aleph_{\alpha+1}$. Let $(\tau_\xi)_{\xi<\lambda}$ be the increasing sequence of elements of X; clearly $\lambda \in \mathrm{LIM}^*$ and $\lim_{\xi<\lambda}\tau_\xi = \omega_{\alpha+1}$.

From the above lemma we obtain $\lambda \geqslant \mathrm{cf}(\lambda) = \mathrm{cf}(\omega_{\alpha+1}) = \omega_{\alpha+1}$, and thus $|X| \geqslant \aleph_{\alpha+1}$. But of course $|X| \leqslant \aleph_{\alpha+1}$.

Corollary $\forall f$ (*f normal* & $\mathrm{REG} \subseteq \mathbf{CR}_f \Rightarrow \mathrm{INT} \subset \mathbf{CR}_f$).

Let us now turn to what is perhaps the weakest of the so-called "strong infinity axioms"; the assertion that every initial ordinal with a nonzero limit index is singular. We wish to give an alternative formulation of this axiom in terms of normal functions.

Definition 7 $\mathrm{REG}^* = \mathrm{REG} - \{\omega\}$. $\mathrm{WIN} = \{\rho \in \mathrm{REG}^*;\ \exists\alpha(\alpha \in \mathrm{LIM}\ \&\ \rho = \omega_\alpha)\}$. Let $\mathbf{H}$ be the following sentence:

$$\forall f\,(f \text{ normal } \&\ \mathrm{REG}^* \cap \mathbf{CR}_f \neq \emptyset \Rightarrow (\mathrm{ON} - \mathrm{INT}) \cap \mathbf{CR}_f \neq \emptyset).$$

Theorem 7 $\mathbf{ZFC} \vdash \mathbf{H} \Leftrightarrow \mathrm{WIN} = \emptyset$.

Proof: Assume $\mathbf{H}$; we wish to show that $\mathrm{WIN} = \emptyset$. Assume the contrary, and put $\kappa = \min \mathrm{WIN}$. Obviously $\kappa \leqslant \omega_\kappa$. Define the normal function A as above; $A(\alpha) = \omega_\alpha$. $\mathbf{CR}_A \subset \mathrm{INT}$. Now if we had $\kappa = \omega_\beta$ for some $\beta < \kappa$, then as $\kappa = \bigcup(A''\beta)$, we would have $\mathrm{cf}(\kappa) < \kappa$, contradicting $\kappa \in \mathrm{REG}$. Thus we must have $\kappa = \omega_\kappa = A(\kappa)$, which contradicts $\mathbf{H}$.

Now assume $\sim\mathbf{H}$, and let f be a normal function such that $\mathsf{REG}^* \cap \mathbf{CR}_f \neq \emptyset$ and $\mathbf{CR}_f \subset \mathsf{INT}$. Take $\rho \in \mathsf{REG}^* \cap \mathbf{CR}_f$. In view of Theorem 6 we cannot have $\rho = \omega_{\alpha+1}$ for any α, and thus must have $\rho = \omega_\beta$ for some $\beta \in \mathsf{LIM}^*$. But since this gives $\rho \in \mathsf{WIN}$, we have proved that $\mathsf{WIN} \neq \emptyset$.

Theorem 8 *If* $\mathbf{ZF}$ *is consistent, then* $\mathbf{ZFC} \nvdash \forall f\ (f \text{ normal} \Rightarrow \mathsf{REG} \cap \mathbf{CR}_f \neq \emptyset)$.

Proof: Let $\mathbf{ZFC}^\circ$ be the theory $\mathbf{ZFC} + \{\mathsf{WIN} = \emptyset\}$. It is well-known that if $\mathbf{ZF}$ is consistent, then so is $\mathbf{ZFC}^\circ$. Thus, our result will be proved if we can show that $\mathbf{ZFC}^\circ \vdash \exists f\ (f \text{ normal} \Rightarrow \mathsf{REG} \cap \mathbf{CR}_f = \emptyset)$. Hence, we work in $\mathbf{ZFC}^\circ$, and consider once again the function A. Suppose that $A(\rho) = \rho$ for some $\rho \in \mathsf{REG}$, i.e., $\omega_\rho = \rho = \mathsf{cf}(\rho)$. But then ([2], Theorem 2, p. 309) $\rho \in \mathsf{WIN}$. Since we are working within $\mathbf{ZFC}^\circ$, this is a contradiction.

Theorem 9 *Let* f', f'' *be normal functions, and define the functions* g, h *by* $g = \min\{f', f''\}$, $h = \max\{f', f''\}$. *Then* g, h *are normal.*

Proof: We consider g first. Take $\alpha < \beta$; by symmetry, we may assume that $g(\alpha) = f'(\alpha)$, $g(\beta) = f''(\beta)$. Then we have $g(\alpha) = f'(\alpha) \leqslant f''(\alpha) < f''(\beta) = g(\beta)$, and so g is increasing. Take $\lambda \in \mathsf{LIM}^*$, let (τ_ξ) be an increasing λ-sequence, and put $\alpha = \lim_{\xi<\lambda}\tau_\xi$. By symmetry again, we may assume $g(\alpha) = f'(\alpha)$; and for $\xi < \lambda$, choose $f_\xi \in \{f', f''\}$ such that $f_\xi(\tau_\xi) = g(\tau_\xi)$. Since g is increasing, we have $g(\alpha) \geqslant \lim_{\xi<\lambda} g(\tau)$, and so it suffices to prove the reverse inequality. Take $\delta < g(\alpha) = f'(\alpha)$; thus $\delta < f'(\tau_\xi)$ for some $\xi < \lambda$, and so if $f_\zeta = f'$ for some $\zeta \geqslant \xi$, we are through. Hence, we may assume that $g(\tau_\zeta) = f''(\tau_\zeta)$ for all ζ with $\xi \leqslant \zeta < \lambda$. But $\delta < f'(\alpha) \leqslant f''(\alpha)$, and since f'' is normal, we must have $\delta < f''(\tau_\zeta)$ for all ζ with $\eta \leqslant \zeta < \lambda$, for some fixed $\eta < \lambda$. Since this also gives $\delta < g(\tau_\zeta)$ for some $\zeta < \lambda$, we have shown that g is continuous and thus normal. The proof that h is increasing mirrors the corresponding proof for g, and so we simply demonstrate that h is continuous.

Take $\alpha = \lim_{\xi<\lambda}\tau_\xi$ as before. Then by our first lemma,

$$\begin{aligned} h(\alpha) = \sup\{f'(\alpha), f''(\alpha)\} &= \sup\{\sup_{\xi<\lambda} f'(\tau_\xi), \sup_{\xi<\lambda} f''(\tau_\xi)\} \\ &= \sup_{\xi<\lambda}\sup\{f'(\tau_\xi), f''(\tau_\xi)\} \\ &= \sup_{\xi<\lambda} h(\tau_\xi) = \lim_{\xi<\lambda} h(\tau_\xi). \end{aligned}$$

Thus h is normal.

Definition 8 $\mathbf{CR} = \{\mathbf{CR}_f : f \text{ normal } \& \operatorname{dom}(f) = \mathsf{ON}\}$.

Theorem 10 $\langle \mathbf{CR}, \cup, \cap \rangle$ *is a lattice with unit element, no zero element, and in which no element is complemented.*

Proof: Take $A, B \in \mathbf{CR}$, and let f_A, f_B be normal functions with $\mathbf{CR}_{f_A} = A$, $\mathbf{CR}_{f_B} = B$. Put $g = \min\{f_A, f_B\}$, $h = \max\{f_A, f_B\}$. We know from Theorem 9 that g, h are normal, and claim that $\mathbf{CR}_g = A \cup B$, $\mathbf{CR}_h = A \cap B$. $g(\alpha) = \alpha \Leftrightarrow \alpha = \min\{f_A(\alpha), f_B(\alpha)\} \Leftrightarrow (f_A(\alpha) = \alpha$ or $f_B(\alpha) = \alpha)$, and $h(\alpha) = \alpha \Leftrightarrow \alpha = \max\{f_A(\alpha), f_B(\alpha)\} \Leftrightarrow \alpha = f_A(\alpha) = f_B(\alpha)$, since f_A, f_B are normal. This establishes our claim and hence closure, and of course the lattice identities are trivial.

Clearly $\mathsf{ON} \in \mathbf{CR}$ is the unit element. Now take $A \in \mathbf{CR}$, put $\alpha = \min A$, and let f be a normal function with $\mathbf{CR}_f = A$. Define the function g by $g(\beta) = \alpha + f(\beta)$. It is easy to check that g is normal and that $g(\beta) = \beta \Rightarrow f(\beta) = \beta$. On the other hand, $g(\alpha) = \alpha 2 > \alpha$, and so $\mathbf{CR}_g \subset A$. Thus $\mathbf{CR}$ has no zero element.

Finally, if $A \in \mathbf{CR}$ then $\mathrm{ON} - A \notin \mathbf{CR}$, since $A \cap (\mathrm{ON} - A) = \emptyset$.

Lemma *Let f be normal, put $\alpha = \min \mathbf{CR}_f$, and suppose that $\alpha \neq 0$. Then $\alpha = \omega^\beta$ for some β such that* $\mathrm{cf}(\alpha) = \omega$.

Proof: Since $\alpha = \mathrm{C}_f(0) = \lim_{n<\omega} f^n(0) > 0$, it is obvious that $\mathrm{cf}(\alpha) = \omega$. Let ρ be the smallest positive remainder of α, and assume that $\rho < \alpha$. Then of course $\alpha = \tau + \rho$ for some $\tau \geqslant \rho$, and $f(\tau) > \tau$. We may assume without loss of generality that the smallest positive remainder θ of τ is such that $\theta \geqslant \rho$. Let X be an ordered set of type α, let Y be an initial segment of X of type τ, and let Z be the corresponding final segment (of type ρ). From our assumptions we deduce the existence of an order-preserving map $F: X \to X$ such that $(F''Y) \cap Z \neq \emptyset$. Put $T = (F''Y) \cap Z$: then the order-type $\mathrm{o}(T)$ of T is such that $\mathrm{o}(T) \geqslant \rho = \mathrm{o}(Z)$, and thus we must have $T = Z$. But this gives the immediate contradiction $F''Z = \emptyset$, which proves our result.

Theorem 11 **CR** *is neither $\bigcup$-nor $\bigcap$-complete. Specifically:*

(1) *There is an indexed set $\{A_n\}_{n<\omega}$ of elements of* **CR** *such that* $\bigcup\limits_{n<\omega} A_n \notin \mathbf{CR}$.

(2) *There is an indexed set $\{B_\gamma\}_{\gamma<\omega_1}$ of elements of* **CR** *such that* $\bigcap\limits_{\gamma<\omega_1} B_\gamma \notin \mathbf{CR}$.

Proof: (1) For $n < \omega$, define the normal function f_n by $f_n(m) = m$, $m \leqslant n$, $f_n(\alpha) = \omega + \alpha$, $\alpha > n$, and put $A_n = \mathbf{CR}_{f_n}$, $A = \bigcup\limits_{n<\omega} A_n$. Since $n \in A_n$ for each n, we have $\omega \subseteq A$. However, $\omega \notin A$, and so $A = \mathbf{CR}_g$ for no normal g.

(2) For $\gamma < \omega_1$, define the normal function g_γ by $g_\gamma(\alpha) = \gamma + \alpha$, put $B_\gamma = \mathbf{CR}_{g_\gamma}$, and $B = \bigcap\limits_{\gamma<\omega_1} B_\gamma$. Then it is easy to see that $\min B = \omega_1$, and so by the preceding lemma, $B \notin \mathbf{CR}$.

CR, although not $\bigcap$-complete, does have the property that every nonempty set of elements of **CR** has a lower bound in **CR** (the corresponding property for $\bigcup$ is trivial, since $\mathrm{ON} \in \mathbf{CR}$).

Theorem 12 *Let $\{A_i\}_{i\in I}$ be a nonempty indexed set of elements of* **CR**. *Then there is $B \in$* **CR** *with* $B \subseteq \bigcap\limits_{i\in I} A_i$.

Proof: For each $i \in I$, let f_i be a normal function with $\mathbf{CR}_{f_i} = A_i$. We define first of all an auxiliary function f^* by $f^*(\alpha) = \sup_{i\in I} f_i(\alpha)$, and we show that f^* is semi-normal. The only thing that really requires checking is continuity, since the other two requirements for semi-normality are obviously satisfied. Thus we take $\lambda \in \mathrm{LIM}^*$, let (τ_ξ) be an increasing λ-sequence, and put $\alpha = \lim_{\xi<\lambda}\tau_\xi$. Then

$$\begin{aligned} f^*(\alpha) = \sup_{i\in I} f_i(\alpha) &= \sup_{i\in I}\sup_{\xi<\lambda} f_i(\tau_\xi) \\ &= \sup_{\xi<\lambda}\sup_{i\in I} f_i(\tau_\xi) \\ &= \sup_{\xi<\lambda} f^*(\tau_\xi), \end{aligned}$$

where we have made use of our first lemma. Thus f^* is semi-normal. For each α we now define δ_α by $\delta_\alpha = \min\{\gamma;\ f^*(\gamma) = f^*(\alpha)\}$, and we let (μ_ξ) be the sequence of the δ_α in order of magnitude; since $f^*(\beta) \geqslant \beta$ for all β, it is clear that (μ_ξ) is an increasing ON-sequence, and we now define a function g by $g(\xi) = f^*(\mu_\xi)$.

We wish to prove g normal; it is obvious from the definition that g is increasing, and so it suffices to prove continuity. As a preliminary to proving this, we show that if $\alpha = \lim_{\xi<\lambda}\tau_\xi$, with (τ_ξ) an increasing λ-sequence, $\lambda \in \mathrm{LIM}^*$, then $\mu_\alpha = \lim_{\xi<\lambda}\mu_{\tau_\xi}$. Clearly $\mu_\alpha \leqslant \lim_{\xi<\lambda}\mu_{\tau_\xi}$, and so we take $\rho < \mu_\alpha$. Thus $f^*(\rho) < f^*(\mu_\alpha)$, and so $f^*(\rho) = f^*(\mu_\beta)$ for some $\beta < \alpha$. But of course there is $\eta < \lambda$ with $\tau_\eta > \beta$, whence we have $f^*(\mu_{\tau_\eta}) > f^*(\mu_\beta) = f^*(\rho)$, from which we conclude that $\lim_{\xi<\lambda}\mu_{\tau_\xi} \geqslant \mu_{\tau_\eta} > \rho$. Thus $\mu_\alpha = \lim_{\xi<\lambda}\mu_{\tau_\xi}$. But now, since f is semi-normal, we obtain, for α as above, $g(\alpha) = f^*(\mu_\alpha) = \lim_{\xi<\lambda}f^*(\mu_{\tau_\xi}) = \lim_{\xi<\lambda}g(\tau_\xi)$, and so g is normal.

Put $B = \mathbf{CR}_g$, and take $\alpha \in B$. Thus we have $f^*(\mu_\alpha) = \alpha$. As f^* is semi-normal we conclude that $\mu_\alpha \leqslant \alpha$, and hence $\mu_\alpha = \alpha$. Therefore, $\alpha = f^*(\alpha) = \sup_{i\in I}f_i(\alpha)$, and so $\alpha = f_i(\alpha)$, $i \in I$. Thus $B \subseteq {}_{i\in I}A_i$.

We conclude by answering a question raised earlier in this paper.

Theorem 13 *There is a semi-normal function g such that* $\mathbf{CR}_g = \mathbf{CR}_f$ *for no normal function f.*

Proof: For $\gamma < \omega_1$, define g_γ by $g_\gamma(\gamma) = \gamma + \alpha$, and put $B = \bigcap_{\gamma<\omega_1}\mathbf{CR}_{g_\gamma}$. We have already seen that there is no normal f with $\mathbf{CR}_f = B$. Define g by $g(\alpha) = \sup_{\gamma<\omega_1}g_\gamma(\alpha)$: just as in the proof of Theorem 12, we can show that g is semi-normal, and we claim that $B = \mathbf{CR}_g$. First, if $g(\alpha) = \alpha$, then $g_\gamma(\alpha) = \alpha$ for all $\gamma < \omega_1$, since the g_γ are normal, and so $\alpha \in B$. On the other hand, if $g_\gamma(\alpha) = \alpha$ for all $\gamma < \omega_1$, then obviously $g(\alpha) = \alpha$. This proves our result.

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To be continued

Institute of Advanced Studies
Australian National University
Canberra, Australia