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ADDITIONAL EXTENSIONS OF S4

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1 The purpose of this paper is to explore the modal systems resulting from appending

I1 $ALCLpqCLMLqp$

to either the basis of S4 or any of its known extensions. All of the matrices employed are taken from Sobociński in [4]. In order to make it possible for our proofs to proceed with greater facility in the subsequent discussion, we shall make use of a Fitch-style natural deduction system for S4. Such systems are familiar enough for it to be recognized that the list of derivation rules given below constitute a natural deduction system for S4:

Negative Necessity Elimination (**NLE**)

$$\begin{array}{r|ll} \cdot & \cdot & \\ \cdot & \cdot & \\ n & NLX & \\ \cdot & \cdot & \\ p & MNX & \qquad n, \mathbf{NLE} \end{array}$$

Negative Necessity Introduction (**NLI**)

$$\begin{array}{r|ll} \cdot & \cdot & \\ \cdot & \cdot & \\ n & MNX & \\ \cdot & \cdot & \\ p & NLX & \qquad n, \mathbf{NLI} \end{array}$$

Negative Possibility Elimination (**NME**)

$$\begin{array}{r|ll} \cdot & \cdot & \\ \cdot & \cdot & \\ n & NMX & \\ \cdot & \cdot & \\ p & LNX & \qquad n, \mathbf{NME} \end{array}$$

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Negative Possibility Introduction (**NMI**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
n & LNX & \\
\cdot & \cdot & \\
p & NMX & \quad n, \mathbf{NMI}
\end{array}$$

Necessity Elimination (**LE**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
n & LX & \\
\cdot & \cdot & \\
p & X & \quad n, \mathbf{LE}
\end{array}$$

Necessity Introduction (**LI**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & & \\
n & L\begin{array}{|l} \cdot \\ X \\ \cdot \end{array} & \\
\cdot & & \\
p & LX & \quad n, \mathbf{LI}
\end{array}$$

Strict Implication Elimination ($\mathfrak{C}$**E**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
m & \mathfrak{C}XY & \\
\cdot & \cdot & \\
n & X & \\
\cdot & \cdot & \\
p & Y & \quad m, n, \mathfrak{C}\mathbf{E}
\end{array}$$

Strict Implication Introduction ($\mathfrak{C}$**I**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
m & L\begin{array}{|l} \underline{X} \\ \cdot \\ Y \end{array} & \\
\cdot & & \\
n & & \\
p & \mathfrak{C}XY & \quad m - n, \mathfrak{C}\mathbf{I}
\end{array}$$

Strict Equivalence Elimination ($\mathfrak{E}$**E**)

$$\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
m & \mathfrak{E}XY & \\
\cdot & \cdot & \\
n & X & \\
\cdot & \cdot & \\
p & Y & \quad m, n, \mathfrak{E}\mathbf{E}
\end{array}
\qquad
\begin{array}{c|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
m & \mathfrak{E}YX & \\
\cdot & \cdot & \\
n & X & \\
\cdot & \cdot & \\
p & Y & \quad m, n, \mathfrak{E}\mathbf{E}
\end{array}$$

Strict Equivalence Introduction ($\mathfrak{E}$I)

$$
\begin{array}{l|llll}
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
k & & L & \underline{X} & \\
\cdot & & & \cdot & \\
1 & & & Y & \\
m & & L & \underline{Y} & \\
\cdot & & & \cdot & \\
n & & & X & \\
p & \mathfrak{E}XY & & & k - 1,\ m - n, \mathfrak{E}\mathrm{I}
\end{array}
$$

Possibility Introduction (MI)

$$
\begin{array}{l|ll}
\cdot & \cdot & \\
\cdot & \cdot & \\
n & X & \\
\cdot & \cdot & \\
p & MX & n,\ \mathrm{MI}
\end{array}
$$

It is customary to formulate the rule of Possibility Elimination in the following fashion:

$$
\begin{array}{l|llll}
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
1 & MX & & & \\
\cdot & \cdot & & & \\
m & & L & \underline{X} & \\
\cdot & & & \cdot & \\
n & & & Y & \\
\cdot & \cdot & & & \\
p & MY & & & 1,\ m - n,\ \mathrm{ME}
\end{array}
$$

However, we shall adopt a formulation due to William A. Wisdom in [6]. It appears to me that more facility in constructing proofs is afforded by his formulation. It goes like this:

Possibility Elimination (ME)

$$
\begin{array}{l|llll}
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
k & MX & & & \\
\cdot & \cdot & & & \\
1 & & L & \underline{X} & \\
\cdot & & & \cdot & \\
m & & & Y & \\
\cdot & & & \cdot & \\
n & & & NY & \\
\cdot & \cdot & & & \\
p & Z & & & k,\ 1 - n,\ \mathrm{ME}
\end{array}
$$

Strict Reiteration (**R′**)

LX may occur in a strict subordinate proof if *LX* occurs earlier in the proof to which it is subordinate. Schematically, we represent **R′** thus:

$$\begin{array}{c|l}
\cdot & \cdot \\
\cdot & \cdot \\
n & LX \\
\cdot & \qquad L\ \begin{array}{|l} \cdot \\ LX \\ \cdot \end{array} \\
p & \\
\cdot &
\end{array}$$

2 It is an easy matter to show that

I1 *ALCLpqCLMLqp*

is not a thesis of S4 since matrix $\mathfrak{M}4$ verifies S4 but falsifies **I1** for $p/5$ and $q/2$: *CNLCL*52*CLML*25 = *CNLC*52*CLM*65 = *CNL*2*CLM*65 = *CNL*2*CL*15 = *CN*6*C*15 = *C*3*C*15 = *C*35 = 5. In fact, $\mathfrak{M}4$ also verifies S4.3.1; thus it is clear that **I1** is not a thesis of any of the following systems: S4.3.1, S4.3, S4.2.1, S4.2, S4.1, S4.02, and S4.01. Now if we append **I1** to the basis of S4, we obtain a new modal system which I call S4.03. Obviously, this system is not contained in any of the above systems; however, it is contained in the remaining systems up to and including S4.4. First we show that it is contained in S4.3.2 and hence S4.4 by demonstrating that **F1**, the proper axiom of S4.3.2, inferentially entails **I1** in the field of S1:

(1)	*ALCLpqCMLqp*	**F1**
(2)	*CNLCLpqCMLqp*	1, Implication
(3)	*CMLqCNLCLpqp*	2, Permutation
(4)	*CLpp*	S1
(5)	*CLMLqMLq*	4, p/MLq
(6)	*CLMLqCNLCLpqp*	3, 5, Syllogism
(7)	*CNLCLpqCLMLqp*	6, Permutation
(8)	*ALCLpqCLMLqp*	7, Implication

Before showing that **I1** is a thesis of S4.04 (and hence S4.1.2), we first demonstrate that

L4 *CpLCMLpp*

may also serve as the proper axiom of S4.04. Assume **L4** and the field of S1:

(1)	*CpLCMLpp*	**L4**
(2)	*CLCpqCLpLq*	S1
(3)	*CLCMLppCLMLpLp*	2, p/MLp, q/p
(4)	*CpCLMLpLp*	1, 3, Syllogism
(5)	*CLMLpCpLp*	4, Permutation

Clearly **L4** inferentially entails **L1** in the field of S1. Now we prove that **L1** inferentially entails **L4** in the field of S4. In order to accomplish this, we shall make use of our natural deduction system for S4. We shall also utilize the following formula:

CLMLCMLppCCMLppLCMLpp

This formula is merely a substitution instance (*p/CMLpp*) of **L1**.

```
 1 |CLMLCMLppCCMLppLCMLpp                         L1, p/CMLpp
 2 | |p                                          Hyp
 3 | | |NLMLCMLpp                                Hyp
 4 | | |MNMLCMLpp                                3, NLE
 5 | | | L |NMLCMLpp                            Hyp
 6 | | |   |LNLCMLpp                            5, NLE
 7 | | |   |NLCMLpp                             6, LE
 8 | | |   |MNCMLpp                             7, NLE
 9 | | |   | L |NCMLpp                          Hyp
10 | | |   |   |LNLCMLpp                        6, R'
11 | | |   |   | |MLp                           Hyp
12 | | |   |   | |LNLCMLpp                      10, R'
13 | | |   |   | | L |Lp                        Hyp
14 | | |   |   | |   |LNLCMLpp                  12, R'
15 | | |   |   | |   |NLCMLpp                   14, LE
16 | | |   |   | |   |MNCMLpp                   15, NLE
17 | | |   |   | |   | L |NCMLpp              Hyp
18 | | |   |   | |   |   |Lp                  13, R'
19 | | |   |   | |   |   | |MLp               Hyp
20 | | |   |   | |   |   | |Lp                18, R
21 | | |   |   | |   |   | |p                 20, LE
22 | | |   |   | |   |   |CMLpp               19-21, CI
23 | | |   |   | |   |NLp                     16, 17-22, ME
24 | | |   |   | |p                             11, 13-23, ME
25 | | |   |   |CMLpp                           11-24, CI
26 | | |   |MLCMLpp                             8, 9-25, ME
27 | | |LMLCMLpp                                4, 5-26, ME
28 | |NNLMLCMLpp                                 3-27, NI
29 | |LMLCMLpp                                   28, NE
30 | |CLMLCMLppCCMLppLCMLpp                      1, R
31 | |CCMLppLCMLpp                               30, 29, CE
32 | | |MLp                                     Hyp
33 | | |p                                       2, R
34 | |CMLpp                                      32-33, CI
35 | |LCMLpp                                     31, 34, CE
36 |CpLCMLpp                                      2-35, CI
```

Having proved that **L4** may also serve as the proper axiom of modal system S4.04, we now show that **L4** inferentially entails **I1** in the field of S4. This time we shall employ

CCLpqLCMLCLpqCLpq

in our proof. Note that this is merely a substitution instance of **L4**.

1	$CCLpqLCMLCLpqCLpq$	L4, $p/CLpq$
2	$NALCLpqCLMLqp$	Hyp
3	$CCLpqLCMLCLpqCLpq$	1, R
4	$NLCLpq$	Hyp
5	$CCLpqLCMLCLpqCLpq$	3, R
6	$LMLq$	Hyp
7	$NLCLpq$	4, R
8	$CCLpqLCMLCLpqCLpq$	5, R
9	Np	Hyp
10	$CCLpqLCMLCLpqCLpq$	8, R
11	Lp	Hyp
12	Np	9, R
13	Nq	Hyp
14	Lp	11, R
15	p	14, LE
16	Np	12, R
17	NNq	13-16, NI
18	q	17, NE
19	$CLpq$	11-18, CI
20	$LCMLCLpqCLpq$	10, 19, CE
21	$LMLq$	6, R
22	$NLCLpq$	7, R
23	$MNCLpq$	22, NLE
24	L $NCLpq$	Hyp
25	$LMLq$	21, R′
26	$LCMLCLpqCLpq$	20, R′
27	$NMLCLpq$	Hyp
28	$LNLCLpq$	27, NME
29	$LMLq$	25, R
30	MLq	29, LE
31	L Lq	Hyp
32	$LNLCLpq$	28, R′
33	L Lp	Hyp
34	Lq	31, R′
35	q	34, LE
36	$CLpq$	33-35, CI
37	$LCLpq$	36, LI
38	$NLCLpq$	32, LE
39	$MLCLpq$	30, 31-38, ME
40	$NNMLCLpq$	27-39, NI
41	$MLCLpq$	40, NE
42	$CMLCLpqCLpq$	26, LE
43	$CLpq$	42, 41, CE
44	p	23, 24-43, ME
45	NNp	9-44, NI
46	p	45, NE

47	| | *CLMLqp*	6-46, **CI**
48	| | *ALCLpqCLMLqp*	47, **AI**
49	| | *NALCLpqCLMLqp*	2, **R**
50	| *NNLCLpq*	4-49, **NI**
51	| *LCLpq*	50, **NE**
52	| *ALCLpqCLMLqp*	51, **AI**
53	*NNALCLpqCLMLqp*	2-52, **NI**
54	*ALCLpqCLMLqp*	53, **NE**

Using the matrices of Sobociński ([4], p. 350), we find

1. $\mathfrak{M}7$ verifies **I1**, but rejects S4.02 ([3], p. 381) and hence also S4.04, S4.1.2, S4.1, S4.2.1, S4.3.1, and S4.4.
2. $\mathfrak{M}5$ verifies **I1**, but rejects S4.2 ([4], p. 354) and hence also S4.3 and and S4.3.2.
3. $\mathfrak{M}11$ verifies **I1**, but rejects S4.01 ([1], p. 569).

These considerations demonstrate that S4.03 is a proper extension of S4, properly contained in S4.04, S4.1.2, S4.3.2, and S4.4, and independent of S4.01, S4.02, S4.1, S4.2, S4.3, S4.2.1, and S4.3.1.

We may now wonder whether the addition of **I1** to the basis of any of the other Lewis extensions of S4 independent of S4.03 will also yield additional modal systems. But before directing our attention to this task, we first show that

I2 *CKLMLpqLApMq*

may also serve as the proper axiom of S4.03. Assume **I1** and the field of S1:

(1)	*ALCLpqCLMLqp*	**I1**
(2)	*ACLMLqpLCLpq*	1, Commutation
(3)	*CNCLMLqpLCLpq*	2, Implication
(4)	*CKLMLqNpLCLpq*	3, Implication
(5)	*CKLMLqNpLANLpq*	4, Implication
(6)	*CKLMLqNpLAMNpq*	5, Modal Exchange
(7)	*CKLMLqNpLAqMNp*	6, Commutation
(8)	*CKLMLpNNqLApMNNq*	7, *q/p*, *p/Nq*
(9)	*CKLMLpqLApMq*	8, Double Negation

Quite obviously, this proof may also be carried out in reverse; thus **I2** also inferentially entails **I1** in the field of S1.

Having proved that **I1** and **I2** are inferentially equivalent in the field of S1, we now show that the addition of **I2** to the basis of S4.02 yields S4.04. We prove this by showing that **I2** and **Ł1** together inferentially entail **L1** in a field at least as weak as S4:

1	$CKLMLpNCpLpLApMNCpLp$	I2, $q/NCpLp$
2	$LCLCLCpLppCLMLpp$	Ł1
3	$LMLp$	Hyp
4	$CKLMLpNCpLpLApMNCpLp$	1, R
5	$LCLCLCpLppCLMLpp$	2, R
6	p	Hyp
7	$LMLp$	3, R
8	$CKLMLpNCpLpLApMNCpLp$	4, R
9	$LCLCLCpLppCLMLpp$	5, R
10	NLp	Hyp
11	$LMLp$	7, R
12	$CKLMLpNCpLpLApMNCpLp$	8, R
13	$LCLCLCpLppCLMLpp$	9, R
14	p	6, R
15	$CpLp$	Hyp
16	p	14, R
17	Lp	15, 16, CE
18	NLp	10, R
19	$NCpLp$	15-18, NI
20	$KLMLpNCpLp$	11, 19, KI
21	$LApMNCpLp$	12, 20, CE
22	L $LCpLp$	Hyp
23	$LApMNCpLp$	21, R′
24	$ApMNCpLp$	23, LE
25	p	Hyp
26	$MNCpLp$	Hyp
27	$LCpLp$	22, R
28	Np	Hyp
29	$MNCpLp$	26, R
30	$NLCpLp$	20, NLI
31	$LCpLp$	27, R
32	NNp	28-31, NI
33	p	32, NE
34	p	24, 25-25, 26-33, AE
35	$CLCpLpp$	22-34, CI
36	$LCLCpLpp$	35, LI
37	L $LCLCLCpLppCLMLpp$	9, R′
38	$CLCLCpLppCLMLpp$	37, LE
39	$LMLp$	7, R′
40	$LCLCpLpp$	36, R′
41	$CLMLpp$	38, 40, CE
42	p	41, 39, CE
43	Lp	42, LI
44	$NNLp$	10-43, NI
45	Lp	44, NE
46	$CpLp$	6-45, CI
47	$CLMLpCpLp$	3-46, CI

Now because **Ł1** is a thesis of S4.1, it is obvious that appending either **I1** or **I2** to the basis of S4.1 would yield S4.1.2 since S4.1.2 = {S4.1; **L1**}.

Now we show that **I1** inferentially entails **F1**, the proper axiom of S4.3.2, in the field of S4.2:

(1)	*ALCLpqCLMLqp*	**I1**
(2)	*CNLCLpqCLMLqp*	1, Implication
(3)	*CLMLqCNLCLpqp*	2, Permutation
(4)	*CMLqLMLq*	**G2**
(5)	*CMLqCNLCLpqp*	3, 4, Syllogism
(6)	*CNLCLpqCMLqp*	5, Permutation
(7)	*ALCLpqCMLqp*	6, Implication

Clearly then, appending **I1** to either S4.2 or S4.3 yields S4.3.2.

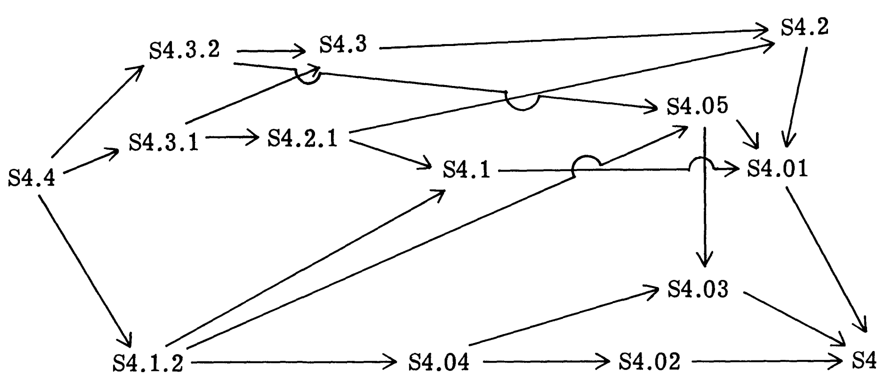
In [9], p. 297, Zeman demonstrates that appending **F1** to S4.1 yields S4.4. Consequently, it follows that since S4.2.1 and S4.3.1 both contain S4.2, appending **I1** to either S4.2.1 or S4.3.1 would also yield S4.4.

Adding formula **I1** to the basis of S4.01 does, however, result in another new modal system. I call this new extension of S4 modal system S4.05. In [1], R. I. Goldblatt shows that S4.01 is properly contained in every known extension of S4 except S4.02 and S4.04. He also demonstrates that S4.01 is independent of both S4.02 and S4.04. Now again consider the following matrices:

1. $\mathfrak{M}7$ verifies both **Γ1**, the proper axiom of S4.01, and **I1**, but rejects S4.02 ([3], p. 381) and hence also S4.04, S4.1.2, S4.1, S4.2.1, S4.3.1, and S4.4.
2. $\mathfrak{M}5$ verifies both **Γ1** and **I1** but rejects S4.2 ([4], p. 354) and hence also S4.3 and S4.2.
3. $\mathfrak{M}11$ verifies **I1**, but rejects **Γ1** ([1], p. 569).
4. $\mathfrak{M}4$ verifies **Γ1**, but rejects **I1**.

These considerations establish that S4.05 is a proper extension of S4, S4.01, and S4.03, properly contained in S4.1.2, S4.3.2, and S4.4, and independent of S4.2, S4.3, S4.1, S4.2.1, S4.3.1, S4.02, and S4.04.

The following diagram enables us to visualize the relationships existing among S4.03, S4.05, and the other Lewis extensions of S4 at the time of writing:



3 In this section we consider the relationship of formula **I1** to modal family $\mathcal{Z}$. We have already observed that $\mathfrak{M}4$ falsifies **I1**; but this matrix verifies system Z7. Consequently, **I1** is not a thesis of any of the following systems: Z7, Z6, Z4, Z5, Z3, and Z1. Now we have also observed that **I1** is a thesis of S4.04, S4.1.2, S4.3.2, and S4.4; thus it must be the case that **I1** is a thesis of both Z8 and Z2.

Appending **I1** to Z1 generates a new system to be called Z1.5. Now consider the following matrices:

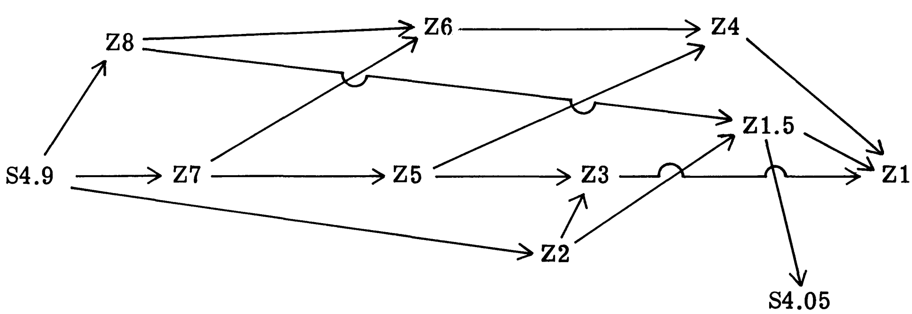
1. $\mathfrak{M}8$ verifies both **I1** and **Γ1**, but rejects **Z1** ([7], p. 354).
2. $\mathfrak{M}5$ verifies both **I1** and **Z1** but rejects **G1** ([5], pp. 310-311), and hence Z4, Z5, Z6, Z7, and Z8.
3. $\mathfrak{M}6$ verifies both **I1** and **Z1** but falsifies **N1** ([9], pp. 296-298), and hence Z3 and Z2.

Now in view of Goldblatt's finding (*cf.* [1], p. 568) that **Γ1** is entailed by **Z1** and the considerations given above, we may conclude that modal system Z1.5 is a proper extension of S4.05 and Z1, properly contained in Z8 and Z2, and independent of Z3, Z4, Z5, Z6, and Z7.

In the last section, we proved that appending **I1** to either S4.2 or S4.3 yields S4.3.2. Thus appending **I1** to either Z4 or Z6 must yield Z8. Again in the last section, we saw that appending **I1** to S4.1 gives S4.1.2; hence it follows that adding it to Z3 yields Z2.

In [2], Sobociński constructs a system which he calls Z9 by appending **Z1** to the basis of S4.4. However, in [8], Zeman proves that Z9 is S4.9. Hence appending **Z1** to S4.4 yields S4.9. Now we observed, in the last section, that adding **I1** to either S4.2.1 or S4.3.1 gives S4.4. Consequently, appending **I1** to either Z5 or Z7 would yield S4.9.

The relationships holding between Z1.5 and the other systems of family $\mathcal{Z}$ are exhibited by the following diagram:



4 Let us now consider **I1** with respect to family $\mathcal{K}$. Now $\mathfrak{M}4$ verifies the entire basis of K3.1, but, as we have seen, falsifies **I1**. Thus it is clear that **I1** is not a thesis of any of the following systems: K3.1, K3, K2.1, K2, K1.1, and K1. Now since **I1** is a thesis of S4.04, S4.3.2, and S4.4, it follows that it is also a thesis of K1.2, K3.2, and K4.

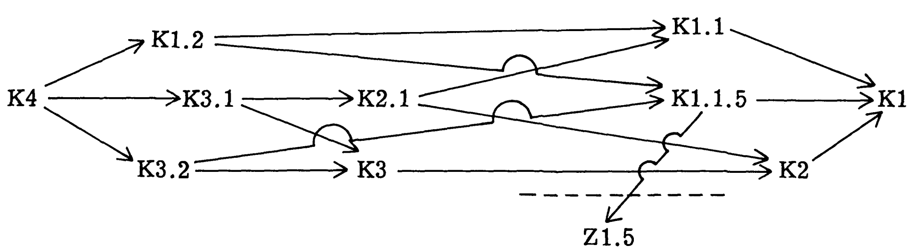
The addition of **I1** to K1 also generates a new modal system to be called K1.1.5. Now consider the following:

1. $\mathfrak{M}2$ verifies both **I1** and **Z1**, but rejects, as is well-known, **K1** and hence K1.1.5.
2. $\mathfrak{M}5$ verifies both **I1** and **K1**, but rejects **G1** ([5], pp. 310-311), and hence K2, K3, K3.2, K2.1, K3.1, and K4.
3. $\mathfrak{M}6$ verifies both **I1** and **K1**, but rejects **N1** ([9], pp. 296-298), and hence K1.1 and K1.2.

These considerations show that K1.15 is a proper extension of K1 and and Z1.5, properly contained in K1.2, K3.2, and K4, and independent of K1.1, K2, K3, K2.1, and K3.1.

Now, as we have already remarked, appending **I1** to S4.1 yields S4.1.2, hence appending it to K1.1 would generate K1.2. Adding **I1** to either S4.2 or S4.3 gives S4.3.2, hence appending it to either K2 or K3 would yield K3.2. It is well-known that {S4.4; **K1**} = K4, and, as we have already pointed out, {S4; **N1**; **F1**} = S4.4, hence appending **I1** to either K2.1 or K3.1, since they both contain S4.2, would yield K4.

The diagram given below exhibits the relationships holding between K1.1.5 and the other systems of family $\mathcal{K}$ at the time of writing:



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