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VARIATIONS OF ZORN'S LEMMA, PRINCIPLES OF COFINALITY, AND HAUSDORFF'S MAXIMAL PRINCIPLE. PART II: CLASS FORMS

JUDITH M. HARPER and JEAN E. RUBIN

5 By varying the ordering relation, we obtain a large number of maximal principles. Some are equivalent to the axiom of choice, some are weaker but do not follow from the axioms of set theory, some are provable from the other axioms, and the negations of some are provable from the other axioms. In Part I, [1],* we considered the set forms of maximal principles. In this paper we consider the class or strong forms. The results for class forms are similar to those for sets, but frequently the Axiom of Regularity, **AR**, is used to insure that at various stages of the proofs sets occur and not proper classes.

Section **6** of this paper deals with class forms of Zorn's Lemma and Principles of Cofinality; section **7**, class forms of Hausdorff's maximal principle; and in section **8** we give a list of the statements used in the paper. The notation used is similar to that used in [1]. For convenience, we shall repeat some of the definitions here.

5.1 $\mathbf{NBG}^{\circ}$ denotes von Neumann-Bernays-Gödel set theory excluding the Axiom of Regularity, **AR**, and the Axiom of Choice. $\mathbf{NBG} = \mathbf{NBG}^{\circ} + \mathbf{AR}$. All proofs are in $\mathbf{NBG}^{\circ}$ unless specifically stated otherwise.

5.2 If R partially orders a class X, $y \in X$, and $S = \{u \in X: uRy\}$ then S is called the *R-initial segment of X generated by y* and is denoted by $\overleftarrow{y}$.

5.3 A class X is *ramified* by a relation R iff R partially orders X such that every R-initial segment $\overleftarrow{y}$ of X is linearly ordered by R.

5.4 A class X is a *forest* under the relation R iff R partially orders X such that every R-initial segment $\overleftarrow{y}$ of X is well ordered by R.

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5.5 A class X is a *tree* under the relation R iff X is a forest and every finite subset of X has an R-lower bound.

5.6 A subclass Q of a partially ordered class $\langle X, R\rangle$ is *quasi-cofinal* in X iff Q has no strict upper bound in X. Q is *cofinal* in X iff Q is linearly ordered by R and for each $y \in X$ there is a $z \in Q$ such that yRz.

For a class X, let

$$W_X = \{\langle t, w\rangle: t \subseteq X \;\&\; w \subseteq t \times t \;\&\; w \text{ well orders } t\}.$$

Let I be the relation defined on W_X such that

$$\langle t, w\rangle\, I\langle t', w'\rangle \text{ iff } t \subseteq t' \;\&\; w = w' \cap (t \times t) \;\&\; t \times (t' \sim t) \subseteq w'.$$

I will be called the *initial segment relation* and $\langle W_X, I\rangle$, the *tree of well ordered subsets of X.*

5.7 Let $R_0 = \emptyset$, $R_{\alpha+1} = R_\alpha \cup \mathcal{P}(R_\alpha)$ for all ordinals α, and $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$ if α is a limit ordinal. It is well known that in **NBG°**, **AR** is equivalent to $V = \bigcup_{\alpha \in \mathrm{On}} R_\alpha$.

Consequently, assuming **AR** we can define the *rank* of a set x by $\rho(x)$ = the smallest ordinal number α such that $x \subseteq R_\alpha$.

5.8 Finally, as in [1], we use the following symbols for types of relations:

A:	arbitrary
TR:	transitive
AS:	antisymmetric
C:	connected
P:	partially ordered
W:	well ordered
L:	linearly ordered
D:	directed (upwards)
R:	ramified
F:	forest
T:	tree
AS & C:	antisymmetric and connected
TR & C:	transitive and connected

Most other symbols and notation used here and not defined above are defined in [1].

6 *Zorn's Lemma and Principles of Cofinality* In [1], $\mathbf{Z}(Q, U)$ stands for the statement:

Every non-empty Q-ordered set, in which every U-ordered subset has an upper bound, has a maximal element.

The natural way to obtain a class (or strong) form of Zorn's Lemma is to change the word "set" to "class" in both places where it occurs in $\mathbf{Z}(Q, U)$. Thus, we obtain the statement $\mathbf{ZC}(Q, U)$:

Every non-empty Q-ordered class, in which every U-ordered subclass has an upper bound, has a maximal element.

However, this form is sometimes difficult to work with because it is difficult to satisfy its hypothesis. (See, for example, Theorem 6.3 and the dicussion following 6.3.)

Another variation which is considered in [2] is the following, **ZS**(Q, U):

Every non-empty Q-ordered class, in which every U-ordered subset has an upper bound, either has a maximal element or contains a U-ordered subclass which is a proper class.

To see that the second part of the conclusion is necessary, take the class of ordinal numbers, On, which is well ordered by $\subseteq$. Any subset of On has its union as least upper bound, and yet On has no maximal element. Also, any subclass which is a proper class is well ordered and cofinal in On.

We consider another class form which is a weakening of **ZS**(Q, U), but is of interest because of its relationship to a Principle of Cofinality.

ZRS(Q, U): *Every non-empty Q-ordered class, in which each initial segment generated by an element is a set and in which each U-ordered subset has an upper bound, either has a maximal element or contains a U-ordered subclass which is a proper class.*

Finally we consider

ZRC(Q, U): *Every non-empty Q-ordered class, in which each initial segment generated by an element is a set and in which each U-ordered subclass has an upper bound, has a maximal element.*

However, it turns out that **ZRS**$(Q, U) \equiv$ **ZRC**(Q, U).

Lemma 6.1 **ZRS**$(Q, U) \equiv$ **ZRC**(Q, U) *for all Q and U.*

Proof: Clearly, **ZRC**$(Q, U) \rightarrow$ **ZRS**(Q, U). Conversely, suppose **ZRS**(Q, U) holds and that X satisfies the hypothesis of **ZRC**(Q, U). Then, by **ZRS**(Q, U), either X has a maximal element or X has a U-ordered subclass Y which is a proper class and has an upper bound b. Clearly, $Y \subseteq \overleftarrow{b}$. By hypothesis, $\overleftarrow{b}$ is a set, and Y, being a proper class, cannot be a subclass of a set. Thus, X has a maximal element. Q.E.D.

Consequently, in what follows we shall omit consideration of **ZRC**(Q, U).

We easily see that for all Q and U

$$\mathbf{ZC}(Q, U) \rightarrow \mathbf{ZS}(Q, U) \rightarrow \mathbf{ZRS}(Q, U) \rightarrow \mathbf{Z}(Q, U).$$

Therefore, if **Z**(Q, U) is false in **NBG**$^\circ$ so is each of the corresponding class forms. Also, if $Q \rightarrow U$, each of the corresponding class forms is provable in **NBG**$^\circ$. Moreover, as for the set forms, we have the rules:

If $U_1 \rightarrow U_2$ then **ZT**$(Q, U_1) \rightarrow$ **ZT**(Q, U_2) for all Q and for **T** = **C**, **S**, or **RS**.
If $Q_1 \rightarrow Q_2$ then **ZT**$(Q_2, U) \rightarrow$ **ZT**(Q_1, U) for all Q and for **T** = **C**, **S**, or **RS**.

Consequently, we see that each of the following class forms are false in **NBG**° (see [1], section 2):

ZC(Q, U) for Q = A, AS, C, or AS & C, and U = TR & C, P, L, D, R, W, F, or T,
ZC(Q, U) for Q = A or AS, and U = TR, C, or AS & C,
ZC(C, U) for U = AS, TR, or AS & C,
ZC(A, AS), and **ZC**(AS & C, TR).

Similarly for **ZS**(Q, U), **ZRS**(Q, U), and **Z**(Q, U).

The following class forms are provable in **NBG**°:

ZC(Q, U) for Q = U,
ZC(Q, A) for all Q,
ZC(Q, AS) for Q = AS & C, P, L, D, R, W, F, or T,
ZC(Q, TR) for Q = TR & C, P, L, D, R, W, F, or T,
ZC(Q, C) for Q = AS & C, TR & C, L, or W,
ZC(Q, AS & C) for Q = L, or W,
ZC(Q, TR & C) for Q = L, or W,
ZC(Q, P) for Q = L, D, R, W, F, or T,
ZC(Q, L) for Q = W,
ZC(Q, D) for Q = L or W,
ZC(Q, R) for Q = L, W, F, or T,
ZC(Q, U) for Q = W, F, or T, and U = F or T.

Similarly for **ZS**(Q, U), **ZRS**(Q, U) and **Z**(Q, U).

We see also that **ZC**(TR, W) is the strongest of the remaining class forms of Zorn's Lemma. That is, in **NBG**°, **ZC**(TR, W) implies each of the remaining class forms. We shall show next that there is a class form of the Well Ordering Theorem which implies **ZC**(TR, W). Let **WOS** be the statement:

Each proper class is equipollent to On.

Theorem 6.2 **WOS** $\rightarrow$ **ZC**(TR, W).

Proof: Let X be a non-empty class and R a transitive relation on X such that each R-well ordered subclass of X has an R-upper bound. Suppose F: On $\approx X$. Define a function G on On such that

$$G(0) = F(0),$$
$$G(\alpha) = \begin{cases} F(\alpha) \text{ if } (\forall\beta < \alpha)(G(\beta)\ R\, F(\alpha)), \\ F(0) \text{ otherwise.} \end{cases}$$

The range of G is an R-well ordered quasi-cofinal subclass of X. Consequently, an R-upper bound for the range of G is an R-maximal element of X. Q.E.D.

Next, let

A* = {**ZS**(Q, U): Q = TR, P, R, F, or T, and U = C, AS & C, TR & C, L, D, or W},
B* = {**ZS**(Q, U): Q = L or R, and U = F or T; or Q = L and U = W},
C* = {**ZS**(Q, R): Q = P or D},
D* = {**ZS**(Q, U): Q = P or D, and U = F or T},
E* = {**ZS**(TR, U): U = P or AS},
F* = {**ZS**(P, U): U = C, AS & C, TR & C, or L},
G* = {**ZS**(TR & C, U): U = W, F, or T},
H* = {**ZS**(TR & C, U): U = AS, AS & C, P, L, D, or R},
I* = {**ZS**(TR, U): U = F or T},
J* = {**ZS**(TR, R)}.

Using the same arguments as used for the corresponding set forms, it is easy to show that each pair of statements in each of the sets B*-J* are equivalent in **NBG°**. (The same remark is true if "**ZS**" is replaced by "**ZC**" or "**ZRS**".)

To show that each pair of statements in A* are equivalent in **NBG°**, we show first that the proof that **Z**(P, L) → **Z**(P, W) ([1], Theorem 2.2) can be modified to show that **ZS**(P, L) → **ZS**(P, W).

Theorem 6.3 **ZS**(P, L) → **ZS**(P, W).

Proof: Let X be a class partially ordered by R in which every well ordered subset has an upper bound. Let $W = \{\langle s, R/s\rangle : s \subseteq X \,\&\, R/s \text{ well orders } s\}$. W is partially ordered by I, the initial segment relation, and every linearly ordered subset has an upper bound. By **ZS**(P, L), W has a maximal element or a linearly ordered subclass which is a proper class. If W has a maximal element, m, then an upper bound for m is a maximal element of X. Otherwise, W has a linearly ordered subclass C, which is a proper class. $\bigcup(\mathfrak{D}(C))$ is well ordered by R and is a proper class. Thus, **ZS**(P, L) → **ZS**(P, W). Q.E.D.

Similarly, it can be shown that **ZS**(TR, C) → **ZS**(TR, W), **ZS**(R, L) → **ZS**(R, W), **ZS**(P, D) → **ZS**(P, W), and **ZS**(Q, U) → **ZS**(TR, W) for Q = R, F, or T, and U = W, L, C, or D. Consequently, the 30 statements in the set A* are pairwise equivalent in **NBG°**. The same statement holds if "**ZS**" is replaced by "**ZRS**". However, the proof of 6.3 cannot be used to prove **ZC**(P, L) → **ZC**(P, W) because the class W in the proof does not necessarily satisfy the hypothesis of **ZC**(P, L). In fact, we do not know whether the statements in A* with "**ZS**" replaced by "**ZC**" are pairwise equivalent in **NBG°**. However, we do have the following diagram for these statements:

$$\begin{tikzcd}
ZC(T,W) \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & ZC(T,L) \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & ZC(T,AS\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & ZC(T,TR\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & ZC(T,C) \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & ZC(T,D) \arrow[d, phantom, "\equiv"] \\
ZC(F,W) \arrow[r, phantom, "\equiv"] & ZC(F,L) \arrow[r, phantom, "\equiv"] & ZC(F,AS\,\&\,C) \arrow[r, phantom, "\equiv"] & ZC(F,TR\,\&\,C) \arrow[r, phantom, "\equiv"] & ZC(F,C) \arrow[r, phantom, "\equiv"] & ZC(F,D) \\
ZC(R,W) \arrow[r] \arrow[u] & ZC(R,L) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(R,AS\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(R,TR\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(R,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(R,D) \arrow[u] \\
ZC(P,W) \arrow[r] \arrow[u] & ZC(P,L) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(P,AS\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(P,TR\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(P,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(P,D) \arrow[u] \\
ZC(TR,W) \arrow[r] \arrow[u] & ZC(TR,L) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(TR,AS\,\&\,C) \arrow[r] \arrow[u] \arrow[dr] & ZC(TR,TR\,\&\,C) \arrow[r, phantom, "\equiv"] \arrow[u] & ZC(TR,C) \arrow[u] & \\
 & & & ZC(TR,D) \arrow[uurr, bend right] & &
\end{tikzcd}$$

Figure 6.1

Consequently, to prove pairwise equivalence it would be sufficient to prove that **ZC**(T, W) $\rightarrow$ **ZC**(TR, W). However, we can prove **ZRS**(P, L) $\rightarrow$ **WOS** in **NBG**, and this is sufficient to prove that each statement in A* with "**ZS**" replaced by "**ZC**" or "**ZRS**" implies **WOS** in **NBG**.

It is easy to see that **WOS** is equivalent to the statement **WOS**$'$:

The universe can be well ordered in such a way that each initial segment is a set.

In [2], Theorem 3.28S, there is a proof that **ZS**(P, L) $\rightarrow$ **WOS**$'$. The axiom of regularity is used in the proof. This proof is easily modified to show **ZRS**(P, L) $\rightarrow$ **WOS**$'$. We can also prove a class form of Theorem 2.7, [1].

Theorem 6.4 **NBG** $\vdash$ **ZRS**(D, W) $\rightarrow$ **WOS**$'$.

Proof: Let

$A = \{\langle x, w\rangle: w \subseteq x \times x,\ w \text{ well orders } x,\ \&\ (\forall s, t)(s \in x\ \&\ \rho(t) < \rho(s) \rightarrow t \in x)\}$.

Since (**ZRS**(D, W) $\rightarrow$ **Z**(D, W) & (**Z**(D, W) $\equiv$ **WO**), A is a proper class. Order A as follows: $\langle x, w\rangle\ S\langle x', w'\rangle$ iff $x \subset x'$ or $\langle x, w\rangle = \langle x', w\rangle$. S partially orders A and every initial segment is a set. Moreover, if $\langle x, w\rangle$ and $\langle x', w'\rangle$ are not S-related then there exists a β such that $x \cup x' \subseteq \{u: \rho(u) \leqslant \beta\}$. Let α be the least such β. Let $y = x \cup x'$ and order y by $w^* = w \cup (w' \cap (x' \sim x)^2) \cup (x \times (x' \sim x))$. Then $\langle y, w^*\rangle \in A$. Since it could be that $x = x'$ but $w \neq w'$, we need to extend y by one more element. So if $\{u: \rho(u) \leqslant \alpha\} \sim y \neq \emptyset$, let u' be an element of this set. Otherwise choose some u' of rank $\alpha + 1$. Then $y \cup \{u'\}$ ordered by $w^* \cup ((y \cup \{u\}) \times \{u\})$ yields an upper bound for $\langle x, w\rangle$ and $\langle x', w'\rangle$. Thus A is directed by S. If $w = \{\langle x_\alpha, w_\alpha\rangle: \alpha < \lambda\}$ is a well ordered subset of A, then we can construct an upper bound for w in the following manner. Let $x = \bigcup\limits_{\alpha<\lambda} x_\alpha$. For $u \in x$ let $O(u)$ = the least α such that u is in x_α. Order x as follows: uRv iff $O(u) < O(v)$ or $[O(u) = O(v) = \alpha\ \&\ u\, w_\alpha v]$. Clearly, $\langle x, R\rangle$ is in A and each $x_\alpha \subseteq x$ for $\alpha < \lambda$. If for some α, $x_\alpha = x$ then we can extend x by one more appropriately chosen element so that we surely have an upper bound for w. So by **ZRS**(D, W), A has a well ordered subclass, T, which is a proper class. We can now construct a well ordering of V such that every initial segment is a set using T and a method similar to that just used to construct an upper bound for w. Q.E.D.

The set form of the principle of cofinality is **C**(Q, U):

Every Q-ordered set contains a quasi-cofinal U-ordered subset.

We shall consider two class forms of this principle.

CC(Q, U): *Every Q-ordered class has a quasi-cofinal U-ordered subclass.*

CRS(Q, U): *Every Q-ordered class, in which every initial segment generated by an element is a set, has a quasi-cofinal U-ordered subclass.*

For set forms, a variation of Zorn's Lemma is equivalent to a corrrsponding principle of cofinality and the situation is similar for class forms:

Theorem 6.5 *If Q is at least a transitive order and U is a property on classes which holds for any singleton, then*

(a) **ZC**$(Q, U) \equiv$ **CC**(Q, U),

and

(b) **ZRS**$(Q, U) \equiv$ **CRS**(Q, U).

Proof: We will prove (b), the proof of (a) is similar. Assume **ZRS**(Q, U), and let X be a Q-ordered class under the relation R in which every initial segment is a set. If X has a maximal element m, then $\{m\}$ is a quasi-cofinal U-ordered subset. If X has a U-ordered subclass Y which is a proper class, Y must be quasi-cofinal. For if not, then Y would have an upper bound, u, and $Y \subseteq \overleftarrow{u}$. This is impossible since $\overleftarrow{u}$ is a set. The other possibility is that X has a U-ordered subset which has no upper bound. In this case X has a quasi-cofinal U-ordered subset. Therefore, **ZRS**$(Q, U) \to$ **CRS**(Q, U).

Assume **CRS**(Q, U) and let X ordered by R satisfy the hypotheses of **ZRS**(Q, U). Then X has a quasi-cofinal U-ordered subclass, Y. If Y is a set, then Y has an upper bound which is clearly a maximal element. If Y is a proper class the result clearly follows. Therefore, **CRS**$(Q, U) \to$ **ZRS**(Q, U), and the equivalence is established. Q.E.D.

Next, we shall show that there is a class form of Felgner's Theorem (2.4, [1]). To do this we use a class form of the order extension principle **OE**. Let **OES** be the statement:

Every partial order of a class can be extended to a linear order,

and let **OER** be:

Every partial order of a class, such that every initial segment generated by an element is a set, can be extended to a linear order such that every initial segment generated by an element (with respect to the extended order) is a set.

Lemma 6.6 **NBG** $\vdash$ **OES** $\to$ **OER**.

Proof: Let X be a class partially ordered by R in which every $\overleftarrow{y}$ is a set. Define $S(O) = \{x\colon x \in X \ \&\ (\rho(x) = O \text{ or } (\exists y)(y \in X \ \&\ \rho(y) = O \ \&\ xRy))\}$. By **AR** (and either there are no individuals or the class of individuals is a set) and every $\overleftarrow{y}$ is a set, $S(O)$ is a set. By **OES** there is a linear order R^* on X which extends R. Let $f(0) = R^*/S(O)$. In general, for $\alpha > 0$, define $S(\alpha) = \{x\colon x \in X \ \&\ (\rho(x) = \alpha \text{ or } (\exists y)(y \in X \ \&\ \rho(y) = \alpha \ \&\ xRy))\} \sim \bigcup_{\beta<\alpha} (S(\beta))$. As for $S(O)$ each $S(\alpha)$ is a set. Let $f(\alpha)$ be the restriction of R^* to $S(\alpha)$. Now define L on X by: xLy iff $(\exists\alpha)(\exists\beta)(x \in S(\alpha) \ \&\ y \in S(\beta) \ \&\ \alpha < \beta)$ or $(\exists\gamma)(x, y \in S(\gamma) \ \&$

$xf(\gamma)y$). L is clearly an extension of R. It is also clear that $\bigcup_{\alpha\in\mathrm{On}} (S(\alpha)) = X$ and $\{S(\alpha)\colon \alpha\in\mathrm{On}\}$ is a disjoint collection. It follows that L is connected on X.

Now suppose xLy and yLz. Let $t(x) = \alpha$ where α is the unique ordinal number such that $x\in S(\alpha)$.

Case i: $t(x) < t(y) < t(z)$. Then clearly xLz.
Case ii: $t(x) < t(y) = t(z) = \alpha$, and $yf(\alpha)z$. Then $t(x) < t(z)$. Therefore, xLz.
Case iii: $\alpha = t(x) = t(y) < t(z)$ and $zf(\alpha)y$. Then $t(x) < t(z)$. Therefore, xLz.
Case iv: $\alpha = t(x) = t(y) = t(z)$ and $xf(\alpha)y$ and $yf(\alpha)z$. Then $xf(\alpha)z$ by the transitivity of $f(\alpha)$. Therefore, xLz.

Thus L is transitive on X. It is also clear that L is antisymmetric. That every $\overleftarrow{y}$ is a set follows because if $t(y) = \alpha$ then $\overleftarrow{y} \subseteq \bigcup_{\beta\leq\alpha} (S(\beta))$ which is a set. This completes the proof of the lemma. Q.E.D.

Theorem 6.7 **NBG** + **OES** $\vdash$ **CRS**(L, W) $\rightarrow$ **WOS′**.

Proof: Let A be as in the proof of Theorem 6.4. (Again, we assume that there are no individuals or the class of individuals is a set.) In **NBG°** + **OE** we have **CRS**(L, W) $\rightarrow$ **C**(L, W) and **C**(L, W) $\rightarrow$ **AC**. Thus, A is a proper class. Let I be the initial segment relation on A, so that A is a tree as ordered by I. Let $O(\langle x, w\rangle)$ = the ordinal number of x as well ordered by w. Let P be a linear order of A (for instance one which extends =). Define I^* on A by: $\langle x,w\rangle\ I^*\langle x',w'\rangle$ iff $O(\langle x,w\rangle) < O(\langle x',w'\rangle)$ or $[O(\langle x,w\rangle) = O(\langle x',w'\rangle)$ & $\langle x,w\rangle\ P\langle x',w'\rangle]$. Now let Q be a linear order of $\mathcal{P}(V) = V$ which extends $\subseteq$ and such that every proper initial segment $\overleftarrow{y}$ is a set. This can be done by using **OER**. Now define I^{**} on A by: $\langle x,w\rangle\ I^{**}\langle x',w'\rangle$ iff ($x = x'$ & $\langle x,w\rangle\ I^*\langle x',w\rangle$) or xQx'. A ordered by I^{**} is a linearly ordered proper class such that every proper I^{**}-initial segment $\overleftarrow{y}$ is a set, and I^{**} extends I. (That $\overleftarrow{y}$ is a set follows because $\overleftarrow{y}$ is a subset of $\{\langle x',w'\rangle\colon x'Qx$ & $w'\subseteq x'\times x'$ & w' well orders $x'\}$ where $y = \langle x,w\rangle$. The set of well orderings of a set x is dominated by $\mathcal{P}(x\times x)$ which is a set. This along with every initial Q-segment $\overleftarrow{s}$ of $\mathcal{P}(V)$ being a set, shows that $\overleftarrow{y}$ is a set.)

By **CRS**(L, W), A ordered by I^{**} has a well ordered cofinal subclass W. Clearly every proper initial I^{**}-segment of W is a set. Therefore, either $W\approx\lambda$ for some $\lambda\in\mathrm{On}$ or $W\approx\mathrm{On}$. If W is a set we use the method of Felgner's proof (see [1], Theorem 2.4) to obtain a well ordering of V, and conclude that V is a set. This is false, so we have that W is a proper class and $W\approx\mathrm{On}$. So $W = \{\langle x_\alpha, w_\alpha\rangle\colon \alpha\in\mathrm{On}\}$. Define $s_0 = x_0$ and $s_\alpha = \bigcup_{\beta<\alpha} x_\beta$. Note each s_α is a set and also satisfies the formula:

$$(*)\qquad (\forall t)(\forall u)(t\in s\ \&\ \rho(u) < \rho(t) \rightarrow u\in s).$$

This follows because each x_α satisfies this property. Define $w^* = w_0$, and $w^*_\alpha = w_\alpha \cap \left(x_\alpha \sim \bigcup_{\beta<\alpha} x_\beta\right)^2$ for $\alpha > 0$. Now define, $v_0 = w^*_0$, and $v_\alpha = \bigcup_{\beta<\alpha} v_\beta\ \cup$

$\left(\bigcup_{\beta<\alpha} x_\beta \times \left(x_\alpha \sim \bigcup_{\beta<\alpha} x_\beta\right)\right) \cup w_\alpha^*$. Each v_α is a set and well orders s_α. Because s_α satisfies (*), for each α, $\langle s_\alpha, v_\alpha\rangle \in A$. Moreover, by construction, if $\beta < \alpha$, $\langle s_\beta, v_\beta\rangle\, I \langle s_\alpha, v_\alpha\rangle$. Therefore, $\{\langle s_\alpha, v_\alpha\rangle\colon \alpha \in \mathsf{On}\}$ is a well ordered subclass of A. Consider $S = \bigcup_{\alpha\in\mathsf{On}} s_\alpha$ as ordered by $\bigcup_{\alpha\in\mathsf{On}} v_\alpha = V^*$. It is clear that V^* well orders S and that every proper initial V^*-segment, $\overleftarrow{y}$, is a set. Also S is a proper class. Therefore, S contains elements of arbitrarily high rank. Moreover, because each s_α satisfies (*), so does S. Thus, if $z \in V$, there exists an α such that $\rho(z) = \alpha$. Now there exists $z' \in S$ such that $\rho(z) < \rho(z')$. But this implies $z \in S$ by (*). Hence, $S = V$. Q.E.D.

To complete the analogy between class and set forms of Zorn's Lemma, we give a class form of Theorem 2.9, [1]. Let $\mathbf{AC}^{\mathbf{WOS}}$ be the statement:

There is a choice function on any class X of non-empty sets such that $X \approx \mathsf{On}$.

Using a simple modification of the proof of Theorem 2.9, we obtain its class form:

Theorem 6.8 $\mathbf{ZRS}(\mathsf{TR}\ \&\ \mathsf{C},\ \mathsf{P}) \to \mathbf{AC}^{\mathbf{WOS}}$.

We summarize our results in Figure 6.2. (See also Figure 2.4 in [1].) Figure 6.2 remains valid if "**ZS**" is replaced by "**ZRS**". However, we do not have the same results for $\mathbf{ZC}(Q, U)$ because we were not able to prove that the statements in A* with "**ZS**" replaced by "**ZC**" are pairwise equivalent in $\mathbf{NBG}^\circ$. But it does follow from the Theorems 6.2 and 6.4 that these forms are equivalent in **NBG**. Thus, we have a somewhat weaker result for variations of Zorn's Lemma of the form $\mathbf{ZC}(Q, U)$.

$$\begin{tikzcd}
A^* \arrow[r] \arrow[dd] & F^* \arrow[r] & C^* \arrow[r] & H^* \arrow[r] & \mathbf{AC}^{\mathbf{WOS}} & \\
 & J^* \arrow[r] \arrow[ur] & E^* \arrow[ur] & & & \\
I^* \arrow[r] \arrow[ur] & D^* \arrow[r] \arrow[uur] & G^* \arrow[r] & B^* \arrow[rr, "\mathbf{OES}+\mathbf{AR}"] & & A^*
\end{tikzcd}$$

A* = $\{\mathbf{ZS}(Q, U)$: Q = TR, P, R, F, or T, and U = C, AS & C, TR & C, L, D, or W$\}$,
B* = $\{\mathbf{ZS}(Q, U)$: Q = L or R, and U = F or T; or Q = L and U = W$\}$,
C* = $\{\mathbf{ZS}(Q, \mathrm{R})$: Q = P or D$\}$,
D* = $\{\mathbf{ZS}(Q, U)$: Q = P or D, and U = F or T$\}$,
E* = $\{\mathbf{ZS}(\mathsf{TR}, U)$: U = P or AS$\}$,
F* = $\{\mathbf{ZS}(\mathsf{P}, U)$: U = C, AS & C, TR & C, or L$\}$,
G* = $\{\mathbf{ZS}(\mathsf{TR}\ \&\ \mathsf{C}, U)$: U = W, F, or T$\}$,
H* = $\{\mathbf{ZS}(\mathsf{TR}\ \&\ \mathsf{C}, U)$: U = AS, AS & C, P, L, D, or R$\}$,
I* = $\{\mathbf{ZS}(\mathsf{TR}, U)$: U = F or T$\}$,
J* = $\{\mathbf{ZS}(\mathsf{TR}, \mathsf{R})\}$.

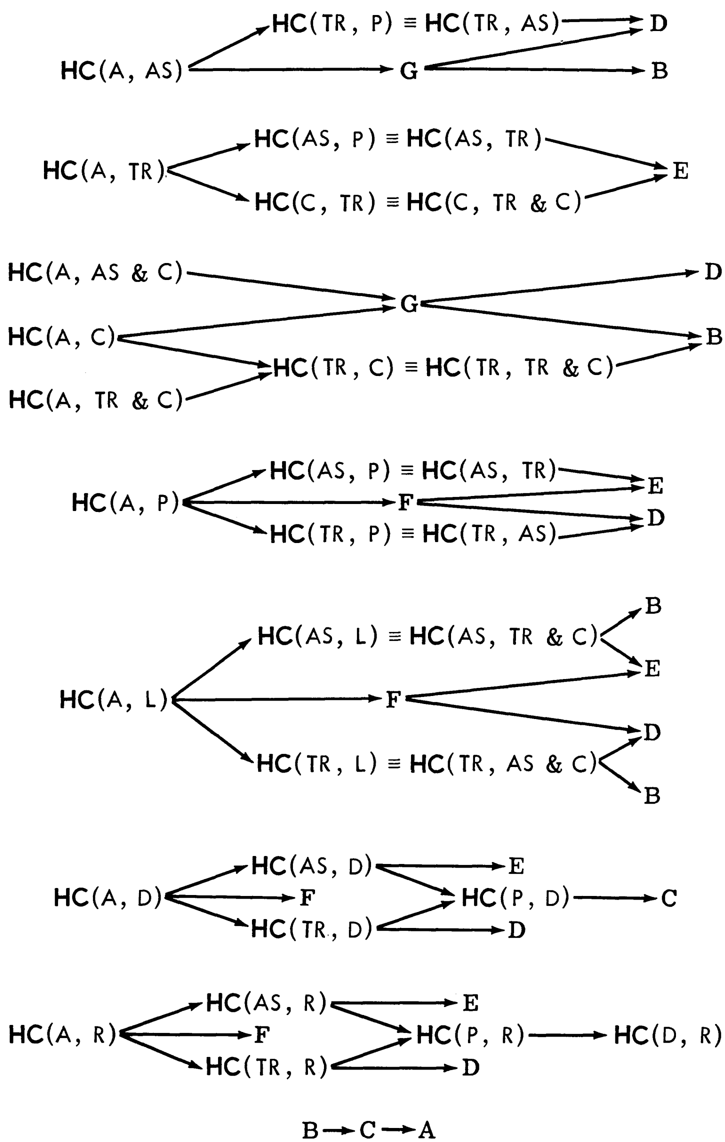
Not ($\mathbf{NBG}^\circ \vdash \mathbf{AC}^{\mathbf{WOS}} \to H^*$) $\qquad$ $\mathbf{NBG}^\circ \vdash \mathbf{WOS} \to A^*$
Not ($\mathbf{NBG}^\circ \vdash B^* \to A^*$) $\qquad$ $\mathbf{NBG} \vdash A^* \to \mathbf{WOS}$

Figure 6.2

7 *Hausdorff's Maximal Principle* Let **HC**(Q, U) denote the statement:

Every Q-ordered class has a $\subseteq$-maximal U-ordered subclass.

We consider the same thirteen possibilities for Q and U as before. As with the set forms, we are left with 73 variations which are not simply provable in **NBG**$^\circ$ or false in **NBG**$^\circ$. In fact, exactly the same relationships hold in **NBG**$^\circ$ between these 73 class forms as hold between the corresponding set forms. See Figure 7.1. In each of the sets A-G in Figure 7.1, each pair of statements are equivalent in **NBG**$^\circ$.



A = {**HC**(Q, U): Q = F or T, and U = C, AS & C, TR & C, L, D, or W},
B = {**HC**(Q, U): Q = P or D, and U = C, AS & C, TR & C, or L},
C = {**HC**(R, U): U = C, AS & C, TR & C, L, or D},
D = {**HC**(TR & C, U): U = AS, AS & C, P, L, D, or R},
E = {**HC**(AS & C, U): U = TR, TR & C, P, L, D, or R},
F = {**HC**(C, U): U = P, L, D, or R},
G = {**HC**(C, AS), **HC**(C, AS & C), **HC**(AS, C), **HC**(AS, AS & C)}.

Figure 7.1

Let **FCS** be the class form of the principle of finite character:

For every class X and every property $\mathcal{P}$ of finite character there exists a $\subseteq$-maximal subclass of X with the property $\mathcal{P}$.

As with the set forms,

$$\mathbf{FCS} \rightarrow \mathbf{HC}(\mathsf{A}, U)$$

for U = AS, TR, AS & C, C, TR & C, P, L, D, and R. Consequently, **FCS** implies each of the 73 variations of **HC**(Q, U) given in Figure 7.1.

Lemma 7.1 *If Q is at least a partial order and U = L, D, or W then* **HC**(Q, U) $\rightarrow$ **CC**(Q, U).

Proof: A $\subseteq$-maximal U-ordered subclass of a Q-ordered class is quasi-cofinal. Q.E.D.

(It is also true that **HC**(TR & C, AS) $\rightarrow$ **ZC**(TR & C, AS). The proof is the same as for the set forms. See [1], section **3**.)

It was shown in Theorem 6.5(a) that if Q is at least a transitive order and U holds for every singleton then **ZC**(Q, U) $\equiv$ **CC**(Q, U) and it was also shown above that **ZC**(Q, U) $\rightarrow$ **WOS**$'$ in **NBG** for Q = TR, P, R, F, or T, and U = C, AS & C, TR & C, L, D, or W. Therefore, it follows that each statement in the set A in Figure 7.1 implies **WOS**$'$ in **NBG** and each statement given in Figure 7.1 which implies A also implies **WOS**$'$ in **NBG**. Moreover, **HC**(D, R) $\rightarrow$ **WOS**$'$ in **NBG** because the class A of Theorem 6.4 ordered by $\check{I}$, the converse of I, is directed, and if Y is a $\subseteq$-maximal ramified subclass and $y \in Y$ then $Z = \overleftarrow{y} \cap Y$ is a quasi-cofinal well ordered subclass in A as ordered by I. Then $\bigcup \mathcal{D}(Z) = V$ and is well ordered by $\bigcup \mathcal{R}(Z)$ such that every proper initial segment is a set. (See Theorem 3.5 in [1].)

Next, we shall show that Theorems 3.14 and 3.15 of [1] also have class forms. Let **ACS** be the statement:

There is a choice class for each non-empty class of pairwise disjoint non-empty sets and let **AC**$^{\mathbf{LOS}}$ *be* **ACS** *for a linearly ordered class of non-empty pairwise disjoint sets.*

Then it is clear that we can prove

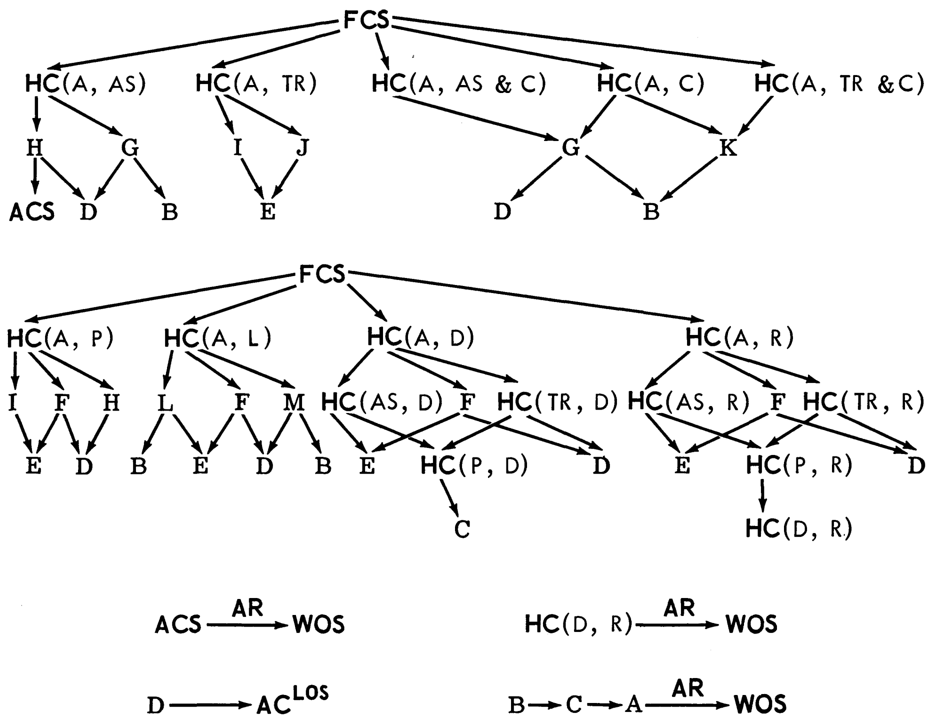
$$\mathbf{HC}(\mathsf{TR}, \mathsf{P}) \rightarrow \mathbf{ACS},$$

and

$$\mathsf{HC}(\mathsf{TR}\ \&\ \mathsf{C},\ \mathsf{L}) \to \mathsf{AC}^{\mathsf{LOS}},$$

similarly to the proofs of 3.14 and 3.15, respectively. It is shown in [2] that **ACS** → **WOS** in **NBG**. We summarize our results for the class forms of Hausdorff's maximal principle in Figure 7.2. In each of the sets A-M in Figure 7.2, each pair of statements are equivalent in **NBG°**.

(We leave it as an exercise for the reader to consider forms of Hausdorff's maximal principle in which the ordering relation has the property that each initial segment generated by an element is a set.)



See Figure 7.1 for the definitions of the sets A-G.

H = {HC(TR, P), HC(TR, AS)},
I = {HC(AS, P), HC(AS, TR)},
J = {HC(C, TR), HC(C, TR & C)},
K = {HC(TR, C), HC(TR, TR & C)},
L = {HC(TR, L), HC(TR, AS & C)},
M = {HC(TR, L), HC(TR, AS & C)}.

Figure 7.2

8. Appendix

The following is a list of statements used in this paper with the abbreviations used to denote them.

ACS: *There is a choice class for each non-empty class of pairwise disjoint non-empty sets.*

AC^{LOS}: *There is a choice class for each non-empty linearly ordered class of pairwise disjoint non-empty sets.*

AC^{WOS}: *There is a choice function on every class X of non-empty sets such that $X \approx$ On.*

AR: *For each non-empty class X there is a $y \in X$ such that $y \cap X = \emptyset$.*

CC (Q, U): *Every Q-ordered class has a quasi-cofinal U-ordered subclass.*

CRS (Q, U): *Every Q-ordered class, in which every initial segment generated by an element is a set, has a quasi-cofinal U-ordered subclass.*

FCS: *For every class X and every property of finite character $\mathcal{P}$ there is a $\subseteq$-maximal subclass of X with the property $\mathcal{P}$.*

HC (Q, U): *Every Q-ordered class has a $\subseteq$-maximal U-ordered subclass.*

OER: *Every partial order on a class, such that every initial segment generated by an element is a set, can be extended to a linear order such that every initial segment generated by an element (with respect to the extended order) is a set.*

OES: *Every partial order on a class can be extended to a linear order.*

WOS: *Each proper class is equipollent to On.*

WOS′: *The universe can be well ordered such that each initial segment is a set.*

ZC (Q, U): *Every non-empty Q-ordered class, in which every U-ordered subclass has an upper bound, has a maximal element.*

ZRC (Q, U): *Every non-empty Q-ordered class, in which each initial segment generated by an element is a set and in which each U-ordered subclass has an upper bound, has a maximal element.*

ZRS (Q, U): *Every non-empty Q-ordered class, in which each initial segment generated by an element is a set and in which each U-ordered subset has an upper bound, either has a maximal element or contains a U-ordered subclass which is a proper class.*

ZS (Q, U): *Every non-empty Q-ordered class, in which every U-ordered subset has an upper bound, either has a maximal element or contains a U-ordered subclass which is a proper class.*

REFERENCES

[1] Harper, J. M., and J. E. Rubin, "Variations of Zorn's lemma, principles of cofinality, and Hausdorff's maximal principle. Part I: Set forms," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 565-588.

[2] Rubin, H., and J. E. Rubin, *Equivalents of the Axiom of Choice*, North-Holland Publishing Co., Amsterdam (1963).

Purdue University
West Lafayette, Indiana