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Every Quotient Algebra for C_1 is Trivial

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1 In recent years, a number of different types of logics have been proposed with the intention of avoiding the various paradoxes of material implication, particularly the property that from a contradiction anything may be deduced. Two such types of logics are the relevance logics of Anderson and Belnap [1], and the paraconsistent logics in the vicinity of C_1. The logic C_1 has primitives $\neg$, $\supset$, &, v, and is given axiomatically below. In the opinion of this author, C_1 has various unsatisfactory features, two of which are that it lacks the theorem $A \supset \neg\neg A$, and that the rule of replacement ($\vdash A \equiv B$ implies $\vdash C(A) \equiv C(B)$, for any context C; $A \equiv B$ being defined as usual by $(A \supset B)$ & $(B \supset A)$) does not hold for C_1.

To date, there has been an outstanding problem (raised, for example, in [10], p. 508) about C_1: how to "algebraise" it. The aim of this paper is to contribute to the solution of that problem by proving that on certain very minimal assumptions C_1 has no nontrivial quotient algebra. We will say presently what it means for a quotient algebra to be trivial. It is suggested that the present result, in addition to "solving" the algebraisation problem, exhibits a further unsatisfactory feature of C_1, namely that C_1 lacks a proper biconditional. We hope to amplify this point in a later paper.

The present enterprise is to investigate the consequences of partitioning the formula algebra of C_1 into a quotient algebra of equivalence classes by some relation $\sim$ holding between formulas. The relation $\sim$ need *not* necessarily be syntactic, i.e., definable by a formula in the operators $\neg$, $\supset$, &, v. We impose the following four requirements on any such relation $\sim$ and quotient algebra: (a) $\sim$ is an equivalence relation, i.e., $A \sim A$, $A \sim B$ implies $B \sim A$, and $A \sim B$ and $B \sim C$ imply $A \sim C$. (b) The formula algebra is homomorphic to the quotient algebra (with corresponding operations) obtained from the equivalence relation; i.e., $A \sim B$ implies $C(A) \sim C(B)$, for any context C. (c) If $A \sim B$ and $\vdash A$ then $\vdash B$ (where '$\vdash$' means provability in C_1). This is necessary to prevent including

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nontheorems and theorems in the same equivalence class, which would have the consequence that we would be unable both to designate the equivalence class (without making some nontheorems theorems), and not to designate it (without refuting some theorem). Again, if $\vdash A$ and not $\vdash B$ then it is easy to show that A and B are distinguishable by the valuation semantics described below, and we would not want an algebraisation in which $|A|$ and $|B|$ were to be identified by the fact that $A \sim B$ when A and B are distinguishable semantically. We prove in Theorem 2 that conditions (b) and (c) in the context of C_1 imply the condition: if $A \sim B$ then $\vdash A \equiv B$. This fact is crucially used in the proof of our main theorem, Theorem 3. (d) The quotient algebra so obtained is nontrivial. We promised earlier a definition of this term. In one sense there are always at least two equivalence relations: one is 'A is the same formula as B', and another is 'A is a formula and B is a formula'. In order to avoid such trivial cases, we make a definition.

Definition 1 Let $\mathcal{L}$ be a sentential language and let $\sim$ and $\mathcal{L}/\sim$ be, respectively, equivalence relation and quotient algebra satisfying (a)-(c) above. Then $\mathcal{L}/\sim$ is *trivial* iff either $(\forall A \in \mathcal{L})(|A| = \{A\})$ (where $|A|$ is the equivalence class of A), or $(\forall A, B \in \mathcal{L})(|A| = |B|)$. Otherwise, $\mathcal{L}/\sim$ is *nontrivial*.

The fourth requirement, then, is that the quotient algebra determined by $\sim$ be nontrivial.

2 We now present C_1 formally. We begin with a language $\mathcal{L}$ consisting of a denumerable number of sentential variables p_i, $1 \leqslant i < \omega$, closed as usual under $\neg$, &, $\vee$, $\supset$. The operator $\equiv$ is defined as usual (see above), and we have two new defined symbols: A° is an abbreviation for $\neg(A \,\&\, \neg A)$, and $\neg^* A$ is an abbreviation for $\neg A \,\&\, A^\circ$. Capital letters from the beginning of the alphabet are metalinguistic schematic variables.

Definition 2 The logic C_1 is the smallest subset of $\mathcal{L}$ closed under uniform substitution and modus ponens (for $\supset$) and containing all instances of the following schemata:

(1) $A \supset (B \supset A)$
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
(3) $(A \,\&\, B) \supset A$
(4) $(A \,\&\, B) \supset B$
(5) $A \supset (B \supset (A \,\&\, B))$
(6) $A \supset (A \vee B)$
(7) $B \supset (A \vee B)$
(8) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(9) $A \vee \neg A$
(10) $\neg\neg A \supset A$
(11) $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$
(12) $(A^\circ \,\&\, B) \supset ((A \,\&\, B)^\circ \,\&\, (A \vee B)^\circ \,\&\, (A \supset B)^\circ)$.

Definition 3 A C_1-*valuation* is a function v: $\mathcal{L} \to \{1, 0\}$ such that

(1) $v(A) = 0 \Rightarrow v(\neg A) = 1$
(2) $v(\neg\neg A) = 1 \Rightarrow v(A) = 1$

(3) $v(B^\circ) = v(A \supset B) = v(A \supset \neg B) = 1 \Rightarrow v(A) = 0$
(4) $v(A) = 0$ or $v(B) = 1 \Longleftrightarrow v(A \supset B) = 1$
(5) $v(A) = v(B) = 1 \Longleftrightarrow v(A \,\&\, B) = 1$
(6) $v(A) = 1$ or $v(B) = 1 \Longleftrightarrow v(A \vee B) = 1$
(7) $v(A^\circ) = v(B^\circ) = 1 \Rightarrow v((A \vee B)^\circ) = v((A \,\&\, B)^\circ) = v((A \supset B)^\circ) = 1$.

A formula A is *true in a valuation* v iff $v(A) = 1$.

Theorem 1 *C_1 is sound and complete with respect to the class of all C_1-valuations; i.e., all and only theorems of C_1 are true in all C_1-valuations* (da Costa and Alves [11]). *C_1 is decidable* (Fidel. See [11], p. 627).

We use the decidability of C_1 extensively below, so we outline Fidel's decision procedure. A *quasi matrix for a formula A* is constructed as follows:

1. List all the propositional constants of A (in a horizontal line) and, as in truth tables for classical propositional calculus, list all possible assignments of 1 and 0 to them.

2. List all denials of propositional constants (to the right of the former list) of A and assign values as follows: if the propositional constant was assigned 0 its denial is assigned 1. If the constant was assigned 1, bifurcate the line on which the 1 occurs, and on one half the denial is assigned 0 and on the other 1.

3. List all remaining subformulas of A and negations of proper subformulas of A and proceed as follows:

(3.1) If the major connective of any such formula is &, $\vee$, or $\supset$, its value is determined from the values of its two components as in classical logic.

(3.2) If the formula is of the form $\neg B$ and B was assigned 0, assign B the value 1.

(3.3) If the formula is of the form $\neg B$ and B was assigned 1, then there are several subcases

(3.3.1) B is of the form $\neg C$ and C was assigned 0. Assign $\neg B$ (i.e., $\neg\neg C$) the value 0.

(3.3.2) B is of the form $\neg C$ and C assigned 1. Bifurcate the line and assign $\neg B$ the value 0 on one bifurcation and 1 on the other.

(3.3.3) B is of the form $C \,\&\, \neg C$ or $\neg C \,\&\, C$. Assign $\neg B$ the value 0.

(3.3.4) B is of the form $C \circ D$ (where $\circ$ is &, $\vee$, or $\supset$) but not of the form 3.3.3. If the value of C is different from the value of $\neg C$ and the value of D is different from the value of $\neg D$, assign $\neg B$ the value 0. Otherwise, bifurcate the line and assign $\neg B$ the value 0 on one half and 1 on the other half.

We can now state the outcomes of this decision procedure: if some line of the quasi matrix of A assigns 0 to A, then for some C_1-valuation v, $v(A) = 0$, and so A is not valid and not a theorem. Otherwise, for all C_1-valuations v, $v(A) = 1$ and A is valid and a theorem.

3 In Section 1 the claim was made that the conditions (b) and (c) we imposed on equivalence relations imply that if $A \sim B$ then $\vdash A \equiv B$. In this section, we prove that result.

Theorem 2 *If an equivalence relation $\sim$ satisfies the conditions:* (b) *if $A \sim B$ then $C(A) \sim C(B)$ for any context C, and* (c) *if $A \sim B$ and $\vdash A$ then $\vdash B$, then it necessarily satisfies the condition: if $A \sim B$ then $\vdash A \equiv B$.*

Proof: Suppose the antecedent of the theorem, and suppose that $A \sim B$. We need to prove that $\vdash A \equiv B$. If either $\vdash A$ or $\vdash B$ then by the antecedent of the theorem $\vdash B$ and $\vdash A$, respectively. Hence, by the properties of C_1 (Axioms (1) and (5)), $\vdash A \equiv B$. So suppose neither $\vdash A$ nor $\vdash B$, and suppose, for contradiction, $\nvdash A \equiv B$. If $\nvdash A \equiv B$ then, as is well known, there is a C_1-valuation v such that $v(A) \neq v(B)$. Suppose that $v(A) = 1$, $v(B) = 0$. It follows from the properties of C_1-valuations that $v(\neg{}^*A) = 0$, $v(A \vee \neg{}^*A) = 1$, and $v(B \vee \neg{}^*A) = 0$. From $v(B \vee \neg{}^*A) = 0$, we have $\nvdash B \vee \neg{}^*A$. Now by condition (b) of the theorem, if $A \sim B$ then $(A \vee \neg{}^*A) \sim (B \vee \neg{}^*A)$. But it is also a fact that $\vdash A \vee \neg{}^*A$. Hence, by condition (c), $\vdash B \vee \neg{}^*A$. Contradiction. Hence $\vdash A \equiv B$. This proves the theorem.

4 We now proceed to our main theorem. First, we need some lemmas.

Lemma 1 *Let $\vdash A \equiv B$. The following are sufficient conditions for the truth of $\vdash \neg A \equiv \neg B$:*

(1) *$A = B$ (A is the same formula as B)*
(2) *for all C_1-valuations v, $v(A) = 0$ (equivalently, $v(B) = 0$)*
(3) *for some C_1-valuation v, $v(A) = v(B) = 1$ and for all C_1-valuations v_1, v_2, $v_1(A) = 1$ implies $v_1(\neg A) = 0$ and $v_2(B) = 1$ implies $v_2(\neg B) = 0$.*

The following are sufficient conditions for the truth of (3)*: for some v, $v(A) = v(B) = 1$, together with any condition from List One together with any condition from List Two.*

List One:

(i) *A is of the form $\neg C$ and $(\forall v)(v(A) = 1$ implies $v(C) = 0)$*
(ii) *A is of the form $C \circ D$ (where $\circ$ is &, $\vee$, $\supset$) and C is $\neg D$ or D is $\neg C$*
(iii) *A is of the form $C \circ D$ (where $\circ$ is &, $\vee$, $\supset$) and $(\forall v)(v(C \circ D) = 1$ implies $v(C) \neq v(\neg C)$ and $v(D) \neq v(\neg D))$.*

List Two: as for List One with 'B', 'E', 'F', replacing 'A', 'C', 'D', respectively.

Proof: Clearly (1) is sufficient. If $(\forall v)(v(A) = 0)$ and $\vdash A \equiv B$ then $(\forall v)(v(B) = 0)$. But $v(A) = 0$ implies $v(\neg A) = 1$, and similarly for B, so $(\forall v)(v(\neg A) = v(\neg B))$ and so, by the conditions for & and $\supset$, $(\forall v)(v(\neg A \equiv \neg B) = 1)$, i.e., $\vDash \neg A \equiv \neg B$, i.e., $\vdash \neg A \equiv \neg B$. Hence (2) is sufficient. As to (3), either $v(A) = 0$ or $v(A) = 1$. If $v(A) = 0$ then $v(\neg A) = 1$, and by hypothesis if $v(A) = 1$ then $v(\neg A) = 0$. But $\vdash A \equiv B$ so A and B have the same values. But again by our hypothesis, if $v(B) = 1$ then $v(\neg B) = 0$, and clearly if $v(B) = 0$ then $v(\neg B) = 1$. Hence $\neg A$ and $\neg B$ have the same values in all valuations and so, as above, $\vDash \neg A \equiv \neg B$, i.e., $\vdash \neg A \equiv \neg B$. Hence (3) is sufficient.

Now we show that List One combined with List Two are sufficient conditions for (3). It is sufficient to show that List One gives sufficient conditions for A to satisfy (3), as the proof for B is identical. Clearly (i) is sufficient: Suppose $v(A) = v(\neg C) = 1$. Then from (i) $v(C) = 0$, but then by condition (2) of Definition 3, $v(\neg\neg C) = 0 = v(\neg A)$.

Ad (ii) Construct a quasi matrix for $\neg A$, i.e., $\neg(C \& \neg C)$. Whenever A is assigned 1, by 3.3.3 of the definition of a quasi matrix above, $\neg A$ is assigned 0. Hence there is no C_1-valuation in which A is assigned 1 and also $\neg A$ assigned 1.

Ad. (iii). As for (ii).

Lemma 2 *Let $\vdash A \equiv B$. If none of the above sufficient conditions (1)-(3) obtain, then $\nvdash \neg A \equiv \neg B$. If none of (i)-(iii) obtain, (3) does not obtain.*

Proof: Construct a quasi matrix for $A \equiv B$, which also involves giving values to $\neg A$ and $\neg B$. Since $\vdash A \equiv B$, A and B receive the same value on all lines. If none of (1)-(3) obtain, then we must bifurcate at least one of the lines for $\neg A$, $\neg B$. But this will ensure that there is a valuation where $v(\neg A) \neq v(\neg B)$. We now simply extend the quasi matrix to a quasi matrix for $\neg A \equiv \neg B$ by calculating the values of $\neg A \supset \neg B$, $\neg B \supset \neg A$, $\neg(\neg A \supset \neg B)$, $\neg(\neg B \supset \neg A)$, and $(\neg A \supset \neg B)$ & $(\neg B \supset \neg A)$. The values of $\neg A \supset \neg B$ and $\neg B \supset \neg A$ are calculated directly, the values of their respective denials being irrelevant. At least one of $\neg A \supset \neg B$, $\neg B \supset \neg A$ is zero on any of the above lines where $\neg A$ and $\neg B$ have different values, and so their conjunction calculates to 0, i.e., $\not\models \neg A \equiv \neg B$, i.e., $\nvdash \neg A \equiv \neg B$.

By inspection of the conditions for construction of a quasi matrix, provided that for some v, $v(A) = 1$, if none of (i)-(iii) apply we must bifurcate the table for $\neg A$. But then we can conclude that there is a C_1-valuation v such that $v(A) = 1$ and $v(\neg A) = 1$, i.e., (3) is false as required.

We can now prove the promised result, which we state as follows

Theorem 3 *No equivalence relation for C_1 satisfying the above conditions (a)-(c) of Section 1 for equivalence relations determines a nontrivial quotient algebra.*

Proof: We prove this by proving that for any such relation $\sim$, if $A \sim B$ then $A = B$ (A is the same formula as B). Suppose $A \sim B$. By condition (b) for $\sim$, $C(A) \sim C(B)$, and so in particular $\neg A \sim \neg B$, $(A \vee p_1) \sim (B \vee p_1)$ and $\neg(A \vee p_1) \sim \neg(B \vee p_1)$, where p_1 is the first propositional variable. Hence, by Theorem 2, $\vdash A \equiv B$, $\vdash \neg A \equiv \neg B$, $\vdash (A \vee p_1) \equiv (B \vee p_1)$, and $\vdash \neg(A \vee p_1) \equiv \neg(B \vee p_1)$. We show that these four formulas are theorems iff $A = B$. Clearly the four theorems hold if $A = B$, because $\vdash A \supset A$. So suppose $A \neq B$. If $\vdash A \equiv B$ and $\vdash \neg A \equiv \neg B$, then we have both $\vdash (A \vee p_1) \equiv (B \vee p_1)$, and also from Lemma 2 that conditions (2) and (3) of Lemma 1 hold with respect to A and B. Construct a quasi matrix for $\neg(A \vee p_1) \equiv \neg(B \vee p_1)$, and consider the (bifurcated) line on which p_1 receives value 1. This line bifurcates giving $\neg p_1$ the value 1 on one half and $\neg p_1$ the value 0 on the other half. Consider the half on which $\neg p_1$ has the value 1. Compute the quasi matrix, including the values of A, $\neg A$, B, $\neg B$, $A \vee p_1$, $B \vee p_1$, $\neg(A \vee p_1)$, $\neg(B \vee p_1)$. Now, since p_1 has 1, $A \vee p_1$ and $B \vee p_1$ both have 1. But now the value of p_1 is the same as the value of $\neg p_1$ on the lines in question, and the values of $A \vee p_1$ and $B \vee p_1$ are 1. So, taking the 'A' and 'B' of Lemma 1 to be $A \vee p_1$ and $B \vee p_1$, respectively, the sufficient conditions (i)-(iii) of Lemma 1 for the truth of (3) of Lemma 1 (i.e., for all v_1, $v_1(A \vee p_1) = 1$ implies $v_1(\neg(A \vee p_1)) = 0$; and for all v_2, $v_2(B \vee p_1) = 1$ implies $v_2(\neg(B \vee p_1)) = 0$) fail. So, by Lemma 2, (3) fails. But (1) also fails because if $A \neq B$ then $A \vee p_1 \neq B \vee p_1$, and (2) fails since neither $A \vee p_1$, nor $B \vee p_1$ have the value 0. Hence all of the sufficient conditions (1)-(3) of Lemma 1 for the truth of $\vdash \neg(A \vee p_1) \equiv \neg(B \vee p_1)$ fail, and so by Lemma 2, $\nvdash \neg(A \vee p_1) \equiv \neg(B \vee p_1)$. Contradiction. Thus, if $\vdash A \sim B$, then $A = B$.

In a sequel, we hope to study the algebraic properties of systems in the neighbourhood of C_1 which have nontrivial algebras. These systems cannot be any weaker than C_1, of course: if we can show that $A \sim B$ implies $A = B$ for C_1, then the same must hold for any system with weaker deductive resources than C_1. Thus for instance we have the corollary

Corollary *All of the systems C_2, C_3, . . . of [11] have only trivial quotient algebras.*

5 That is not quite an end to the question of the algebraisation of C_1, however. There are equivalence relations which partition the formula algebra of C_1 into such trivial quotient algebras. The relation of Theorem 3 will do: $A \sim B$ iff $\vdash(A \equiv B)$ & $(\neg A \equiv \neg B)$ & $((A \vee p_1) \equiv (B \vee p_1))$ & $(\neg(A \vee p_1) \equiv \neg(B \vee p_1))$. Now nothing so far established shows that there might not be interesting (though perhaps bizarre) partial orders which can be imposed on this quotient algebra. Such partial orders will, as usual, be reflexive, antisymmetric, and transitive; the antisymmetry property in question being $|A| \leqslant |B|$ and $|B| \leqslant |A|$ implies $|A| = |B|$. In the light of Theorem 3, this becomes $|A| \leqslant |B|$ and $|B| \leqslant |A|$ implies $A = B$.

There do exist such partial orders. One is: $|A| \leqslant |B|$ iff $\vdash A \supset B$ and either $\nvdash B \supset A$ or $A = B$. The proof that it is a partial order is not difficult. The algebra obtained by imposing this partial order on the (trivial) quotient algebra with singleton equivalence classes is, as might be expected after Theorem 3, rather strange. Some of its properties are: (i) If $|A| \leqslant |B|$, then if $\vdash B$ then either $\nvdash A$ or $A = B$. In fact, every theorem is a maximal element. No non-theorem is maximal, for if $\nvdash A$, then $|A| \leqslant |A \vee \neg A|$. (ii) If $|A| \leqslant |B|$ then if $\vdash A \equiv \neg^*(C \vee \neg C)$, then either $\nvdash B \equiv \neg^*(C \vee \neg C)$ or $A = B$. Everything which is equivalent to a negation* of a theorem is a minimal element. (iii) The algebra is not a lattice with respect to $\vee$ and &: $|A|$, $|C| \leqslant |A \vee C|$ fails for the case where $A = C$, and dually for &.

Another partial order, of little interest, is determined by "A is a subformula of B". There may be other such partial orders, though it is a fair conjecture that any such will turn out to be equally uninteresting or strange. It would be desirable to find some set of conditions for a "reasonable" partial order, according to which it could be shown that there are no reasonable partial orders on quotient algebras for C_1.

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