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A Three-Valued Free Logic for Presuppositional Languages

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Peter Strawson revived the topic of presupposition in 1950.* In subsequent writings, he proposed a notion of semantic presupposition along the following lines: a statement A presupposes a statement B if and only if A is neither true nor false unless B is true. Now, it is a simple matter to show that this (semantic) relation is distinct from the more traditional notion of semantic entailment (sometimes called "logical implication"). It is another matter to show that presupposition exists in English, though I think this is also easily done. This, however, is not to say that Strawson was right on all—or even some—of the cases he considered, for what any particular statement presupposes is a further matter still; this last matter is to be decided by what I call one's "theory of presuppositions". It is not my aim to offer or to defend any particular theory of presuppositions here. Instead, I am concerned with the formalization of a logic in which the relation of presupposition is a nontrivial relation.

With these things in mind, the problem this paper is addressed to is this: Can a logical calculus be developed which will allow some statements to be neither true nor false if they have a presupposition which fails to be true? I present in this paper an axiomatic, first-order logic; a semantics is provided, and the axioms are proved to be sound and complete with respect to the semantics. The logic allows a statement to be true, false, or neither-true-nor-false. In every case, if all the statements being considered have a truth-value (true or false), the logic gives the same results that classical logic does. More specifically, what is wanted here is a logic that does three things: it must allow

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statements to be truth-valueless when they have a presupposition that fails to be true; it must be what Meyer and Lambert have called "universally free";[1] and it must otherwise agree with classical two-valued logic. (Classical two-valued logic will hereafter be referred to as "*C2*".) Furthermore, just as the traditional concept of semantic entailment is *representable* (*not expressible*) in *C2* by the validity of an appropriate conditional ("If-then") statement, so I wish to be able to represent in my logic the concept of semantic presupposition by the validity of another statement. (Of course, I still wish to be able to represent semantic entailment in this logic by the validity of an appropriate conditional.) That is, I wish to develop a first-order logic in which the notion of semantic presupposition is taken just as seriously as semantic entailment is taken in *C2*. The logic presented here does all of these things.

The logic (called '*F3*' here) is presented in Section 1, and strong soundness and completeness theorems are proved there. Section 2 contains a discussion of some of the novel features of *F3*.

1 I dub this logic "*F3*" since it is a free three-valued logic. Strictly speaking, the language of *F3* presented here is a *language-form*; in spite of this I sometimes refer to the "language of *F3*". The primitive vocabulary of *F3* includes: (i) denumerably many individual variables, 'x_1', 'x_2', 'x_3', . . . (referred to by 'X', 'Y', 'Z'; these capital letters are metalinguistic variables); (ii) denumerably many individual parameters, 'a_1', 'a_2', 'a_3', . . . (referred to by 'P', 'P'');[2] (iii) denumerably many d-place predicate parameters (for every d from zero on), 'f_1^d', 'f_2^d', 'f_3^d', . . . (referred to by 'F_i^d', $i \geq 1$);[3] (iv) a one-place predicate constant 'E!';[4] (v) two sentence connectives '$\sim$' and '$\supset$'; (vi) one quantifier letter '$\forall$'; and (vii) the punctuation symbols '(', ')', and ','.

We will refer to the individual signs (the individual variables and individual parameters) by means of 'I', 'I'', 'I''', etc. A formula of *F3* is any string of symbols from the vocabulary; these will be referred to by means of 'A', 'B', 'C', and 'D'.

We will need a substitution notation. Where A is a formula[5] of *F3*, $A(I'/I)$ is to be the result of substituting I' for I, for every occurrence of I in A; and $A(I_1', I_2', \ldots, I_n'/I_1, I_2, \ldots, I_n)$ is to be the result of *simultaneously* substituting I_i' $(1 \leq i \leq n)$ for every occurrence of I_i in A. We will call $A(P_1', P_2', \ldots, P_n'/P_1, P_2, \ldots, P_n)$ a *parametric variant* of A.[6]

The *well-formed formulas* (*wffs*) of *F3* are: (i) any d-place predicate letter followed by d individual parameters—these are the *atomic wffs*; (ii) $\sim A$ and $(A \supset B)$, if A and B are wffs—these are, respectively, the *negation* of A and a *conditional*; and (iii) $(\forall X)A$, if $A(P/X)$ is a wff—$(\forall X)A$ is a (universal) *quantification*.

A *quasi-wff* is $A(X/P)$ if A is a wff and P occurs in A but X does not occur in A. A wff of the form E!P is an *existence* statement. Any set of wffs to which infinitely many individual parameters are foreign will be called *infinitely extendible*. The *length* of a wff A ($l(A)$) is: (i) 1, if A is atomic; (ii) $l(B) + 1$, if A is a negation $\sim B$; (iii) $l(B) + l(C) + 1$, if A is a conditional $(B \supset C)$; and (iv) $l(B(P/X)) + 1$, if A is a quantification $(\forall X)B$. The *subformulas* of a wff or quasi-wff A are: (i) A itself; (ii) B, if A is a negation $\sim B$; (iii) B and C, if A is a conditional $(B \supset C)$; (iv) every one of $B(a_1/X)$, $B(a_2/X)$,

$B(a_3/X), \ldots$, if A is a quantification $(\forall X)B$; and (v) every subformula of a subformula of A. The wffs of *F3* are presumed to be alphabetically ordered.[7]

There are several defined signs that will prove to be of some interest:[8]

$$
\begin{aligned}
(A \vee B) &= ((A \supset B) \supset B)\\
(A \,\&\, B) &= \sim(\sim A \vee \sim B)\\
(A \equiv B) &= ((A \supset B) \,\&\, (B \supset A))\\
(A \mathrel{\mathrm{I}} B) &= (A \supset (A \supset B))\\
-A &= (A \supset \sim A)\\
J_1(A) &= \sim(A \supset \sim A)\\
J_3(A) &= \sim(\sim A \supset A)\\
J_2(A) &= \sim(J_1(A) \vee J_3(A))\\
(A \rightarrow B) &= ((J_1(A) \vee J_3(A)) \supset J_1(B))\\
(A \Rightarrow B) &= ((A \rightarrow B) \,\&\, (J_2(A) \supset -B))\\
(\exists X)A &= \sim(\forall X)\sim A.
\end{aligned}
$$

The axioms of *F3* are any wffs of any of the following kinds:

A1 $A \supset (B \supset A)$
A2 $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
A3 $((A \supset \sim A) \supset A) \supset A$
A4 $(\sim A \supset \sim B) \supset (B \supset A)$
A5 $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$
A6 $A \supset (\forall X)A$
A7 $E!P \supset ((\forall X)A \supset A(P/X))$
A8 $(\forall X)E!X$
A9 $-(\exists X)(A \supset A) \supset \sim(\exists X)(A \supset A)$

and any wff of the sort $(\forall X)A$ where, for some P foreign to $(\forall X)A$, $A(P/X)$ is an axiom.[9]

Rule. From A and $(A \supset B)$, infer B. (B is the *ponential* of A and $(A \supset B)$.)

Proofs are defined in the expected way. If S is a set of wffs of *F3* and A is a wff of *F3*, we say that *A is derivable from S* ($S \vdash A$) if there is a proof of A from S. And, as a special case, *A is provable*, or *A is a theorem* ($\vdash A$), if A is derivable from the empty set.

A set S of wffs of *F3* is *syntactically consistent* if there is no wff A such that $S \vdash A$ and $S \vdash \sim A$; S is *syntactically inconsistent* otherwise. S is *maximally consistent* if S is syntactically consistent and for any wff A not derivable from S, $S \cup \{A\}$ is syntactically inconsistent. S is *omega-complete relative to a set* Σ of individual parameters if for any quantification $(\forall X)A$, $S \vdash (\forall X)A$ iff $S \vdash A(P/X)$ for every P in Σ. Σ may be empty; if it is, S is *omega-complete relative to* ϕ if every (universal) quantification is derivable from S.

1.1 The semantics of F3 A *truth-value assignment*, α, is to be any function from the atomic wffs of *F3* into $\{1, 2, 3\}$. Σ is to be any (possibly empty) set of individual parameters of *F3*. Then, (α, Σ) is a *truth-value pair*, and will be referred to as 'α_Σ'. A truth-value pair (α, Σ) is *existence-normal* if: (i) $\alpha_\Sigma(E!P) = 1$ for every P in Σ, and (ii) $\alpha_\Sigma(E!P) = 3$ for every other P. From this point on, we will restrict our attention to existence-normal truth-value pairs, and we will usually call them just "tv-pairs".

The value of a wff A on an existence-normal tv-pair (α, Σ) is:

(i) if A is atomic, $\alpha_\Sigma(A) = \alpha(A)$
(ii) if A is a negation $\sim B$, $\alpha_\Sigma(A) = 4 - \alpha_\Sigma(B)$
(iii) if A is a conditional $(B \supset C)$, $\alpha_\Sigma(A) = \max(1, \alpha_\Sigma(C) - \alpha_\Sigma(B) + 1)$
(iv) if A is a quantification $(\forall X)B$,

$$\alpha_\Sigma(A) = \begin{cases} 1 & \text{if } \alpha_\Sigma(B(P/X)) = 1 \text{ for every } P \text{ in } \Sigma \\ 3 & \text{if } \alpha_\Sigma(B(P/X)) = 3 \text{ for some } P \text{ in } \Sigma \\ 2 & \text{otherwise.} \end{cases}$$

With this done, we say that A *is true on* (α, Σ) if $\alpha_\Sigma(A) = 1$; A *is false on* (α, Σ) if $\alpha_\Sigma(A) = 3$; otherwise, A *is neither-true-false on* (α, Σ).[10]

The resultant truth-tables for the sentence connectives are presented below.

A	$\sim A$	$-A$	$J_1(A)$	$J_2(A)$	$J_3(A)$
1	3	3	1	3	3
2	2	1	3	1	3
3	1	1	3	3	1

$\supset$	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

I	1	2	3
1	1	2	3
2	1	1	1
3	1	1	1

$\rightarrow$	1	2	3
1	1	3	3
2	1	1	1
3	1	3	3

$\Rightarrow$	1	2	3
1	1	3	3
2	3	1	1
3	1	3	3

$\vee$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

&	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

$\equiv$	1	2	3
1	1	2	3
2	2	1	2
3	3	2	1

A set S of wffs of *F3* is *tv-verifiable* if there is a tv-pair (α, Σ) on which every member of S is true. (A wff A is tv-verifiable if $\{A\}$ is.) S is *semantically consistent* if there is a tv-verifiable set S' to which S is isomorphic.[11] S is *semantically inconsistent* if it is not semantically consistent. A is a *semantic consequence* of S (S *entails* A; $S \models A$) if $S \cup \{-A\}$ is semantically inconsistent. A is *valid* (*logically true*, $\models A$), if $\alpha_\Sigma(A) = 1$ for every α_Σ. (This will be so iff $\{-A\}$ is semantically inconsistent.) A is *logically false* if $\alpha_\Sigma(A) = 3$ for every α_Σ. And, A is *logically indeterminate* if A is neither logically true nor logically false. A and B are *logically equivalent* if $(A \equiv B)$ is logically true (so that $\alpha_\Sigma(A) = \alpha_\Sigma(B)$ for every α_Σ).

Before proceeding, a few comments on the semantics of *F3* are in order; we wish to point out some of the novelties of the semantics. (i) Since Σ may be empty, there may be nothing; i.e., every singular term may fail to denote and every existential quantification may fail to be true. In particular, $(\exists x)(f(x) \vee \sim f(x))$ is not a logical truth, and we are, therefore, free of Russell's lament.[12] (ii) Any atomic wff (except an existence statement) containing a nondenoting singular term *may be true*. It is not required that all such atomic wffs be either neither-true-nor-false or false. This, I think, is desirable, for a sentence of English such as 'Pegasus is a winged-horse' is, I think, true even though 'Pegasus' fails to denote. Also, such a sentence may be false, as Russell urged with his famous "King of France" example; or, it may be neither-true-nor-false as Strawson sometimes argued.[13] (iii) Not only are there valid arguments in *F3*, but there are valid statements as well. This is one of the notable points of difference between *F3* and the work of Professor Woodruff.[14] (iv) Any conjunction is "as false as" any of its conjuncts, and any disjunction is "as true as" any of its disjuncts. Thus, $(p \ \& \sim p)$ is never true, but it is not necessarily false, for if 'p' and '$\sim p$' are neither-true-nor-false, so is $(p \ \& \sim p)$. But, if any conjunct of some conjunction is false, so is the entire conjunction. The case is similar for disjunctions: $(p \vee \sim p)$ is not necessarily true, but it is never false. However, if any disjunct of a disjunction is true, the entire disjunction is true.[15] Finally, we note that even though $(p \vee \sim p)$ is not valid, $(p \vee -p)$ *is* valid; '$-p$' is the exclusion-negation of 'p'. (v) By the construction of α_Σ, $\alpha_\Sigma((\forall X)A) = 1$ iff $\alpha_\Sigma(A(P/X)) = 1$ for every P such that $\alpha_\Sigma(\text{E}!P) = 1$. And, $\alpha_\Sigma((\exists X)A) = 1$ iff $\alpha_\Sigma(A(P/X)) = 1$ for some P such that $\alpha_\Sigma(\text{E}!P) = 1$.[16] (vi) $\alpha_\Sigma(\text{E}!P) = 1$ or $\alpha_\Sigma(\text{E}!P) = 3$, since it is presumed that a statement is neither-true-nor-false only if it has a presupposition that fails to be true, and $\text{E}!P$ never has a presupposition that fails to be true. (vii) Neither $(\forall X)A \supset A(P/X)$ (universal instantiation) nor $A(P/X) \supset (\exists X)A$ (existential generalization) is valid, but both $\text{E}!P \supset ((\forall X)A \supset A(P/X))$ and $\text{E}!P \supset (A(P/X) \supset (\exists X)A)$ are valid. (viii) The table for $(A \rightarrow B)$ is the same as Woodruff's table for his $(A \Rightarrow B)$.[17]

1.2 ***Preliminary lemmas*** This section contains various theorems concerning *F3*. Many of these will be useful in proving the soundness and completeness of our axioms. Some are very familiar results; others are a bit unusual. We begin with some results on provability. (Where a proof is relatively straightforward, only the numbers of the theorems needed for the proof are listed to the right of the theorem.)

T1 (a) *If $S \vdash A$, then $S' \vdash A$ for at least one finite subset S' of S.*
(b) *If $S \vdash A$, then $S \cup S' \vdash A$ for every set S' of wffs of F3.*
(c) *If A is a member of S, then $S \vdash A$.*
(d) *If A is an axiom of F3, then $S \vdash A$ for every set S of wffs of F3.*
(e) *If $S \vdash A$ and $S \vdash A \supset B$, then $S \vdash B$.*
(f) *If either $S \vdash A$ or $S \vdash \sim A \supset \sim B$, then $S \vdash B \supset A$.*
(g) *If $S \vdash \sim A$, then $S \vdash A \supset B$.*
(h) *If $S \vdash A$, then $S \vdash (\forall X)A$, so long as X is foreign to A.*
(i) *If $S \vdash (\forall X)(A \supset B)$, then $S \vdash (\forall X)A \supset (\forall X)B$.*
(j) *If $S \vdash (\forall X)(A \supset B)$ and $S \vdash (\forall X)A$, then $S \vdash (\forall X)B$.*

Proof: (a)-(j) are just T2.1.1(a)-(g), (o), (p), (q), of [7]. The proofs there will suffice.

T2 (a) *If* $S \vdash A \supset B$, *then* $S \vdash (B \supset C) \supset (A \supset C)$.
(b) *If* $S \vdash A \supset B$ *and* $S \vdash B \supset C$, *then* $S \vdash A \supset C$.
(c) *If* $S \vdash A \supset B$, *then* $S \vdash (C \supset A) \supset (C \supset B)$.
(d) $S \vdash A \supset A$.
(e) *If* $S \vdash A \mathrm{I} (B \supset C)$ *and* $S \vdash A \mathrm{I} B$, *then* $S \vdash A \mathrm{I} C$.
(f) *If* $S \cup \{A\} \vdash B$, *then* $S \vdash A \mathrm{I} B$. (The Deduction Theorem for *F3*.)
(g) *If* $S \vdash A \supset B$ *and* $S \vdash B \mathrm{I} C$, *then* $S \vdash A \mathrm{I} C$.
(h) *If* $S \vdash A \supset B$ *and* $S \cup \{B\} \vdash C$, *then* $S \cup \{A\} \vdash C$.
(i) $S \vdash \sim\sim A \supset A$.
(j) $S \vdash A \supset \sim\sim A$.
(k) *If* $S \vdash \sim\sim A$, *then* $S \vdash A$.
(l) *If* $S \vdash A$, *then* $S \vdash \sim\sim A$.
(m) *If* $S \vdash A \supset B$, *then* $S \vdash \sim B \supset \sim A$.
(n) *If* $S \vdash A \supset B$ *and* $S \vdash \sim B$, *then* $S \vdash \sim A$.
(o) *If* $S \vdash \sim(A \supset B)$, *then* $S \vdash A$ *and* $S \vdash \sim B$.
(p) *If* $S \vdash A$ *and* $S \vdash \sim B$, *then* $S \vdash \sim(A \supset B)$.
(q) *If* $S \vdash --A$, *then* $S \vdash A$.
(r) *If* $S \vdash A \mathrm{I} -A$, *then* $S \vdash -A$.
(s) *If* $S \vdash -A$ *and* $S \vdash -\sim B$, *then* $S \vdash A \supset B$.

Proof: See T10.1.1(a)-(s) of [7].

T3 (a) *If* $S \vdash A \supset B$, *then* $S \cup \{A\} \vdash B$. [T1(c), T1(b), T1(e)]
(b) *If* $S \cup \{A\} \vdash B$ *and* $S \vdash A$, *then* $S \vdash B$. [T2(f), T1(e)]
(c) $S \vdash \sim(f(P) \supset f(P)) \supset A$. [T2(d), T2(l), T1(g)]
(d) *If* $S \vdash \sim A \supset B$, *then* $S \vdash \sim B \supset A$. [T2(m), T2(i), T2(b)]
(e) *If* $S \vdash A \supset \sim B$, *then* $S \vdash B \supset \sim A$. [T2(m), T2(j), T2(b)]

T4 (a) *If* $S \vdash A$ *and* $S \vdash A \mathrm{I} B$, *then* $S \vdash B$. [T1(e)]
(b) *If* $S \vdash A \supset B$, *then* $S \vdash A \mathrm{I} B$. [T1(f)]
(c) *If* $S \vdash A \mathrm{I} B$, *then* $S \cup \{A\} \vdash B$. [T1(b), T1(c), T4(a)]

T5 (a) $S \vdash A \& B$ *iff* $S \vdash A$ *and* $S \vdash B$. [T2(o), T2(n), T2(k), T1(f), T2(m), T2(p)]
(b) $S \vdash A \equiv B$ *iff* $S \vdash A \supset B$ *and* $S \vdash B \supset A$. [T5(a)]
(c) *If* $S \vdash A \equiv B$ *and* $S \vdash A$, *then* $S \vdash B$. [T5(b), T1(e)]
(d) *If* $S \vdash A \equiv B$ *and* $S \vdash B$, *then* $S \vdash A$. [T5(b), T1(e)]

T6 (a) $S \vdash (A \supset (B \supset C)) \equiv (B \supset (A \supset C))$.
(b) $S \vdash (\forall X)A \supset (\mathrm{E!}P \supset A(P/X))$.[18]
(c) $S \vdash (\forall Y)((\forall X)A \supset A(Y/X))$.
(d) *If* $S \vdash A(P/X)$ *for some* P *foreign to* S *and to* $(\forall X)A$, *then* $S \vdash (\forall X)A$.

Proof: (a) Wajsberg [21] established the weak completeness of A1-A4 for (what comes to the same as) the sentential fragment of *F3*. Thus, since $(A \supset (B \supset C)) \equiv (B \supset (A \supset C))$ is valid, it is provable. Hence, (a) by T1(b).

(b) $S \vdash (\mathrm{E!}P \supset ((\forall X)A \supset A(P/X))) \equiv ((\forall X)A \supset (\mathrm{E!}P \supset A(P/X)))$ by (a). But, by T1(d), $S \vdash \mathrm{E!}P \supset ((\forall X)A \supset A(P/X))$. Hence, (b) by T5(c).

(c) Since $E!P \supset ((\forall X)A \supset A(P/X))$ is an axiom, so is $(\forall Y)(E!Y \supset ((\forall X)A \supset A(Y/X)))$. Hence, by T1(d), $S \vdash (\forall Y)(E!Y \supset ((\forall X)A \supset A(Y/X)))$. Thus, $S \vdash (\forall Y)E!Y \supset (\forall Y)((\forall X)A \supset A(Y/X))$ by T1(i). But since $(\forall Y)E!Y$ is an axiom, $S \vdash (\forall Y)E!Y$ by T1(d). Hence, (c) by T1(e).

(d) The proof of (d) is like that of T2.1.1(t) in [7]. Note in particular that T2.1.1(s) is our T6(c).

T7 (a) *S is syntactically inconsistent iff $S \vdash A$ for every wff A.*
(b) *S is syntactically inconsistent iff $S \vdash \sim(p \supset p)$.*
(c) *If S is syntactically inconsistent, then so is every superset of S.*
(d) *S is syntactically inconsistent iff at least one finite subset of S is syntactically inconsistent.*
(e) *If $S \vdash A$ and $S \vdash -A$, then S is syntactically inconsistent.*
(f) *If $S \cup \{A\}$ is syntactically inconsistent, then $S \vdash -A$.*
(g) *If $S \cup \{-A\}$ is syntactically inconsistent, then $S \vdash A$.*
(h) *If $S \vdash A$, then $S \cup \{-A\}$ is syntactically inconsistent.*
(i) *A is valid iff $\{-A\}$ is not tv-verifiable.*

Proof: (a)-(d) are just T2.1.2(a)-(d) of [7]; (e)-(i) are just T10.1.5(a)-(e) of [7]. The proofs there will suffice.

T8 *If S is syntactically consistent, then at least one of $S \cup \{A\}$ and $S \cup \{-A\}$ is syntactically consistent.*

Proof: Suppose both $S \cup \{A\}$ and $S \cup \{-A\}$ are syntactically inconsistent. Then, by T7(f) and (g), $S \vdash -A$ and $S \vdash A$. Hence, by T7(e), S is syntactically inconsistent. Therefore, T8.

T9 (a) *If $S \vdash A \text{ I } B$ and $S \vdash -B$, then $S \vdash -A$.* [T3(a), T1(c), T1(e), T1(b), T1(e), T7(f)]
(b) *If $S \vdash A \supset B$ and $S \vdash -B$, then $S \vdash -A$.* [T4(b), T9(a)]
(c) *$S \vdash -(A \text{ I } B)$ iff $S \vdash A$ and $S \vdash -B$.*

Proof: (c) $S \vdash -(A \text{ I } B) \text{ I } (A \ \& -B)$ by a proof like that of T6(a). So, suppose $S \vdash -(A \text{ I } B)$. Then $S \vdash A$ and $S \vdash -B$ by T4(a) and T5(a). Next, suppose that $S \vdash A$ and $S \vdash -B$. Then, since $S \vdash (A \ \& -B) \text{ I } -(A \text{ I } B)$ (proof like that of T6(a)), (c) by T5(a) and T4(a).

T10 (a) *If $S \cup \{A_1(P/X), A_2(P/X), \ldots, A_n(P/X)\} \vdash B(P/X)$, then $S \cup \{(\forall X)A_1, (\forall X)A_2, \ldots, (\forall X)A_n\} \vdash (\forall X)B$, so long as P is foreign to S, $(\forall X)A_1, (\forall X)A_2, \ldots, (\forall X)A_n$, and to $(\forall X)B$.*
(b) *If $S \cup \{A(P/X)\} \vdash B(P/X) \supset C(P/X)$, then $S \cup \{(\forall X)A\} \vdash (\forall X)B \supset (\forall X)C$, so long as P is foreign to S, $(\forall X)A$, $(\forall X)B$, and $(\forall X)C$.*
(c) *$S \cup \{(\forall X)(A \supset B)\} \vdash (\exists X)A \supset (\exists X)B$.*
(d) *$S \vdash (\exists X)A \supset A$.*
(e) *$S \cup \{(\forall X)(A \supset B)\} \vdash (\exists X)A \supset B$.*
(f) *If $S \vdash (\forall X)(A \supset B)$, then $S \vdash A \supset (\forall X)B$, so long as X is foreign to A.*
(g) *$S \vdash (\forall Y)A(Y/X) \supset (\forall X)A$.*
(h) *$S \vdash (\exists X)(A \supset A) \supset (\exists Y)E!Y$.*[19]

Proof: (c)-(g) are proved as T5.3.8(e), (g), (h), (j), and (k) of [7]. (a) is proved in a way similar to T5.3.8(c) of [7]; see [1], pp. 80-81, T2.3.15(c) for complete details.

(b) Let P be foreign to S, $(\forall X)A$, $(\forall X)B$, and $(\forall X)C$. Suppose $S \cup \{A(P/X)\} \vdash B(P/X) \supset C(P/X)$. Then (b) by (a) and T1(i).

(h) $S \vdash (\forall X)(\mathrm{E!}X \supset ((A \supset A) \supset \mathrm{E!}X))$ by T1(d). Hence, $S \vdash (\forall X)\mathrm{E!}X \supset (\forall X)((A \supset A) \supset \mathrm{E!}X)$ by T1(i). So, by T1(d) and T1(e), $S \vdash (\forall X)((A \supset A) \supset \mathrm{E!}X)$. But, $S \cup \{(\forall X)((A \supset A) \supset \mathrm{E!}X)\} \vdash (\exists X)(A \supset A) \supset (\exists X)\mathrm{E!}X$ by T10(c). So, $S \vdash (\exists X)(A \supset A) \supset (\exists X)\mathrm{E!}X$ by T3(b). But, $S \vdash (\forall Y){\sim}\mathrm{E!}Y \supset (\forall X){\sim}\mathrm{E!}X$ by T2(d) if X and Y are the same, and by T10(g) if X and Y are different. So, by T2(m), $S \vdash (\exists X)\mathrm{E!}X \supset (\exists Y)\mathrm{E!}Y$. Thus, (h) by T2(b).

T11 (a) $S \vdash (\forall X)A \supset (\forall Y)A(Y/X)$.
(b) $S \vdash (\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$.

Proof: (a) Since $(\forall Y)A(Y/X)$ is presumed to be well-formed, Y is sure to be foreign to $(\forall X)A$; hence, (a) by T6(c) and T10(f).

(b) $\{\mathrm{E!}P, \mathrm{E!}P'\} \vdash (\forall X)(\forall Y)A \supset (\forall Y)A(P/X)$, and $\{\mathrm{E!}P, \mathrm{E!}P'\} \vdash (\forall Y)A(P/X) \supset (A(P/X))(P'/Y)$, by A7, T3(a), and T1(d). Hence, by T2(b), $\{\mathrm{E!}P, \mathrm{E!}P'\} \vdash (\forall X)(\forall Y)A \supset (A(P/X))(P'/Y)$. Hence, by T10(a), $\{(\forall Z)\mathrm{E!}Z, \mathrm{E!}P'\} \vdash (\forall Z)((\forall X)(\forall Y)A \supset (A(P'/Y))(Z/X))$. Hence, by T1(d) and T3(b), $\{\mathrm{E!}P'\} \vdash (\forall Z)((\forall X)(\forall Y)A \supset (A(P'/Y))(Z/X))$. And so, by T10(f), $\{\mathrm{E!}P'\} \vdash (\forall X)(\forall Y)A \supset (\forall Z)(A(P'/Y))(Z/X)$. Hence, by (a) and T2(b), $\{\mathrm{E!}P'\} \vdash (\forall X)(\forall Y)A \supset (\forall X)A(P'/Y)$. Hence, by T10(a), $\{(\forall Z)\mathrm{E!}Z\} \vdash (\forall Z)((\forall X)(\forall Y)A \supset (\forall X)A(Z/Y))$. So, by T1(d) and T3(b), $\vdash (\forall Z)((\forall X)(\forall Y)A \supset (\forall X)A(Z/Y))$. Hence, by T10(f), $\vdash (\forall X)(\forall Y)A \supset (\forall Z)(\forall X)A(Z/Y)$. And so, by (a) and T2(b), $\vdash (\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$. Hence, (b) by T1(b).

T12 (a) $S \vdash {\sim}(\exists x)(f(x) \supset f(x)) \supset (\forall X)A$.
(b) $S \cup \{{\sim}(\exists x)(f(x) \supset f(x))\} \vdash (\forall X)A$.

Proof: (a) Let P be foreign to $(\forall X)A$. $\vdash {\sim}(f(P) \supset f(P)) \supset A(P/X)$ by T3(c). Hence, $\vdash (\forall X)({\sim}(f(X) \supset f(X)) \supset A)$ by T6(d). Hence, by T1(i), $\vdash (\forall X){\sim}(f(X) \supset f(X)) \supset (\forall X)A$. Hence, $\vdash (\forall x){\sim}(f(x) \supset f(x)) \supset (\forall X)A$ by T10(g) and T2(b). Thus, since $\vdash {\sim\sim}(\forall x){\sim}(f(x) \supset f(x)) \supset (\forall x){\sim}(f(x) \supset f(x))$ by T2(i), (a) by T2(b) and T1(b).

(b) Proof is by (a) and T3(a).

Next we prove presupposition-free varieties of the familiar principles Universal Instantiation and Existential Generalization.

T13 (a) $S \cup \{\mathrm{E!}P\} \vdash (\forall X)A \supset A(P/X)$. [T1(d), T3(a)]
(b) $S \cup \{\mathrm{E!}P\} \vdash A(P/X) \supset (\exists X)A$. [T13(a), T3(e)]

Finally, we state that the isomorphism relation has the necessary properties for the proof of the soundness and completeness theorems.

T14 (a) *S is isomorphic to S.*

(b) *If S' is isomorphic to S, then S is isomorphic to S'.*

Proof: These are T2.1.3(a), (b) of [7]. The theorems are proved there.

1.3 ***The soundness of F3*** Demonstration of the Soundness Theorem for *F3* is rather straightforward, but somewhat tedious. It is, though, fairly obvious because of the simplicity of our deductive apparatus.

T15 *If A is an axiom of F3, then so is any parametric variant of A.*

Proof: By cases; it involves just tedious substitutions.

T16 *If A is an axiom of F3, then A is valid.*

Proof: By cases for each of A1-A9. The generalization axiom is proved with the aid of T15.

T17 *If S is tv-verifiable, then S is syntactically consistent.*

Proof: Let the column consisting of $A_1, A_2, \ldots, A_n$ be a proof of $\sim(p \supset p)$ from S, and suppose S is true on some tv-pair (α, Σ). It is easily shown by induction on i that for each i from 1 through n, A_i would be true on (α, Σ). Hence, in particular, A_n $(= \sim(p \supset p))$ would be true on (α, Σ). But this is impossible. Hence, S is true on no tv-pair, and is therefore not tv-verifiable. Thus, T17 by T7(b).

T18 *If $\vdash A$, then A is valid* ($\vDash A$). (Weak Soundness)

Proof: Suppose $\vdash A$. Then $\{-A\}$ is syntactically inconsistent by T7(h), and so, by T17, $\{-A\}$ is not tv-verifiable. Thus, A is valid by T7(i).

T19 (a) *If S' is isomorphic to S and S' is syntactically consistent, then S is syntactically consistent.*
(b) *If S' is isomorphic to S and S' is syntactically inconsistent, then S is syntactically inconsistent.*

Proof: See T2.2.6(a)-(b) of [7].

T20 *If S is semantically consistent, then S is syntactically consistent.*

Proof: Suppose S is syntactically inconsistent. Then by T19(b), so is every set to which S is isomorphic. Hence, by T17, none is tv-verifiable. Hence, S is semantically inconsistent.

T21 *If $S \vdash A$, then $S \vDash A$.* (Strong Soundness)

Proof: Suppose $S \vdash A$. Then $S \cup \{-A\}$ is syntactically inconsistent by T7(h). Hence, by T20, $S \cup \{-A\}$ is semantically inconsistent. Thus, $S \vDash A$.

1.4 ***The completeness of F3*** The proof of the Completeness Theorem employed here is of the Henkin variety. Thus, we will proceed toward the construction of a maximally consistent set of wffs from any syntactically consistent and infinitely extendible set of wffs. First, we define the *Lindenbaum extension of a syntactically consistent set of wffs.*

Let S be a syntactically consistent set of wffs. Let A_n be the alphabetically n^{th} wff of *F3*. Let the *Lindenbaum extension of S* be $\bigcup_{n=0}^{\infty} S_n$ $(= S_\infty)$ where:

(i) $S_0 = S$, and

(ii) $S_n(n > 0) = \begin{cases} S_{n-1} \cup \{A_n\}, \text{ if } S_{n-1} \cup \{A_n\} \text{ is syntactically consistent} \\ S_{n-1}, \text{ if } S_{n-1} \cup \{A_n\} \text{ is syntactically inconsistent.} \end{cases}$

Then show in the usual way that: (a) for each n from 0 on, S_n is syntactically consistent; (b) S_∞ is syntactically consistent; and (c) S_∞ is maximally consistent. Thus,

T22 *The Lindenbaum extension of a syntactically consistent set of wffs of F3 is a maximally consistent set.*

T23 *Let S be a maximally consistent set of wffs of F3. Then, for any wff A of F3, exactly one of A and* $-A$ *is derivable from S.*

Proof: "At most one" is by the syntactic consistency of S and T7(e). "At least one" is by a supposition to the contrary, the maximal consistency of S, and T8.

T24 *If S is syntactically consistent, then at least one of* $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ *and* $S \cup \{(\exists x)(f(x) \supset f(x))\}$ *is syntactically consistent.*

Proof: Suppose both $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ and $S \cup \{(\exists x)(f(x) \supset f(x))\}$ are syntactically inconsistent. Then $S \vdash -(\exists x)(f(x) \supset f(x))$ and $S \vdash - \sim(\exists x)(f(x) \supset f(x))$ by T7(f). Hence, $S \vdash \sim(\exists x)(f(x) \supset f(x))$ by A9, T1(d), and T1(e). Hence, S is syntactically inconsistent by T7(e).

Next, we define the *Σ-extension of an infinitely extendible set of wffs.*

Let S be an infinitely extendible set of wffs of *F3*. Let Σ consist of all the individual parameters of *F3* foreign to S.

Case 1. $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ is syntactically consistent. Then the Lindenbaum extension S_∞ of $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ is to be the Σ-extension of S.

Case 2. $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ is syntactically inconsistent. Then, let

$S^0 = S \cup \{(\exists x)(f(x) \supset f(x))\}$

$S^n(n > 0) = S^{n-1} \cup \{\mathrm{E!}P_n, (B_n(P_n/X_n) \text{ I } (\forall X_n)B_n)\}$, where P_n is the alphabetically earliest member of Σ foreign to $\mathrm{E!}P_1$, $(B_1(P_1/X_1) \text{ I } (\forall X_1)B_1), \ldots, \mathrm{E!}P_{n-1}$, $(B_{n-1}(P_{n-1}/X_{n-1}) \text{ I } (\forall X_{n-1})B_{n-1})$, and $(\forall X_n)B_n$, and $(\forall X_n)B_n$ is the alphabetically n^{th} quantification of *F3*

and let $S^\infty = \bigcup_{n=0}^{\infty} S^n$.

Finally, let S_∞, the Lindenbaum extension of S^∞, be the Σ-extension of S.

T25 *Let S be infinitely extendible; let Σ consist of all the individual parameters of F3 foreign to S; and let S_∞ be the Σ-extension of S.*
(a) *If S is syntactically consistent, then S_∞ is maximally consistent.*
(b) *If $\sim(\exists x)(f(x) \supset f(x)) \in S_\infty$, then S_∞ is omega-complete relative to $\emptyset$.*
(c) *If $\sim(\exists x)(f(x) \supset f(x)) \notin S_\infty$, then S_∞ is omega-complete relative to Σ.*

Proof: (a) Let S be syntactically consistent.

Case 1 $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ is syntactically consistent. Then by T22, S_∞ is maximally consistent.

Case 2. $S \cup \{\sim(\exists x)(f(x) \supset f(x))\}$ is syntactically inconsistent. There are four parts:

(I) Show S^0 is syntactically consistent. By T24.

(II) Show S^n $(n > 0)$ is syntactically consistent. Suppose S^n is syntactically inconsistent. Then by T7(c)-(d), there is sure to be some i $(i > 0)$ such that $S^0 \cup \{\mathrm{E!}P_1, (B_1(P_1/X_1)\ \mathrm{I}\ (\forall X_1)B_1), \ldots, \mathrm{E!}P_i, (B_i(P_i/X_i)\ \mathrm{I}\ (\forall X_i)B_i)\}$ is syntactically inconsistent. Thus by T7(f), $S^0 \cup \{\mathrm{E!}P_1, B_1(P_1/X_1)\ \mathrm{I}\ (\forall X_1)B_1, \ldots, \mathrm{E!}P_{i-1}, B_{i-1}(P_{i-1}/X_{i-1})\ \mathrm{I}\ (\forall X_{i-1})B_{i-1}, \mathrm{E!}P_i\} \vdash -(B_i(P_i/X_i)\ \mathrm{I}\ (\forall X_i)B_i)$. Call the set before the turnstile "$S^{i-1} \cup \{\mathrm{E!}P_i\}$". Thus, by T9(c),

(i) $S^{i-1} \cup \{\mathrm{E!}P_i\} \vdash B_i(P_i/X_i)$ and
(ii) $S^{i-1} \cup \{\mathrm{E!}P_i\} \vdash -(\forall X_i)B_i$.

Now, pursuing (i), we find that $S^{i-1} \cup \{(\forall X_i)\mathrm{E!}X_i\} \vdash (\forall X_i)B_i$, by T10(a). Thus, by T2(f), $S^{i-1} \vdash (\forall X_i)\mathrm{E!}X_i\ \mathrm{I}\ (\forall X_i)B_i$. But $S^{i-1} \vdash (\forall X_i)\mathrm{E!}X_i$, by A8 and T1(d). Thus,

(i*) $S^{i-1} \vdash (\forall X_i)B_i$ by T1(e) twice.

Next, pursuing (ii), it is found that $S^{i-1} \vdash \mathrm{E!}P_i\ \mathrm{I}\ -(\forall X_i)B_i$, by T2(f). Let Y be an individual variable foreign to $-(\forall X_i)B_i$. Then, $(\mathrm{E!}Y\ \mathrm{I}\ -(\forall X_i)B_i)(P_i/Y)$ is just $(\mathrm{E!}P_i\ \mathrm{I}\ -(\forall X_i)B_i)$. So, $S^{i-1} \vdash (\mathrm{E!}Y\ \mathrm{I}\ -(\forall X_i)B_i)(P_i/Y)$. Therefore, since P_i is foreign to S^{i-1} and to $-(\forall X_i)B_i$, it is found by the definition of 'I', T6(d), and T1(i), that $S^{i-1} \vdash (\forall Y)\mathrm{E!}Y \supset (\forall Y)(\mathrm{E!}Y \supset -(\forall X_i)B_i)$. But $S^{i-1} \vdash (\forall Y)\mathrm{E!}Y$ by A8 and T1(d). So, $S^{i-1} \vdash (\forall Y)(\mathrm{E!}Y \supset -(\forall X_i)B_i)$ by T1(e). However, $S^{i-1} \cup \{(\forall Y)(\mathrm{E!}Y \supset -(\forall X_i)B_i)\} \vdash (\exists Y)\mathrm{E!}Y \supset -(\forall X_i)B_i$ by T10(e). Thus, $S^{i-1} \vdash (\exists Y)\mathrm{E!}Y \supset -(\forall X_i)B_i$, by T3(b). Remember though that $\{(\exists x)(f(x) \supset f(x))\} \subseteq S^0 \subseteq S^{i-1}$. Hence, $S^{i-1} \vdash (\exists Y)\mathrm{E!}Y$ by T1(c), T10(h), and T1(e). Thus,

(ii*) $S^{i-1} \vdash -(\forall X_i)B_i$ by T1(e).

Thus, by (i*) and (ii*), S^{i-1} is syntactically inconsistent by T7(e).

Now, the same argument repeated $i - 1$ more times will show that S^0 is syntactically inconsistent, which contradicts Part I. Thus, S^n $(n \geqslant 0)$ must be syntactically consistent.

(III) Show S^∞ is syntactically consistent. This is done in the usual way by supposing S^∞ is not syntactically consistent. Hence, by T7(d) at least one finite subset $S^{\infty\prime}$ of S^∞ would be syntactically inconsistent. Hence, there would be an n such that every member of $S^{\infty\prime}$ would be in S^n, and this set will also be syntactically inconsistent by T7(c). But this contradicts Part II. Hence, S^{∞} is syntactically consistent.

(IV) Show S_{∞} is maximally consistent. By (III) and T22. Hence, (a).
(b) Suppose $\sim(\exists x)(f(x) \supset f(x)) \in S_{\infty}$. Then by T12(b), $S_{\infty} \vdash (\forall X)A$ for every quantification $(\forall X)A$ of *F3*. Hence, S_{∞} is omega-complete relative to $\emptyset$.
(c) Suppose $\sim(\exists x)(f(x) \supset f(x)) \notin S_{\infty}$. Let $(\forall X)A$ be an arbitrary quantification of *F3*. Show $S_{\infty} \vdash (\forall X)A$ iff $S_{\infty} \vdash A(P/X)$ for every $P \in \Sigma$. (i) Suppose $S_{\infty} \vdash (\forall X)A$. By T6(b), $S_{\infty} \vdash (\forall X)A \supset (\mathrm{E}!P \supset A(P/X))$, for every P. Thus, $S_{\infty} \vdash \mathrm{E}!P \supset A(P/X)$ for every P. But for every $P \in \Sigma$, $S_{\infty} \vdash \mathrm{E}!P$ by its construction and T1(c). So by T1(e) again, $S_{\infty} \vdash A(P/X)$ for every $P \in \Sigma$. (ii) Suppose $S_{\infty} \vdash A(P/X)$ for every $P \in \Sigma$. Then $S_{\infty} \vdash A(P_n/X)$, where P_n is the n^{th} member of Σ. But $S_{\infty} \vdash A(P_n/X) \mathrm{I} (\forall X)A$ by the construction of S_{∞} and T1(c). So, by T1(e), $S_{\infty} \vdash (\forall X)A$. Thus, (c).

We now show how to construct an existence-normal tv-pair (α, Σ') on which every member of S_{∞} is true. Let (α, Σ') be as follows: α assigns the value 1 to any atomic wff A such that $S_{\infty} \vdash A$ (and hence, by the syntactic consistency of S_{∞}, not $S_{\infty} \vdash \sim A$, and not $S_{\infty} \vdash -A$), and α assigns the value 3 to any atomic wff A such that $S_{\infty} \vdash -A$ (and hence, not $S_{\infty} \vdash A$). By T23, these are all the cases we need to consider. Σ' is either $\emptyset$ or the set Σ of T25(c).

Then show of any wff A of *F3* that:

(i) if $S_{\infty} \vdash A$ (and hence, not $S_{\infty} \vdash \sim A$ and not $S_{\infty} \vdash -A$), $\alpha_{\Sigma'}(A) = 1$
(ii) if $S_{\infty} \vdash -A$ (and hence, not $S_{\infty} \vdash A$), $\alpha_{\Sigma'}(A) = 3$.

Again, by T23, these are all the cases we need to consider.

Proof: The proof is by induction on the length, $l(A)$, of A. The induction is by cases, with atomic wffs forming the basis of the induction. The other cases are negations, conditionals, and universal quantifications. In every case, we first suppose that $S_{\infty} \vdash A$, and show that $\alpha_{\Sigma'}(A) = 1$. Then we suppose that $S_{\infty} \vdash -A$, and show that $\alpha_{\Sigma'}(-A) = 1$. Concerning quantifications, there are two possibilities: (i) $\sim(\exists x)(f(x) \supset f(x)) \in S_{\infty}$; and, (ii) $\sim(\exists x)(f(x) \supset f(x)) \notin S_{\infty}$. Under (i) it is shown that $\alpha_{\Sigma'}((\forall X)B) = 1$ for every quantification $(\forall X)B$, since S_{∞} is omega-complete relative to $\emptyset$ and every quantification $(\forall X)B$ is true on $(\alpha, \emptyset)$, no matter the α. Under (ii), we use the omega-completeness of S_{∞} relative to Σ' (established by T25(c)), and proceed in a way similar to that for negations and conditionals. The complete details are in [1], pp. 146-148.

Thus, since every member of S is a member of S_{∞}, and is, by T1(c), derivable from S_{∞}, every member of S is sure to be true on the existence-normal tv-pair (α, Σ'). Hence,

T26 *If S is syntactically consistent and infinitely extendible, then S is tv-verifiable.*

T27 *If A is valid* ($\vDash A$), *then* $\vdash A$. (Weak Completeness)

Proof: Suppose A is valid. Then $\{-A\}$ is not tv-verifiable by T7(i). Hence, $\{-A\}$ is syntactically inconsistent by T26. And so, by T7(g), $\vdash A$.

T28 *To any set S of wffs of $F3$, there corresponds a set S' of wffs of $F3$ such that S' is infinitely extendible and S is isomorphic to S'.*

Proof: See T2.3.10 of [7].

T29 *If S is syntactically consistent, then there is an infinitely extendible and syntactically consistent set of wffs to which S is isomorphic.*

Proof: By T28 and T19(a).

T30 *If S is syntactically consistent, then S is semantically consistent.*

Proof: By T29 and T26.

T31 *If $S \vDash A$, then $S \vdash A$.* (Strong Completeness Theorem)[20]

Proof: Suppose $S \vDash A$. Then by T30, $S \cup \{-A\}$ is syntactically inconsistent. Hence, by T7(g), $S \vdash A$.

We can summarize the preceding results as follows:

T32 (a) $\vdash A$ *iff* $\vDash A$.
(b) *S is syntactically (in)consistent iff S is semantically (in)consistent.*
(c) $S \vdash A$ *iff* $S \vDash A$.

We close with a compactness theorem:

T33 (a) *S is syntactically consistent iff every finite subset of S is syntactically consistent.*
(b) *S is semantically consistent iff every finite subset of S is semantically consistent.*

Proof: (a) By T7(d).
(b) By (a) and T32(b).

2 Earlier, following Strawson, we characterized the notion of presupposition in the following way: A presupposes B iff A is neither-true-nor-false unless B is true. We will re-express this as:

(i) A presupposes B iff B is a necessary condition for the truth of A and B is a necessary condition for the falsity of A.

Also, we adopt the following informal definition of entailment:

(ii) A entails B iff the conjunction of A and the denial of B is inconsistent.

We use these definitions because they seem to accord best with Strawson's discussion of these two topics.[21] Now, according to Strawson, two statements A and B are inconsistent if they cannot both be true together. Our characterization in Section 1 of a semantically inconsistent set captures this idea exactly. And, when one denies a statement B, one has implicitly asserted that B is not true—he has not necessarily asserted that B is false. Hence, we would represent the denial of B as "$-B$". Thus, we arrive at the following formal definition of entailment (between two wffs) for $F3$:[22]

(iii) *A entails B* iff $\{A, -B\}$ is semantically inconsistent.

On the basis of (iii) it is a simple matter to show that A entails B iff B is a semantic consequence of $\{A\}$. We will show below (T42) that A entails B iff $(A \text{ I } B)$ is valid. Hence, just as in *C2*, we can represent the entailment of B by A by the validity of an appropriate conditional in the object language.

Next, we provide a formal definition of presupposition for *F3* that is similar in structure to that of entailment.

(iv) A presupposes B iff $\{J_1(A) \vee J_3(A), \sim J_1(B)\}$ is semantically inconsistent.[23]

It was mentioned in Section 1 that the table for '$\rightarrow$' is precisely the table one wants; this is claimed to be so on the basis of our informal definition of 'presupposes' ((i) above). We will show below (T43) that A presupposes B iff $(A \rightarrow B)$ is valid. Hence, we will have demonstrated that our formal definition of 'presupposes' (= (iv)) is faithful to the informal one (= (i)).

We begin with some auxiliary lemmas: some will be directly useful in the proof of T42 and T43; several though, are included to display some of the features of *F3*. Again, when convenient, only the numbers of the theorems needed for the proof will be listed to the right of the theorem.

T34 (a) *If* $S \vdash A \vee B$, *then* $S \vdash B \vee A$.
(b) *If* $S \vdash A$ *or* $S \vdash B$, *then* $S \vdash A \vee B$.
(c) *If* $S \vdash A \vee B$ *and* $S \vdash \sim B$, *then* $S \vdash A$.
(d) *If* $S \vdash A \vee B$ *and* $S \vdash \sim A$, *then* $S \vdash B$.
(e) $S \vdash A \vee -A$ *for any wff* A *of F3*.

Proof: $(A \vee B) \supset (B \vee A)$ is valid. Hence (a) by T32(a), T1(b), and T1(e).
(b) Suppose $S \vdash B$. Then $S \vdash (A \supset B) \supset B$ by T1(f). Suppose $S \vdash A$. Then $S \vdash A \vee B$ by T1(f) and (a).
(c) Suppose $S \vdash (A \supset B) \supset B$ and $S \vdash \sim B$. Then $S \vdash \sim(A \supset B)$ by T2(n). Hence, $S \vdash A$ by T2(o).
(d) By (a) and (c).
(e) By A3, T1(d), and (a).

T35 (a) *If* $S \vdash A$, *then* $S \vdash J_1(A)$. [T2(1), T2(p)]
(b) *If* $S \vdash J_1(A)$, *then* $S \vdash A$. [T2(o)]
(c) *If* $S \vdash J_3(A)$, *then* $S \vdash \sim A$. [T2(o)]

T36 (a) $S \vdash J_i(A) \vee \sim J_i(A)$. $(i = 1, 2, 3)$
(b) $S \vdash \sim J_i(A) \equiv -J_i(A)$. $(i = 1, 2, 3)$
(c) $S \vdash -A \equiv \sim J_1(A)$.
(d) $S \vdash (A \rightarrow B) \equiv (\sim A \rightarrow B)$.
(e) $S \vdash (A \rightarrow B) \equiv ((J_1(A) \vee J_3(A)) \text{ I } J_1(B))$.
(f) $S \vdash J_1(A) \vee J_3(A)$ *iff* $S \vdash (A \vee \sim A)$.
(g) $S \vdash ((A \rightarrow B) \,\&\, (B \rightarrow C)) \supset (A \rightarrow C)$.

Proof: (a)-(g) By the validity of the appropriate wff, T32(a), and T1(b).

T37 *If A is a semantic consequence of S, then A is a semantic consequence of any superset, $S \cup S'$, of S.*

Proof: By T32(c) and T1(b).

T38 *A entails B iff B is a semantic consequence of* $\{A\}$.

Proof: By the definition of 'entails' (= (iii)) and the definition of 'semantic consequence'.

T39 (a) $S \vdash A \rightarrow (p \supset p)$.
(b) $S \vdash A \rightarrow (J_1(A) \vee J_3(A))$.
(c) $S \vdash A \rightarrow (A \vee {\sim}A)$.

Proof: (a)-(c) Like that of T36(a) above.

T40 (a) $S \vdash (A \rightarrow B) \supset (-B \supset J_2(A))$.
(b) $S \vdash (A \rightarrow B) \supset ({\sim}J_1(B) \supset J_2(A))$.
(c) $S \vdash (A \rightarrow B) \supset (A \mathrm{I} B)$.

Proof: (a)-(c) Like that of T36(a).

T41 (a) *If* $S \vdash A \rightarrow B$ *and* $S \vdash A$, *then* $S \vdash B$.
(b) *If* $S \vdash A \rightarrow B$, *and either* $S \vdash -B$ *or* $S \vdash {\sim}J_1(B)$, *then* $S \vdash J_2(A)$.
(c) *If* $S \vdash A \rightarrow B$, *then* $S \vdash A \mathrm{I} B$.

Proof: (a) Suppose $S \vdash A \rightarrow B$ and $S \vdash A$. Then $S \vdash J_1(A)$ by T35(a). Hence, $S \vdash J_1(A) \vee J_3(A)$ by T34(a). Hence, $S \vdash J_1(B)$ by T1(e), and so, $S \vdash B$ by T35(b).

(b) Suppose $S \vdash A \rightarrow B$. (i) Suppose further that $S \vdash -B$. Then $S \vdash {\sim}J_1(B)$ by T2(l). Hence, by T2(n), $S \vdash J_2(A)$. (ii) Suppose instead that $S \vdash {\sim}J_1(B)$. Then by T36(c), T5(d), and (i), $S \vdash J_2(A)$.

(c) Suppose $S \vdash A \rightarrow B$. Then by T1(b), $S \cup \{A\} \vdash (J_1(A) \vee J_3(A)) \supset J_1(B)$. But $S \cup \{A\} \vdash A$ by T1(c); hence, $S \cup \{A\} \vdash J_1(A)$ by T35(a). Hence, $S \cup \{A\} \vdash J_1(A) \vee J_3(A)$ by T34(b). Hence, $S \cup \{A\} \vdash J_1(B)$, by T1(e). Thus, $S \cup \{A\} \vdash B$ by T35(b), and so, by T2(f), $S \vdash A \mathrm{I} B$.

T42 (a) *A entails B iff* $\vdash A \mathrm{I} B$.
(b) *A entails B iff* $A \mathrm{I} B$ *is valid.*

Proof: (a) Suppose A entails B. Then, by T38, B is a semantic consequence of $\{A\}$. Hence, by T32(c), $\{A\} \vdash B$. Thus, $\vdash A \mathrm{I} B$ by T2(f). Suppose on the other hand, that $\vdash A \mathrm{I} B$. Then, by T4(c), $\{A\} \vdash B$. Hence, B is a semantic consequence of $\{A\}$, by T32(c). Hence, by T38, A entails B.
(b) By (a) and T32(a).

T43 (a) *A presupposes B iff* $\vdash A \rightarrow B$.
(b) *A presupposes B iff* $A \rightarrow B$ *is valid.*

Proof: (a) Suppose A presupposes B. Then $\{J_1(A) \vee J_3(A), {\sim}J_1(B)\}$ is semantically inconsistent. Hence, so is $\{J_1(A) \vee J_3(A), -J_1(B)\}$. Thus, by T32(c), $\{J_1(A) \vee J_3(A)\} \vdash J_1(B)$. Hence, by T2(f), $\vdash (J_1(A) \vee J_3(A)) \mathrm{I} J_1(B)$. Hence, by T36(e) and T5(d), $\vdash A \rightarrow B$. Suppose for the converse that $\vdash A \rightarrow B$. Then by T36(e) and T5(c), $\vdash (J_1(A) \vee J_3(A)) \mathrm{I} J_1(B)$. Hence, by T4(c), $\{J_1(A) \vee J_3(A)\} \vdash J_1(B)$. Hence, by T32(c), $J_1(B)$ is a semantic consequence of $\{J_1(A) \vee J_3(A)\}$. But ${\sim}J_1(B)$ is logically equivalent to $-J_1(B)$. Hence, $\{J_1(A) \vee J_3(A), {\sim}J_1(B)\}$ is semantically inconsistent, and A presupposes B.
(b) By (a) and T32(a).

T44 *If A presupposes B, then A entails B.*

Proof: Suppose A presupposes B. Then by T43(a), $\vdash A \rightarrow B$. Hence, by T41(c), $\vdash A \text{ I } B$. Hence, A entails B by T42(a).

Theorems 42 and 43 are the results promised in the opening paragraphs of this section. T44 establishes a connection between presupposition and the more familiar notion of entailment; the relation between entailment and presupposition is found to be an inclusion relation. That is, if A presupposes B, then A entails B, but the converse does not hold in general. Thus, we have verified our earlier claim that presupposition and entailment are distinct relations.

This logic is quite neutral with respect to any particular theory of presuppositions. In view of T43, we can gain some insight into the notion of presupposition. By T43 and T39(a), we find that every sentence presupposes every valid wff. This fact may be uninteresting, but it is, nonetheless, to be expected. For, if ever a valid wff were not true, we would certainly be hard-put to say what the truth-value of, say, an atomic wff is. We would hesitate to say it is either true or false, no longer knowing what to expect!

It is an immediate consequence of our definition of presupposition that every sentence presupposes its own bivalence—that it is either true or false. Now, no matter whether we express the bivalence of A as $(J_1(A) \vee J_3(A))$ or as $(A \vee \sim A)$, we see by T39(b)-(c) and T43 that every sentence does presuppose its own bivalence. This is most natural, for if this presupposition is not satisfied (i.e., if A is not bivalent), A is neither-true-nor-false.

Informally, it has been recognized that if A presupposes B and B is not true, A is neither-true-nor-false. By T40(a)-(b), T41(b), and T43, we see that this is provable (valid) in *F3*. Also, since the presuppositions of a sentence are a subset of the entailments of a sentence (this by T44), we look for analogues of modus ponens and modus tollens for '$\rightarrow$'. These are stated respectively in T41(a) and T41(b).

Turning now to '$\Rightarrow$', $(A \Rightarrow B)$ is useful for saying that A presupposes *only* B.[24] Intuitively, $A \Rightarrow B$ is true iff B is both a necessary and sufficient condition for the truth or falsity of A. Then we can prove that:

T45 (a) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash B$, *then* $S \vdash J_1(A) \vee J_3(A)$.
(b) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash B$, *then* $S \vdash A \vee \sim A$.
(c) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash A$, *then* $S \vdash B$.
(d) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash J_2(A)$, *then* $S \vdash -B$.
(e) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash J_2(A)$, *then* $S \vdash \sim J_1(B)$.
(f) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash J_1(B)$, *then* $S \vdash A \vee \sim A$.
(g) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash J_1(B)$, *then* $S \vdash J_1(A) \vee J_3(A)$.
(h) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash -B$, *then* $S \vdash J_2(A)$.
(i) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash -J_1(B)$, *then* $S \vdash J_2(A)$.
(j) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash \sim A$, *then* $S \vdash B$ *and* $S \vdash J_1(B)$.
(k) *If* $S \vdash A \Rightarrow B$ *and* $S \vdash J_3(A)$, *then* $S \vdash B$ *and* $S \vdash J_1(B)$.

Proof: (a)-(k) By the validity of an appropriate conditional, T32(c), T5(a), and T1(e) or T4(a).

Informally, T45 shows that *F3* reflects the following claims:

if A presupposes only B, and B is true, then A is either true or false (a, b, f, g);
if A presupposes only B, and A is true, then B is also true (c);
if A presupposes only B, and A is neither-true-nor-false, then B is not true (d, e);
if A presupposes only B, and B is not true, then A is neither-true-nor-false (h, i);
and, if A presupposes only B, and A is false, then B is true (j, k).

Now, beyond the facts that: (i) every wff presupposes every valid wff (T39(a)), and (ii) every wff presupposes its own bivalence (T39(b), (c)), *F3* is neutral concerning the presuppositions of any particular statements. In general then, in view of the *strong* soundness and completeness theorems for *F3* ($S \vdash A$ iff $S \vDash A$, not just $\vdash A$ iff $\vDash A$), one can add as nonlogical axioms (or axiom-schemata) statements which express one's theory of presuppositions, thereby obtaining a logic "containing" one's theory of presuppositions. It is possible, by the addition of appropriate extra axioms (and the expected revisions of the semantics), to state what the presuppositions of a particular (kind of) statement are by means of '$\rightarrow$', and that these are the *only* presuppositions of that (kind of) statement by means of '$\Rightarrow$'. This logic is, for example, neutral with respect to the Russell-Strawson debate on the truth-conditions for statements containing singular terms. Either view can be accommodated by the addition of one or two (rather obvious) axiom-schemata. In particular, should one feel (contra-Strawson) that valid statements are the *only* statements presupposed by any statement, one can represent this *in* the logic (and not merely in the meta-language). One would add $(A \Rightarrow (p \supset p))$ as an axiom, thereby obtaining as theorems all and only the theorems of Meyer and Lambert's system *FQ*.[25] If, in addition, one is prepared to assume that: (i) every singular term has a referent, and (ii) every domain of discourse is nonempty, one can represent these claims in the logic by the addition of $E!P$, for every P. When this is added to the preceding axiom, the logic obtained is just *C2* with an extra one-place predicate constant 'E!'. (These claims will be proved shortly.)

Let me summarize some of the novelties of *F3*. *F3* does seem to capture many of the features of a Strawsonian notion of presupposition. For example, the Kiparskys [4] have noted that presupposition is transitive, and that it is constant under negation. That is, if A presupposes B and B presupposes C, then A presupposes C; and, A presupposes B if and only if not-A presupposes B. Both of these results are provable *in F3*; they are, respectively, T36(g) and T36(d). Also concerning negation, Strawson [15] distinguished between the contrary and the contradictory of a statement. This distinction is, I argue ([1], ch. 1), just the same distinction which van Fraassen [20] notes as, respectively, the choice-negation and the exclusion-negation of a statement. This distinction is made in *F3* by means of '$\sim$' and '$-$'. Both entailment and presupposition are representable *in F3* by the validity of a particular statement (cf. T42 and T43). And, it was found that if A presupposes B, then A entails B as well, but the converse does not hold in general (cf. T44). Several authors have noticed that it is an immediate consequence of the definition of presupposition that a statement presupposes its own bivalence as well as every valid statement. (van Fraassen is one of these authors; see e.g., [18], p. 146.)

Both of these facts are also provable in *F3*; they are proved at T39(a)-(c). Finally, more than one writer has noticed that presupposition can be defined in terms of what I have been calling "entailment"; others, e.g., Keenan, call it "logical implication". Keenan says ([3], p. 45) that A presupposes B just in case A logically implies B and the negation of A ($\sim A$) also logically implies B. We find in *F3* that both $((A \text{ I } B) \;\&\; (\sim A \text{ I } B)) \text{ I } (A \rightarrow B)$ and $(A \rightarrow B) \text{ I } ((A \text{ I } B) \;\&\; (\sim A \text{ I } B))$ are valid in *F3* and so, by T32(a), both are provable in *F3*. That is, if A entails B and $\sim A$ entails B, then A presupposes B, and conversely.

2.1 Some extensions of F3 We consider some extensions of *F3* to other, perhaps more familiar, logics. We will now prove some of the claims made in the previous paragraphs.

First, we consider *QC*!, the system presented in [5]. This is, in all essentials, Meyer and Lambert's *FQ* of [8], for, as Leblanc notes: "My main concern here will be to outfit Lambert's free logic *FQ*, rechristened for the occasion *QC*!, with a truth-value semantics." ([5], p. 154)

The axioms of *QC*! are:

A1! $A \supset (B \supset A)$
A2! $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3! $(\sim A \supset \sim B) \supset (B \supset A)$
A4! $A \supset (\forall X)A$
A5! $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$
A6! $(\forall X)A \supset (\mathrm{E!}P \supset A(P/X))$
A7! $(\forall X)\mathrm{E!}X$,

and, any wff of the sort $(\forall X)A$, where $A(P/X)$ is an axiom of *QC*! for some P foreign to $(\forall X)A$.[26]

Modus ponens is the one rule of inference.

The primitive wffs of *F3* are the same as those of *QC*!. What I will now show is that when

A10 $A \Rightarrow (p \supset p)$

is added to the axioms of *F3*, for any set S of wffs and any wff A,

(1) $S \vdash A$ in *F3* + A10 iff $S \vdash A$ in *QC*!.[27]

The proof of (1) is by cases. We need to show that: (a) all of the axioms of *QC*! are provable in *F3* + A10, and (b) all of the axioms of *F3* + A10 are provable in *QC*!. Toward (a) we notice that most of the axioms of *QC*! are axioms of *F3* + A10; only two (A2! and A6!) are not. A6! is provable in *F3* (hence, in *F3* + A10) by T6(b). A2! is, therefore, the only case remaining.

Suppose there is an existence-normal tv-pair (α, Σ) on which A2! $\Rightarrow (p \supset p)$ and $-$A2! are true. Then $\alpha_\Sigma(-\text{A2!}) = 1$ and since $\alpha_\Sigma(p \supset p) = 1$ for every α_Σ, $\alpha_\Sigma(\text{A2!}) = 1$ or $\alpha_\Sigma(\text{A2!}) = 3$, by the truth-table for '$\Rightarrow$'. But, by truth-tables we find that $\alpha_\Sigma(\text{A2!}) \neq 3$. So, $\alpha_\Sigma(\text{A2!}) = 1$. But then $\alpha_\Sigma(-\text{A2!}) = 3 \neq 1$, contrary to our earlier assumption. So, there can be no α_Σ on which A2! $\Rightarrow (p \supset p)$ and $-$A2! are true. Hence, $\{\text{A2!} \Rightarrow (p \supset p), -\text{A2!}\}$ is semantically inconsistent, and $\{A2! \Rightarrow (p \supset p)\} \models A2!$. Thus, $\{A2! \Rightarrow (p \supset p)\} \vdash A2!$ by T32(c). But since $A2! \Rightarrow (p \supset p)$ is now (in *F3* + A10) considered as an axiom, we conclude that $\vdash$A2! in *F3* + A10. Hence, (a).

Now for (b)—the converse of (a)—the proof, again, is by cases. A1, A4, A5, A6, and A8 are, respectively, A1!, A3!, A5!, A4!, and A7!. The remaining cases are A2, A3, A7, A9, and A10.

(i) A2 and A3. Interpret these as instructed in [5]. Both are valid in *QC*!; hence, both are provable in *QC*! by the completeness theorem for *QC*!

(ii) A7. By the validity (hence, provability) in *QC*! of $(A \supset (B \supset C)) \equiv (B \supset (A \supset C))$, A7 is provable in *QC*! by A6!

(iii) A9. This is shown to be provable in *QC*! by the same kinds of moves as those in (i) above. Remember that A9 is of the form $(A \supset {\sim}A) \supset {\sim}A$.

(iv) A10. $A \Rightarrow (p \supset p)$ is valid in *QC*! (This is easily verified once it is reduced to primitive notation.) Hence, as in (i) above, it is provable in *QC*!

Hence, (b). Thus, (1).

Next we turn our attention to classical two-valued logic (*C2*) and establish a rather intimate connection between *F3* and *C2*. Add

A11 E!*P*

to *F3* + A10 as an extra axiom (for every *P*).[28] Call this system "*F3* + A10, A11". And, add A11 to *C2*; call it "A11!" there, and call the resulting system "*C2*!". The axiomatization of *C2* that we will use will be that presented in [7], §1.2. The names of these axioms we will preface with '*C*' to distinguish them from the other systems being considered, and we will add '!' as a suffix to each, indicating the addition of A11 to *C2*. Then *CA1*!-*CA5*! are just A1!-A5!; *CA6*! is $(\forall X)A \supset A(P/X)$. The other is the usual generalization clause, and the rule is modus ponens. For the semantics of *C2*!, add the following clause to §1.3 of [7]:

(ix) E!*P* is true on α for every individual parameter *P*.

Now we show:

(2) $S \vdash A$ in *F3* + A10, A11 iff $S \vdash A$ in *C2*!.

The proof, again, is by cases. There is only one case to consider for the necessity clause, since *CA1*!-*CA5*! are provable in *F3* + A10, A11 by (1). *CA6*! is provable in *F3* + A10, A11 by A7, A11, T1(d) and T1(e). The converse is immediate in view of the fact that each of A1-A11 is valid in *C2*!. Hence, by the strong completeness theorem for *C2*!, A1-A11 are provable in *C2*!.[29]

Thus, we have shown a strong connection between *F3* and *C2*, and we have found that the semantic concept of presupposition is representable in the object-language of *F3*. Finally, in view of the theorems proved in the first half of Section 2, and the remarks following those theorems, the concept of presupposition representable in *F3* seems to be a thoroughly "Strawsonian" semantic concept.

NOTES

1. See [8], p. 8, where Meyer and Lambert say that a logic that meets both of the following criteria is universally free: "(1) no existence assumptions are made with respect to individual constants, and (2) theorems are valid in every domain including the empty domain."

2. These may do duty for singular terms; they will occur only free in any well-formed formula. Also, because of our definition of well-formed formulas, the individual variables will occur only bound in well-formed formulas.

3. Usually the superscript and subscript will be omitted in practice.

4. The intended reading of 'E!' is 'exists'.

5. We will, henceforth, allow an expression to be its own name so long as no ambiguity threatens.

6. For a more detailed treatment of these matters, cf., [7], p. 5.

7. The ordering in [7], p. 10, with the addition of '117' for the code number of 'E!' will suffice.

8. 'I' has some of the features of the two-valued implication sign that our '$\supset$' lacks. $-A$ is the exclusion-negation of A, while $\sim A$ is the choice-negation of A. (See [20], pp. 37, 69, fn. 8; see also *ibid*., p. 154.) 'J_1' is something like a truth-operator, 'J_3' something like a falsity-operator, and 'J_2' something like a neither-true-nor-false operator. I do not wish to suggest, though, that 'J_1' be interpreted as 'it is true that', for there are some good reasons for not doing so. The view that I would like to defend is that the truth-table for 'it is true that A' is the same as the table for A. But this is not a matter for this paper.

9. A1-A4 are Wajsberg's axioms [21] for Łukasiewicz's three-valued sentential logic. A5 and A6 are common to many axiomatic systems; it is easily verified that 'X' is sure to be foreign to A in A6. A7 is a permutation of an axiom that first appeared—as far as I know—in [8]. A8 also appeared in that paper. A8 asserts that everything exists, but as Quine has reminded us ([9], p. 1), this doesn't tell us very much about what there is. There is some question about the independence of A9—I don't know whether it is independent. The converse of A9 is provable.

10. If $\Sigma = \emptyset$, every universal quantification is true on (α,Σ) no matter the α; and, an existential quantification is true on (α,Σ) only if $\Sigma \neq \emptyset$. Thus, even if $A(P/X)$ is true on (α,Σ), it does not follow that $(\exists X)A$ is also true on (α,Σ). $(\exists X)A$ is true on (α,Σ) only if, in addition, $P \in \Sigma$. Because of this and the fact that $\alpha_\Sigma(\mathrm{E}!P) = 1$ just in case $P \in \Sigma$, and $\alpha_\Sigma(\mathrm{E}!P) = 3$ just in case $P \notin \Sigma$, we will think of Σ as constituting the set of *designating singular terms*. Hence, we can say that $(\forall X)A$ is true on (α,Σ) if and only if every designating instance of it is true on (α,Σ).

11. Roughly, S' is isomorphic to S if there is a one-to-one function R from the set consisting of all the individual parameters in S into the set consisting of all the individual parameters in S'. If S is infinitely extendible, then S is semantically consistent iff S is

tv-verifiable. The need for the term 'isomorphic' arises from cases like the following: $\{f(a_1), f(a_2), f(a_3), \ldots, \sim(\forall x)f(x)\}$ is syntactically consistent and–in model-theoretic semantics–found to be semantically consistent as well. However, the set is not tv-verifiable. Thus, the lack of tv-verifiability should not amount to a lack of semantic consistency. So, a remedy for such cases is needed. Several remedies are available; the one chosen here is fully described in [7], pp. 11-12. For a brief survey of the alternative "remedies", see [6], pp. 6f.

12. The "lament" that I refer to is expressed by Russell in [11], p. 203: "There does not even seem any logical necessity why there should be even one individual. . . . The primitive propositions of *Principia Mathematica* are such as to allow the inference that at least one individual exists. But I now view that as a defect in logical purity."

13. For Russell's views on this, cf. [10] and [11], ch. XVI. The view of Strawson's mentioned here is stated in [14]; [15], ch. 8, Pt. III, §7; and [16].

14. Incidentally, all the connectives in Woodruff's paper [22] (except his 'u') are definable in *F3*. The sentence connectives of *F3* are not "truth-functionally complete"; that is, not all possible propositional functions are definable using just '$\sim$' and '$\supset$'. Słupecki notes [13] that the function which uniformly takes the value 2, no matter the value of the argument, is not definable on this basis. This function is Woodruff's 'u'. The fact that this function is not definable in *F3* is not regarded as a defect, though, for it has no plausible reading in English. But, since all the other connectives are definable in *F3*, should one find the connectives of *F3* intuitively unappealing, one can easily work with other defined connectives, e.g., Bochvar's connectives (as used, e.g., by Woodruff).

15. These are several of the points of difference between *F3* and the (related) work of others, e.g., [22], [17] and later papers of van Fraassen, and [2].

16. Cf. Note 10, above.

17. See [22], p. 127. By a consideration of cases, one comes to realize that this is precisely the table that one wants for a "presupposition" connective. I will not pause for this here. An argument to support the adoption of this table as well as all the other tables is contained in [1], ch. 1.

18. The wff in (b) is more common as an axiom; cf. [8], [5], and [12]. We chose A7 and not the wff in (b) as an axiom because it greatly simplifies the proof of (c) to do so. And, in view of (a) and (d), no theorems are gained or lost by our choice.

19. I am indebted to an anonymous referee for the proof of (h).

20. *F3* is, therefore, not only "statement complete" (T27) but "argument complete" (T31) as well. The supervaluation logic of van Fraassen [17] is statement-but not argument-complete. There are, of course, other differences between *F3* and the "supervaluation logic" of Professor van Fraassen.

21. See e.g., [15], pp. 16-20, and p. 175. Remember, we are interested in a "Strawsonian" notion of semantic presupposition.

22. The earlier definition of entailment (semantic consequence) was more general; it was for a set S and a wff A. (iii) can be regarded as a special case of the earlier definition where S is a singleton.

23. $\{A \vee \sim A, \neg B\}$ would work just as well.

24. As far as I know, the inclusion of this sentence connective (with its truth-table) as an interesting connective, and the investigation of some of its properties, is unique to the work presented here. It is distinct from '$\rightarrow$'.

25. [8]. *FQ* is the same as Leblanc's *QC*! in [5]; see Note 27 below.

26. We distinguish these axioms from the axioms of *F3* by '!'. The quantification in A4! is vacuous.

27. "*F3* + A10" is, of course, short for "the logic that results from the addition of A10 to *F3*". The intuitive content of A10 is just this: for any wff A, A presupposes all and only valid wffs. Semantically, the consequence of this is that every wff A is either true or false. Deductively, it is that $(A \vee {\sim}A)$ is now provable for every wff A. One way of viewing the Russell-Strawson exchange on the analysis of definite descriptions is that it is an argument over differing theories of presuppositions. Russell might say that every statement is either true or false, while this is not necessarily so on the "Strawsonian" view. Hence, should one feel, along with Russell, that every statement is either true or false, one ought to add A10 to the axioms of *F3*, but not otherwise.

28. One would add A11 to *F3* + A10 if one wanted one's logic to be such that every statement is either true or false (A10) and every singular term designates something. These are just the usual presuppositions that are "packed-into" classical two-valued logic.

    Once A11 is added to *F3*, A8 is redundant, for $(\forall X)\mathrm{E}!X$ is then an axiom by the generalization clause in our axioms.

29. Leblanc, in [5], p. 157, defines "null assignments" as those under which E!P is false for every P, and "standard assignments" as those under which E!P is true for every P. Then, *F3* + A10, A11 provides proof of every wff true under every standard assignment. Cf. [5], pp. 157, and 164-165.

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