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A Shorter Proof of a Recent Result by R. Di Paola

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In a recent paper [2] Di Paola has proven that a formula $F(x)$ exists which is nonextensional in a very strong sense, despite its relatively simple structure (in particular, $F(x)$ is equivalent to $\neg \dot{Thm}(\overline{t(\dot{x})})$, where $\dot{Thm}$ is the standard (extensional) *RE*-formula numerating the set of theorems of *PA* and $t(x)$ is a fixed term). As is shown in [2], the result is relevant for an algebraic approach to incompleteness phenomena; especially when an attempt is made to extend the theory of the so-called diagonalizable algebras by considering structures in which formulas with free variables and quantifiers are representable. (See [2] for general motivations, remarks, and consequences.)

In this paper another proof of the result is presented, which is shorter than Di Paola's; moreover, unlike Di Paola's paper, no prerequisites are required. A generalization is also discussed.

We recall the statement of the theorem.

Theorem *There is a* Π_1 *formula* $F(x)$ *of PA such that*

(i) *there is an infinite recursive set* $\mathfrak{F}$ *of fixed points of* $F(x)$ *in PA and the set* $\mathcal{E} = \{\phi / \phi \in \mathfrak{F} \text{ and } \omega \models \phi\}$ *is not recursive*
(ii) *for every recursively enumerable* Σ_1*-sound extension T of PA and almost all* $\phi \in \mathcal{E}$, ϕ *is undecidable in T*
(iii) *for every T as in* (ii) *and for every* $\phi \in \mathcal{E}$, *almost all sentences* ψ *which are provably equivalent to* ϕ *in T are not fixed points of* $F(x)$ *in T.*

Moreover there is a fixed term $t(x)$ *of a PR-extension* PA^+ *of PA such that* $\vdash_{PA^+} \neg \dot{Thm}(\overline{t(\dot{x})}) \leftrightarrow F(x)$.

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Remark 1: In [2] a relation E_T is mentioned and at first glance the statement there seems to be weaker. Actually, since the following argument applies also to the formula constructed in [2], the two statements are equivalent, even if this is not entirely evident in Di Paola's proof (in particular, the hypothesis stated in [2] that E_T is an r.e. relation is unnecessary).

Proof: We identify sentences with their Gödel numbers; in particular, we assume that 1 is a theorem of PA and 0 its negation.

Let S be a simple set and let $A(x)$ be a Σ_1 formula which numerates it in PA in the sense that $n \in S$ iff $\vdash_{\overline{PA}} A(\bar{n})$. Let $\dot{S}(x)$ be a formula equivalent to $\dot{Thm}(\overline{A(\bar{x})})$ for every x; note that $\dot{S}(x)$ is a Σ_1 formula which numerates S in every T as in (ii) and that $\vdash_T \dot{S}(\bar{n})$ iff $\vDash \dot{S}(\bar{n})$ (iff $n \in S$).

Define $\mathfrak{F}$ as follows: $\mathfrak{F} = \{\neg\dot{S}(\bar{n})/n \in \omega\}$; therefore $\mathcal{E} = \{\neg\dot{S}(\bar{n})/n \in \bar{S}\}$. Note that $\mathcal{E}$ is an immune set since, if W were an infinite r.e. subset of $\mathcal{E}$, $\{n/\neg\dot{S}(\bar{n}) \in W\}$ would be an infinite r.e. subset of $\bar{S}$.

Define a total (primitive) recursive function h as follows:

$$h\phi = \begin{cases} A(\bar{n}) \text{ if } \phi \in \mathfrak{F} \text{ and } \phi = \neg\dot{S}(\bar{n}) \\ 1 \text{ (or any theorem of } PA\text{) if } \phi \notin \mathfrak{F}. \end{cases}$$

Let $H(x,y)$ be a Σ_1 formula which binumerates h (as a set of pairs) in PA and, hence, also in T. Now, define the required Π_1 formula $F(x)$ as follows:

$$F(x) = \forall z(H(x,z) \to \neg\dot{Thm}(z)) .$$

It is easy to verify that for every ϕ the formula $F(\bar{\phi})$ is provably equivalent to $\neg\dot{Thm}(\overline{h\phi})$; so the term $t(x)$ mentioned in the last part of the statement is readily constructed. It follows that

(a) every $\phi \in \mathfrak{F}$ is a fixed point of $F(x)$: if $\phi \in \mathfrak{F}$, by the definition of h, $\neg\dot{Thm}(\overline{h\phi})$ is provably equivalent to ϕ

(b) a sentence ϕ not belonging to $\mathfrak{F}$ is a fixed point of $F(x)$ in T iff it is the negation of a theorem of T: if $\phi \notin \mathfrak{F}$, $F(\bar{\phi})$ is provably equivalent to $\neg\dot{Thm}(\bar{1})$, i.e., to 0.

Moreover, we have

(c) every sentence belonging to $\mathfrak{F} - \mathcal{E}$ is the negation of a theorem of T: indeed, if $\phi \in \mathfrak{F} - \mathcal{E}$, say $\phi = \neg\dot{S}(\bar{n})$, we have $\vDash \dot{S}(\bar{n})$ and hence $\vdash_T \dot{S}(\bar{n})$, i.e., $\vdash_T \neg\phi$. In view of (b), we can conclude
(d) a sentence not belonging to $\mathcal{E}$ is a fixed point of $F(x)$ in T iff it is the negation of a theorem of T.

Now consider a theory T as above and the set $\mathfrak{Z}$ of the theorems of T; by (c) we have $\mathfrak{Z} \cap \mathcal{E} = \mathfrak{Z} \cap \mathfrak{F}$: hence $\mathfrak{Z} \cap \mathcal{E}$ is an r.e. subset of $\mathcal{E}$ and therefore is finite. On the other hand, an element of $\mathcal{E}$ cannot be the negation of a theorem of T since T is Σ_1-sound (if $\phi \in \mathcal{E}$, $\neg\phi$ is a false Σ_1 sentence). So (ii) is proven.

In order to prove (iii), consider a sentence ψ which is provably equivalent to ϕ in T and is a fixed point of $F(x)$ in T. Note that ψ cannot be a refutable sentence: if $\vdash_T \neg\psi$, it would follow that $\vdash_T \neg\phi$, contradicting the fact that T is Σ_1-sound. So, by (d) the considered ψ must belong to $\mathcal{E}$, but the set of fixed points of $F(x)$ which are provably equivalent to ϕ is an r.e. set and, being enclosed in $\mathcal{E}$, is finite.

Remark 2: By (d) it immediately follows that, if T is sound, then $\mathcal{E}$ is the set of all true fixed points of $F(x)$ in T.

Remark 3: As in Di Paola's proof, the described construction is quite general and some variations are possible in order to satisfy further side conditions. In particular the choice of the simple set S (and therefore the choice of the co-r.e. immune set $\mathcal{E}$) is completely arbitrary. On the other hand, if we start from another kind of set (instead of from a simple set), by a quite similar construction we get again a formula $F(x)$ and a corresponding set of fixed points of $F(x)$.

If we limit ourselves to a fixed Σ_1-sound r.e. extension T of PA, statements analogous to Di Paola's theorem are easily found. For instance, let us define the formula $F(x)$ to be provably equivalent to $\neg \dot{Thm}_T(\overline{\bar{x} \neq \bar{c}})$ for every x, where c is the Gödel number of the sentence $\neg \dot{Thm}_T(\bar{0})$ which expresses the consistency of T within the same T. It is readily seen that the set $\mathfrak{F}$ of *all* fixed points of $F(x)$ is constituted by the sentence whose Gödel number is c and by all refutable sentences. So, on the one hand the set $\mathcal{E}$ is recursive (and in fact it is a singleton); but, on the other hand, *every* ψ provably equivalent to the element of $\mathcal{E}$ (but different from it) is not a fixed point of $F(x)$.

The situation is much more complex if nonsound theories are considered also. In this case the formula $F(x)$ cannot be equivalent to $\neg \dot{Thm}(\overline{t(\bar{x})})$ for some term $t(x)$, because it may happen that $\vdash_{\overline{T}} \forall x(\dot{Thm}(x))$. However, if instead of the standard formula $\dot{Thm}(x)$ other extensional formulas numerating the set of theorems of PA are considered, similar constructions are still possible. For instance, let us refer to the variant of Rosser predicate $R^f(x)$ which is defined in [1]: we recall that it is an extensional Σ_1 formula such that $\vdash_{\overline{PA}} R^f(\bar{1})$ and $\vdash_{\overline{PA}} \neg R^f(\bar{0})$.

We can prove the following statement.

There is a Π_1 *formula* $F(x)$ *of PA such that*

(i) *there is an infinite recursive set* $\mathfrak{F}$ *of fixed points of* $F(x)$ *in PA and the set* $\mathcal{E} = \{\phi/\phi \in \mathfrak{F}, \vDash \phi$ *and not* $\vdash_{\overline{PA}} \phi\}$ *is not recursive*
(ii) *for each r.e. consistent extension T of PA and almost all* $\phi \in \mathcal{E}$, ϕ *is undecidable in T*
(iii) *for every T as in* (ii) *and almost all* $\phi \in \mathcal{E}$, *almost all sentences* ψ *which are provably equivalent to* ϕ *in T are not fixed points of* $F(x)$ *in T.*

Moreover there is a fixed term $t(x)$ *of a PR-extension* PA^+ *of PA such that* $\vdash_{\overline{PA}} \neg R^f(\overline{t(\bar{x})}) \leftrightarrow F(x)$.

We only sketch the proof. Consider a maximal set M and apply Friedberg's decomposition theorem to obtain two disjoint r.e. sets A and B which are not recursive and whose union is M. Note that if A' and B' are disjoint r.e. sets containing A and B respectively, then both $A'-A$ and $B'-B$ are finite.

Let $A(x)$ be a Σ_1 formula which exactly separates A and B in PA (that is, $n \in A$ iff $\vdash_{\overline{PA}} A(\bar{n})$ and $n \in B$ iff $\vdash_{\overline{PA}} \neg A(\bar{n})$); let $\dot{S}(x)$ be a formula provably equivalent to $R^f(\overline{A(\bar{x})})$ for every x. Define as in the previous proof the set $\mathfrak{F}$, the function h, and the formula $F(x)$ (replacing $\dot{Thm}(z)$ by $R^f(z)$).

The claim follows. As regards (iii), note that the fixed points of $F(x)$ in T are the elements of $\mathcal{E}$, the negations of the theorems of T, and some theorems of T. So, if $\phi \in \mathcal{E}$ and $\vdash_T \phi$ or $\vdash_T \neg\phi$, in T there exist infinitely many fixed points ψ of $F(x)$ which are provably equivalent to ϕ; but if $\phi \in \mathcal{E}$ and ϕ is undecidable in T (and this is the case for almost all $\phi \in \mathcal{E}$) only finitely many ψ as above can exist.

REFERENCES

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