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Connection Structures

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Abstract B. L. Clarke, following a proposal of A. N. Whitehead, presents an axiomatized calculus of individuals based on a primitive predicate "x is connected with y". In this article we show that a proper subset of Clarke's system of axioms characterizes the complete orthocomplemented lattices, while the whole of Clarke's system characterizes the complete atomless Boolean algebras.

1 Introduction In [2] and [3] Clarke presents an axiomatized calculus of individuals based on a primitive predicate "x is connected with y". Such a calculus represents a revised version of the proposal made by Whitehead in *Process and Reality* and is similar to the calculus proposed by Leonard and Goodman in [5].

In this article we show that a proper subset of Clarke's system of axioms characterizes the complete orthocomplemented lattices, while the whole of Clarke's system characterizes the complete atomless Boolean algebras.

2 Connection structures Let R be a nonempty set and C a binary relation on R, set $C(x) = \{y \in R/xCy\}$ and suppose the following axioms are true of every $x,y \in R$:

A1 xCx;
A2 $xCy \Rightarrow yCx$;
A3 $C(x) = C(y) \Rightarrow x = y$.

We call *regions* the elements of R and, if $x,y \in R$ and xCy, we say that x is *connected* with y. If X is a nonempty subset of R, we say that x is the *fusion* of X just in case for every $y \in R$, xCy iff for some $z \in X$, zCy; in other words, x is the fusion of x provided that

$$C(x) = \cup\{C(z)/z \in X\}. \tag{1}$$

The fusion of the nonempty subsets of R is assured by the following axiom.

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A4 *$X \subseteq R$ and $X \neq \emptyset \Rightarrow$ there exists $x \in R$ such that x is the fusion of X.*

If A1–A4 are satisfied, we say that $\mathcal{R} = (R, C)$ is a *connection structure.*

By A3 and A4 there is a unique fusion of a nonempty class X of regions; we denote it by $f(X)$. A3 implies that the relation $\leq$ defined in R by

$$x \leq y \Leftrightarrow C(x) \subseteq C(y) \tag{2}$$

is a partial ordering. If $x \leq y$ we say also that x is *contained* in y or that x is a *subregion* of y. If a region z exists such that $z \leq x$ and $z \leq y$, we say that *x overlaps y* and we write xOy. Observe that the system obtained by adding to A1–A4 the axiom "the overlapping relation coincides with the connection relation" is equivalent to the system of axioms proposed in [5].

Notice that $(R, \leq)$ admits a maximum 1, namely the fusion of R. If $x \neq 1$ we define the *complement* $-x$ of x by

$$-x = f(\{z \in R/z \text{ is not connected with } x\}). \tag{3}$$

(That is, $C(-x) = \cup\{C(z)/z$ is not connected with $x\}$.) Notice that, if $x \neq 1$, the set $\{z \in R/z$ is not connected with $x\}$ is nonempty; indeed, since $C(1) = R$ and $C(x) \neq C(1)$, there exists a region y not connected with x.

Lemma 1 *For every pair of regions x and y the following hold:*
(a) *$xOy \Rightarrow xCy$;*
(b) *$(R, \leq)$ has a minimum only in the case $R = \{1\}$;*
(c) *for every $x \neq 1$, x is not connected with $-x$;*
(d) *for every $x \neq 1$, $-x \neq 1$.*

Proof: Assume that $C(z) \subseteq C(x)$ and $C(z) \subseteq C(y)$. Since $z \in C(z)$, from $C(z) \subseteq C(x)$ it follows that zCx and therefore that $x \in C(z)$. From $C(z) \subseteq C(y)$ it follows that $x \in C(y)$ and this proves (a). To prove (b), assume that an element 0 exists such that $C(0) \subseteq C(x)$ for every $x \in R$; then $x \in C(0)$ for every $x \in R$ and $C(0) = R = C(1)$. By A3 we have $0 = 1$. To prove (c), assume that $xC-x$; then, since $x \in C(-x) = \cup\{C(z)/z$ is not connected with $x\}$, a suitable z exists such that $x \in C(z)$ and z is not connected with x, a contradiction. Finally, since $xC1$ for every $x \in R$, (d) is a consequence of (c).

To prove that, in a sense, the connection structure theory coincides with the orthocomplemented lattice theory, we associate with every connection structure (R, C) an algebraic structure $(L, \leq, -)$ as follows. Given any arbitrary element 0 not in R, we set $L = R \cup \{0\}$; moreover we set $0 \leq x$ for every $x \in R$, $-1 = 0$ and $-0 = 1$. Also, recall that an *orthocomplemented lattice* is a lattice L equipped with a unary operation $- : L \to L$ such that

$$L1 \;\; --x = x; \qquad L2 \;\; x \wedge -x = 0; \qquad L3 \;\; x \leq y \Leftrightarrow -x \geq -y;$$

we assume also that $0 \neq 1$, i.e. in L there are at least two elements.

Proposition 2 *The structure $(L, \leq, -)$ associated to a connection structure (R, C) is an orthocomplemented complete lattice; the join in L of a nonempty subset of R coincides with its fusion. Moreover, if $x, y \in R$, then*

$$xCy \Leftrightarrow x \nleq -y. \tag{4}$$

Conversely, let $(L,\leq,-)$ *be a complete orthocomplemented lattice, set* $R = L - \{0\}$ *and define* C *by* (4). *Then* (R,C) *is a connection structure whose associated orthocomplemented lattice is* $(L,\leq,-)$.

Proof: If X is a nonempty subset of L it is immediate that the join $\vee X$ coincides with the fusion of $X - \{0\}$ if $X \neq \{0\}$ and with 0 otherwise. This proves that $(L,\leq)$ is a complete lattice. To prove (4), assume $y \neq 1$ and xCy; then, since $y \in C(x)$, from $C(x) \subseteq C(-y)$ we have it that $y \in C(-y)$ and this contradicts Lemma 1(c). This proves that $C(x)$ is not contained in $C(-y)$ and therefore $x \nleq -y$. Conversely, if $x \nleq -y$ then $C(x)$ is not contained in $C(-y)$ and, by (3), xCy.

In the case $y = 1$, since $-y = 0$, (4) is obvious.

Now, we will prove that $(L,\leq,-)$ is orthocomplemented. In the case $x \in \{0,1\}$, *L1*, *L2*, and *L3* are obvious. Assume that $x \notin \{0,1\}$ and $y \notin \{0,1\}$; then to prove *L3* we observe that

$$x \leq y \Leftrightarrow C(x) \subseteq C(y) \Leftrightarrow \{z \in R/z \nleq -x\} \subseteq \{z \in R/z \nleq -y\}$$
$$\Leftrightarrow \{z \in R/z \leq -y\} \subseteq \{z \in R/z \leq -x\} \Leftrightarrow -y \leq -x.$$

To prove *L1* notice that, since x is not connected with $-x$, by (4) we have

$$x \leq --x. \tag{5}$$

By applying (5) to the region $-x$ we obtain that $-x \leq ---x$. Thus, by *L3*, $x \geq --x$ and therefore $x = --x$.

L2 follows from Lemma 1(a) and (c).

Conversely, let $(L,\leq,-)$ be a complete orthocomplemented lattice, define in $R = L - \{0\}$ the relation C by (4) and let $x,y \in R$. To prove that xCx, observe that from $x \leq -x$ it follows that $x = x \wedge x \leq x \wedge -x = 0$.

A2 follows from the equivalences

$$xCy \Leftrightarrow x \nleq -y \Leftrightarrow -x \ngeq --y \Leftrightarrow -x \ngeq y \Leftrightarrow yCx.$$

To prove A3 notice that $C(x) \subseteq C(y) \Leftrightarrow \{z \in R/z \nleq -x\} \subseteq \{z \in R/z \nleq -y\} \Leftrightarrow \{z \in R/z \leq -x\} \supseteq \{z \in R/z \leq -y\} \Leftrightarrow -y \leq -x \Leftrightarrow x \leq y$.

Thus, the order defined in (R,C) coincides with the order of the lattice L and this proves A3.

To prove A4, let X be a nonempty subset of R and $x = \vee X$; we will prove that x is the fusion of X; i.e., $C(x) = \cup\{C(z)/z \in X\}$. Indeed, since $x \geq z$ for every $z \in X$, $C(x) \supseteq \cup\{C(z)/z \in X\}$. Conversely, assume that $y \in C(x)$ and that $y \notin C(z)$ for every $z \in X$. Then y is different from 1, $x \nleq -y$ and $z \leq -y$ for every $z \in X$. This contradicts the fact that x is the lower upper bound of X.

It is immediate that the lattice associated to (R,C) coincides with L. Since

$$-x = \vee\{z \in R/z \leq -x\} = \vee\{z \in R/z \text{ is not connected with } x\},$$

the orthocomplement $-x$ in L coincides with the complement defined in (R,C) by (3).

3 The points in a connection structure A *point* of a connection structure (R,C) is defined by Clarke as a nonempty subset P of R such that

(i) $x \in P, y \in P \Rightarrow xCy$
(ii) $x \in P, y \in P, xOy \Rightarrow x \wedge y \in P$
(iii) $x \in P, y \geq x \Rightarrow y \in P$
(iv) $x \vee y \in P \Rightarrow x \in P$ or $y \in P$.

As usual, we say that a point P *belongs to* a region x and write $P\epsilon x$ provided that $x \in P$; moreover we denote by Π the set of points of (R,C). Clarke proposes the following axiom:

A5 *$xCy \Rightarrow$ a point P exists such that $P\epsilon x$ and $P\epsilon y$.*

Notice that A5 together with (i) assures that two regions are connected iff they contain a common point. In particular, every region contains at least one point.

The following proposition shows that, in a sense, the system A1–A5 characterizes the complete Boolean algebras.

Proposition 3 *If a connection structure (C,R) satisfies A5, then the connection relation coincides with the overlapping relation. Moreover, the orthocomplemented lattice associated to (C,R) is a complete Boolean algebra. Conversely, every complete Boolean algebra is associated to a suitable connection structure satisfying A5.*

Proof: By Lemma 1(a), we have only to prove that xCy implies xOy. Now, if xCy, a point P exists such that $x \in P$ and $y \in P$. In the case $x = 1$ or $y = 1$, it is obvious that xOy. Assume $x \neq 1$ and $y \neq 1$ and set $u = -x \vee -y$. It is $u \neq 1$, otherwise $-x \vee -y \in P$ and so, by (i), either $xC-x$ or $yC-y$ and this contradicts Lemma 1(c). Thus it is $u \neq 1$, since $u \geq -x$ and $u \geq -y$, we have that $-u \leq x$ and $-u \leq y$; i.e., xOy.

To prove that the orthocomplemented lattice L associated to (R,C) is a Boolean algebra, we prove that the map $h: L \to \mathcal{P}(\Pi)$ defined by setting $h(0) = \varnothing$ and $h(x) = \{P \in \Pi / P\epsilon x\}$, for $x \neq 0$, is an injective homomorphism from the orthocomplemented lattice L into the Boolean algebra $\mathcal{P}(\Pi)$.

Assume that $x,y \in R$; then the equality $h(x \vee y) = h(x) \cup h(y)$ is an immediate consequence of (iv) and (iii). Moreover, from (iii) it follows that $h(x \wedge y) \subseteq h(x) \cap h(y)$. To prove $h(x \wedge y) \supseteq h(x) \cap h(y)$, assume that $P \in h(x) \cap h(y)$, i.e. $x,y \in P$; then, since xCy, we have also xOy and (by (ii)) $P \in h(x \wedge y)$. Both the equalities $h(x \vee y) = h(x) \cup h(y)$ and $h(x \wedge y) = h(x) \cap h(y)$ are immediate if $x = 0$ or $y = 0$.

To prove $h(-x) = -h(x)$, assume $x \neq 0$, $x \neq 1$ and $P \in h(-x)$; then, since x is not connected with $-x$, by (i) we have $P \notin h(x)$. Conversely, if $P \notin h(x)$, since $x \vee -x = 1 \in P$, then (iv) entails that $-x \in P$ and therefore $P \in h(-x)$. If $x = 0$ or $x = 1$ it is immediate that $h(-x) = -h(x)$.

To prove that h is injective, assume $h(x) = h(y)$: if $x = 0$ then $h(y) = h(0) = \varnothing$ and by A5 $y = 0$. If $x = 1$ and $y \neq 1$ then every point of $-y$ is not in y and this contradicts the fact that $h(y) = h(x) = \Pi$. In the same way we proceed if $y = 0$ or $y = 1$. Assume that both x and y are different from 0 and 1 and $x \not\leq y$ or, equivalently, $x \not\leq --y$. Then, since $xC-y$, by A5 a point P exists such that $x, -y \in P$. Hence, $P \in h(x)$ and $P \notin h(y)$ and this contradicts the hypothesis that $h(x) = h(y)$. Consequently $x \leq y$, in the same manner one proves that $y \leq x$ and therefore $x = y$. This proves the first part of the proposition.

Assume now that L is a complete Boolean algebra; then, since L is an orthocomplemented lattice, by Proposition 2 a connection structure (R, C) is associated to L. To prove A5, observe that in every Boolean algebra

$$\begin{aligned} x \wedge y = 0 &\Rightarrow -x \vee -y = 1 \\ &\Rightarrow y = (-x \vee -y) \wedge y = (-x \wedge y) \vee (-y \wedge y) = -x \wedge y \Rightarrow y \leq -x. \end{aligned}$$

This means that in (R, C) the relation C coincides with the relation O and the points of (R, C) coincide with the prime filters of L. As a consequence, A5 becomes a well-known property of Boolean algebras.

Notice that Axiom A2.1′ given in [3] becomes

A6 *There is no atom in* $(R, \leq)$.

As a consequence the following proposition holds.

Proposition 4 *The whole of Clarke's system of Axioms A1–A6 characterizes the atomless complete Boolean algebras. The class of the open regular subsets of a euclidean space with respect to the overlapping relation is a model of this system.*

4 Concluding remarks Recall that if $\mathcal{T}$ is a topology on a set X, the set $\mathcal{T}^* = \{x \in \mathcal{T} / x = (\bar{x})^\circ\}$ of the open regular subsets of X is a complete Boolean algebra with respect to the set theoretic inclusion relation. Namely, we have it that, if Y is a nonempty subset of $\mathcal{T}^*$, then

$$\vee Y = ((\cup Y)^-)^\circ, \ \wedge Y = (\cap Y)^\circ$$

and if $x \in \mathcal{T}^*$, then the complement of x is the interior of the set theoretic complement of x. Conversely, every complete Boolean algebra can be obtained in this way (see Halmos [4]). Thus the system A1–A5 of axioms characterizes, in a sense, the structure of the open regular subsets of a topological space. This is all right since the regular open subsets are natural candidates to represent regions. Unfortunately, the coincidence of the connection relation with the overlapping relation is rather unsatisfactory and far from the purpose of Whitehead and Clarke. Indeed the relations "x is externally connected to y" and "x is tangential part of y" proposed in [2] and [3] are satisfied by no pair of regions and the concept of "nontangential part" collapses into that of "part". As a consequence, the question of a suitable modification of the system of axioms proposed by Clarke arises. The new system should still admit as models the class of the nonempty regular open subsets of a topological space $(S, \mathcal{T})$. But in these models the definition of the connection relation should be as follows:

$$xCy \Leftrightarrow \bar{x} \cap \bar{y} \neq \emptyset.$$

The models obtained in this way satisfy A1 and A2. Moreover, if the topological space is regular, the relation $\leq$ defined by (2) coincides with the inclusion relation and this gives, in particular, A3. Indeed, $x \subseteq y$ implies $\bar{x} \subseteq \bar{y}$ and therefore $C(x) \subseteq C(y)$. Conversely, let $C(x) \subseteq C(y)$ and assume that $\bar{x}$ is not contained in $\bar{y}$. Then an element p of S exists such that $p \in \bar{x}$ and $p \notin \bar{y}$. Let z be an open neighborhood of p such that $\bar{z} \cap \bar{y} = \emptyset$ and set $z' = (\bar{z})^\circ$. Then it is well-known that z' is regular; moreover, $z' \in C(x)$ and $z' \notin C(y)$, which contradicts the hypothesis that $C(x) \subseteq C(y)$. Then $\bar{x} \subseteq \bar{y}$ and therefore $x = (\bar{x})^\circ \subseteq (\bar{y})^\circ = y$.

Now, in spite of the completeness of $(R,\leq)$, A4 is not satisfied and there are subsets of R with no fusion. For example, assume that the topological space under consideration is the real line, set $X = \{\,]0,1 - 1/n[\,/n > 1\}$ and assume the existence of a region x for which $C(x) = \cup\{C(]0,1 - 1/n[)/n > 1\}$. Then we have $x \supseteq\,]0,1 - 1/n[$ for every $n > 1$ and hence $1 \in \bar{x}$. As a consequence, the region $]1,2[$ is connected with x in spite of the fact that $]1,2[$ is not connected with $]0,1 - 1/n[$ for every $n > 1$. Thus A4 should be weakened, assuming only the completeness of $(R,\leq)$.

Finally, the definition of point and axiom A5 give rise to some difficulties. Indeed, in view of A5, if $p \in S$ then the class $P = \{x \in R/p \in \bar{x}\}$ should be a point of the connection structure (R,C). Unfortunately this is not true since (ii) is not satisfied. Indeed, two nonconvex regions of a euclidean space can be tangent in p and overlap in a region that does not contain p in its closure.

Perhaps it is possible to avoid such difficulties by considering suitable bases of filters rather than filters. For example, call *representative of a point* every class p of regions such that, for every pair x,y of regions,

(i)′ $x \in p,\ y \in p \Rightarrow xCy$;
(ii)′ $x \in p,\ y \in p,\ xOy \Rightarrow x \wedge y \in p$;
(iii)′ $(\forall z \in p,\ xOz \text{ and } yOz) \Rightarrow xCy$.

Moreover, we can call *point represented by p* the class $P = \{x \in R/x \geq z$ for a suitable $z \in p\}$. At least in the euclidean spaces, this definition of point seems to work well. Indeed, it is matter of routine to prove that, for every $p \in S$, the class of open convex regular subsets whose closure contains p satisfies (i)′, (ii)′, and (iii)′ and therefore represents a point P. With respect to this definition of point, it is easy to see that A5 is satisfied.

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