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New Semantics for the Extensional but Hyper-intensional Part $\mathcal{L}_\alpha$ of the Modal Sense Language $\mathcal{SL}_\alpha^\nu$

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Abstract In a previous paper ("On the interpreted sense calculus $\mathcal{SC}_\alpha^\nu$") the author constructed an interpreted modal sense language $\mathcal{SL}_\alpha^\nu$, in which a certain logical calculus is valid, in order to deal with, e.g., iterated belief sentences whose sense orders are smaller than the (possibly transfinite) ordinal α. It contains descriptions, modal operators, nonlogical operators, and wfe's having both types (of all finite levels) and (arbitrary) sense orders $<\alpha$.

In the semantics of $\mathcal{SL}_\alpha^\nu$ properties are represented by sets of QS's (quasi-senses), and paradoxes are avoided by considering any belief relation B^β sensitive only up to the sense order β $(0 < \beta < \alpha)$. $\mathcal{SL}_\alpha^\nu$ differs from the languages considered by Church, Parsons, and Quine in that, for example, the notions of *possible world* or *the sense* (QS) *of* . . . need not be primitive in $\mathcal{SL}_\alpha^\nu$.

The present work concentrates on the extensional (but hyper-intensional) part $\mathcal{L}_\alpha$ of $\mathcal{SL}_\alpha^\nu$ deprived of nonlogical operators. By two successive changes in $\mathcal{L}_\alpha$'s semantics (and ontology) the interpreted extensional sense languages $\mathcal{L}_\alpha'$ and $\mathcal{L}_\alpha''$ respectively arise. In these the hyper-intensionality axiom $f = g \equiv (\forall x_1, \ldots, x_n).f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)(\Vdash r = q \equiv . \, r \equiv q)$ [the instantiation axiom $(\forall x)F(x) \, . \supset F(\Delta)$ (x free for Δ in $F(x)$)] is valid for more and more [for more] general choices of the sense orders for the wfe's f, g, and x_1 to x_n [x, Δ, and $F(\Delta)$]. In $\mathcal{L}_\alpha''$ these choices are the most general ones for which, according to the present point of view, it is convenient to render these axioms valid.

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N1 Introduction This paper concerns the theory of sense logic developed in Bonotto and Bressan [4] on synonymy for extensional languages, in its extension (Bonotto [3]) to the modal language ML^{ν} presented in Bressan [7], in the memoir of Bonotto and Bressan [5] on generalized synonymy notions and quasi-senses for such a modal language, and especially in Bressan [10] where a general interpreted sense language $\mathcal{SL}_{\alpha}^{\nu}$ is introduced in order to treat, e.g., iterated belief sentences, whose sense orders may be transfinite but smaller than the ordinal α. Furthermore $\mathcal{SL}_{\alpha}^{\nu}$ contains, among other things, descriptions, modal operators, nonlogical operators, and wfe's (well-formed expressions) having both types of all finite levels, and sense orders represented by all ordinals $<\alpha$. Incidentally, $\mathcal{SL}_{\alpha}^{\nu}$ thus embodies all generalized versions of the modal language ML^{ν} based on type systems—see e.g., Bressan [9] and Bressan and Zanardo [14].

Note that, unlike what generally happens with sense languages whose wfe's have sense orders, the well-formedness of every wfe of $\mathcal{SL}_{\alpha}^{\nu}$ is invariant under the replacement of any wfe occurring in it with any other wfe that has the same type but any sense order. This feature of $\mathcal{SL}_{\alpha}^{\nu}$ may be very useful in the construction of an orderless version of $\mathcal{SL}_{\alpha}^{\nu}$, possibly based on the same type system τ_{ν} as $\mathcal{SL}_{\alpha}^{\nu}$.

The semantics for $\mathcal{SL}_{\alpha}^{\nu}$ has, so to speak, a set theoretical character; e.g., a property is represented in it by a set of QS's (quasi senses), i.e., objects representing senses. In connection with this, roughly speaking, the semantics for $\mathcal{SL}_{\alpha}^{\nu}$ (various versions of which are being constructed) is based on the relations between, e.g., hyper-intensional attributes and the senses (or QS's) of the entities for which they hold; and thus this semantics provides an insight into those relations, and in particular it gives a precise picture of a way in which senses can be constructed.

Furthermore, in order to avoid paradoxes in any possible situation, $\mathcal{SL}_{\alpha}^{\nu}$'s semantics complies with, so to speak, the segmentation of languages developed by Tarski to avoid truth paradoxes. In connection with this, every belief relation B^{β} to be considered in $\mathcal{SL}_{\alpha}^{\nu}$ is sensitive only up to a certain fixed sense order β $(0 < \beta < \alpha)$. Through its segments, $\mathcal{SL}_{\alpha}^{\nu}$ affords a general example of how a natural language can be completed and consistently interpreted, as far as everyday or scientific talk involving belief sentences is concerned. Furthermore, using B^{β} $(0 < \alpha < \beta)$ instead of a single belief relation B has the advantage that the use of each B^{β} is clear and presents no problems, whereas problems arise when B is iterated.

The differences between $\mathcal{SL}_{\alpha}^{\nu}$ and some other hyper-intensional languages are briefly discussed in Bressan [10]. Here let us only emphasize that the semantics proposed for $\mathcal{SL}_{\alpha}^{\nu}$ directly in [10], or indirectly here (by considering a certain extensional but hyper-intensional part $\mathcal{L}_{\alpha}$ of $\mathcal{SL}_{\alpha}^{\nu}$), aims at providing a contribution to Fregean semantics, and in particular hopes to perform a complete formalization of the parts of natural languages being treated; in fact no semantical notion such as *possible world* or *the sense of* . . . needs to be primitive in $\mathcal{L}_{\alpha}$ or $\mathcal{SL}_{\alpha}^{\nu}$. In this respect the semantics for $\mathcal{SL}_{\alpha}^{\nu}$ differs from that in, e.g., Church's and Parson's works (see [17], [18] and [21]). To some extent the interpreted languages in these papers stand to $\mathcal{SL}_{\alpha}^{\nu}$ as Bressan's (unusual) extensional language, proposed (and employed) in [6], stands to Bressan's modal language ML^{ν} constructed in [7]. It can be said that the former language also has some advantages, e.g., of a didactical nature, due to its simplicity. However, the latter attains much more general and deeper results.[1]

In Bressan [11] an axiom system valid in $\mathcal{SL}^{\nu}_{\alpha}$ is presented, for which no completeness claim is made. Let me add that, in my opinion, $\mathcal{SL}^{\nu}_{\alpha}$ has strong expression powers, yet certain axioms valid in ordinary extensional calculi and in, e.g., the modal calculus MC^{ν} set up in [7] hold in $\mathcal{SL}^{\nu}_{\alpha}$ only in case the sense orders of certain wfe's occurring in them are equal or satisfy certain simple relations. In some cases this is natural and even mandatory; in others these restrictions (justifiable by the ontology underlying $\mathcal{SL}^{\nu}_{\alpha}$'s semantics) seem to be avoidable by means of suitable changes in this semantics (and ontology). Therefore I have considered the possibility of such changes, in order to improve the general theory presented in [10] and [11].

Since the aforementioned problem concerns (sense) orders, in order to concentrate on it better it is convenient to consider the extensional part of $\mathcal{SL}^{\nu}_{\alpha}$ deprived of nonlogical operators. Let us call $\mathcal{L}_{\alpha}$ the sense language thus obtained.

The main aim of this paper is to present two successive changes in $\mathcal{L}_{\alpha}$'s semantics, which correspond to certain changes in $\mathcal{L}_{\alpha}$'s ontology (see Sections N11 and N12). In the interpreted extensional sense languages $\mathcal{L}'_{\alpha}$ and $\mathcal{L}''_{\alpha}$ that thus (successively) arise, the hyper-intensionality axiom is logically valid in more and more general forms. Briefly, under customary assumptions this axiom (scheme) reads[2]

$$(1.1)\quad f = g \equiv (\forall x_1, \dots, x_n).f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$$
$$(\Vdash p = q \equiv . \; p \equiv q),$$

where f and g are functors or relators of orders not greater than that of x_r $(r = 1, \dots, n)$.

In more detail, both the $\supset$-part of (1.1) and its converse are valid in $\mathcal{L}_{\alpha}$ only in case f and g have the same sense order (see N10). This also holds for $\mathcal{L}'_{\alpha}$, as far as the above converse is concerned (see N11); but the $\supset$-part of (1.1) is valid in $\mathcal{L}'_{\alpha}$, no matter which orders $(<\alpha)$ f and g have (see N11). In addition, the whole wff (well-formed formula) (1.1) is valid in $\mathcal{L}''_{\alpha}$ in this general way (see N13).

Furthermore, *the instantiation axiom*

$$(1.2)\quad (\forall x)F(x) \;.\!\supset F(\Delta), \text{ where } \Delta \text{ is free for } x \text{ in } F(x),$$

is valid in $\mathcal{L}'_{\alpha}$ and $\mathcal{L}''_{\alpha}$ whenever the order of x is larger than that of Δ, and in other cases too (see N14). But the validity of (1.2) in $\mathcal{SL}^{\nu}_{\alpha}$ (which induces the validity in $\mathcal{L}_{\alpha}$) holds only under much more restrictive conditions (see A4.4 in [11]).

A brief synopsis of the present work is as follows. First, the semantics of $\mathcal{L}_{\alpha}$ is presented and discussed; then a few changes on it that turn $\mathcal{L}_{\alpha}$ into $\mathcal{L}'_{\alpha}$ are considered and discussed; $\mathcal{L}''_{\alpha}$ is also introduced in a similar way. This is done for two reasons. First, the semantics of $\mathcal{L}_{\alpha}$, $\mathcal{L}'_{\alpha}$, and $\mathcal{L}''_{\alpha}$ have increasing complexity and the discussion of them allows us to justify the use of $\mathcal{L}''_{\alpha}$. Second, the way chosen to introduce $\mathcal{L}''_{\alpha}$ – viz. by defining the semantics of $\mathcal{L}'_{\alpha}$ ($\mathcal{L}''_{\alpha}$) in terms of the one for $\mathcal{L}_{\alpha}$ ($\mathcal{L}'_{\alpha}$) – seems to me technically efficient and certainly convenient, at least for the first time one considers extensional or modal languages such as these.

We now describe the content of the present work in more detail. In N2 the subject studied here is technically specified. In N3 the formation rules for the extensional sense language $\mathcal{L}_\alpha$ are given. In N4–N9 the semantics for $\mathcal{L}_\alpha$ is presented gradually, starting with some general preliminaries (in N4). Among other things, in N4 some reasons are given for the identification of the extension **F** of the false proposition with the nonexisting object, which is regarded as denoted by the descriptions that don't satisfy the condition of exact uniqueness.

The λ-th segment $\mathcal{L}_\lambda$ of $\mathcal{L}_\alpha$, formed with $\mathcal{L}_\alpha$'s wfe's of orders $<\lambda$ is considered $(0 < \lambda < \alpha)$, $\mathcal{L}_1$ is treated as an ordinary (nonintensional) extensional language, and a QE (quasi-extension) as well as a QS (quasi-sense) is assigned to every wfe of $\mathcal{L}_1$, in N5 and N6.

More generally, in this work every wfe Δ of $\mathcal{L}_\lambda$ is regarded as having an HQE (hyper-QE or order-endowed QE) and a (hyper-) QS (at any c-valuation $\mathcal{G}$ and v-valuation $\mathcal{V}$). More precisely, if Δ is a wfe$^\beta$ (wfe of order β), then Δ has a QE$^\beta$ and a QS$^\beta$ (a QE and a QS of orders $\leq\beta$), which represent Δ's extension and (hyper-) sense, respectively $(\beta < \lambda \leq \alpha)$.

Rules (h_{1-8}) and (ϵ_{1-8}) assign Δ one HQE, $\bar{\Delta} = \mathrm{des}_{\mathcal{G}\mathcal{V}}(\Delta)$, and one QS, $\breve{\Delta} = \mathrm{sens}_{\mathcal{G}\mathcal{V}}(\Delta)$, at any $\mathcal{G}$ and $\mathcal{V}$ for $\mathcal{L}_\lambda$. This is done for $\lambda = 1$ in N5 and N6 (as was said), for $\lambda = 2$ in N7 and N8, and for any λ $(0 < \lambda \leq \alpha)$ in N9. These technical presentations are preceded by some intuitive considerations on the senses of $\mathcal{L}_1$, in N6, and about the general semantics for $\mathcal{L}_\lambda$, especially in the case $\lambda = 2$, in N7. Incidentally, it is explicitly shown that for every type t and ordinal $\beta < \alpha$, we can construct a (new) HQE of type t and order β, different from all HQE's of orders $<\beta$, if and only if t is a function or an attribute-type (see (A) and (B) in N7).

In N10 some examples show the aforementioned limited validity in $\mathcal{L}_\alpha$ of both parts of (1.1). It is concluded that in $\mathcal{L}_\alpha$'s ontology, so to speak, (i) an attribute or function φ is determined by its set theoretical part and (ii) φ's predication or application depends on the order of the wfe that denotes φ.

In N11 it is seen how φ can determine its own rule of predication or application. On the basis of this ontological change—which is somehow similar to the one connected with the notions of function used by analysts and topologists—the semantics for $\mathcal{L}'_\alpha$ is briefly set up and discussed. In N12, by a second ontological and semantic change, $\mathcal{L}'_\alpha$ is turned into a language $\mathcal{L}''_\alpha$ in which a general version of (1.1) is valid; furthermore, this ontological change is compared with a present situation in mathematics connected with real or complex numbers. The aforementioned general version of (1.1) is proved in N13 to be valid, and the analogue for (1.2) is done in N14, together with some discussions. In addition, Theorem 13.2 states a maximality property of the semantics for $\mathcal{L}''_\alpha$ by which, so to speak, proper *general* interpretations are excluded.

N2 On how "sense" is meant in this work In the assertion "Charles is a dormouse" of ordinary language, "dormouse" is used in a figurative sense. In its proper sense "dormouse" expresses a kind of animal. One can therefore speak of an ambiguity in the use of "dormouse"; in different contexts it denotes different entities, and according to logicians it has different extensions. Therefore one might say that "dormouse" has various *ambiguity-senses* or *extensional-senses*, in contraposition with *hyper-intensional senses* (see below).

The theory of senses considered here is not concerned with ambiguities, but with, e.g., the (hyper-intensional) senses of expressions (such as "3", "$\lg_2 8$", and "6/2") that denote the same object. This theory deals with situations of the following type, where "$a \asymp b$" is to be read as "a and b are synonymous":

(2.1.1) $\square[3 = \lg_2 8 \text{ (or } 6/2)]$,
(2.1.2) $\sim[3 \asymp \lg_2 8 \text{ (or } 6/2)]$.

Now, in spite of (2.1.1), by (2.1.2) it may occur that

(a) *Pete knows that* $5 = 2 + 3$, but
(b) *Pete does not know that* $5 = 2 + \lg_2 8$ (or $2 + 6/2$).

Hence, as is well known, the substitution of expressions of necessarily identical entities in some assertions changes the truth values of these, i.e., their extensions. Such contexts are connected with belief sentences, i.e., with sentences such as $\mathrm{B}(\mathfrak{M},p)$ and $\mathrm{K}(\mathfrak{M},p)$, where we write

(2.2) $\mathrm{B}(\mathfrak{M},p)$ for "the man $\mathfrak{M}$ believes that p (holds)"

and

(2.3) $\mathrm{K}(\mathfrak{M},p)$ for "$\mathfrak{M}$ knows that p (holds)".

Incidentally, remember that there is a large variety of hyper-intensional notions which can be used in ordinary or scientific speech, and to which the present theory applies (see N7), and that one often accepts the definition[3]

(2.4) $\mathrm{K}(\mathfrak{M},p) \equiv_{def} \mathrm{B}(\mathfrak{M},p) \wedge p.$

Unlike (2.2)–(2.3) the intuitive belief sentences $\alpha_2, \alpha_3, \ldots$ below are iterated

(2.5) $\alpha_0 \equiv_{def} 2 + \lg_2 8 = 5$ or $\alpha_0 \equiv_{def} 2 + 6/2 = 5$,
(2.6) $\alpha_n \equiv_{def} \mathrm{B}(\mathfrak{M}_n, \alpha_{n-1})$ for $n \in \mathbb{N}_*$ $(= \mathbb{N} - \{0\})$.

More precisely, let us say that α_n has sense order n $(n \in \mathbb{N})$ and iteration order $n - 1$ $(n \in \mathbb{N}_*)$.

Due to the nested characters of the above sentences, from a rigorous point of view it appears incorrect to use the same belief predicate for all of them. Thus it appears convenient to replace (2.6) with

(2.7) $\alpha_n \equiv_{def} \mathrm{B}^n(\mathfrak{M}_n, \alpha_{n-1})$ for $n \in \mathbb{N}_*$,

where $\mathrm{B}^n(\mathfrak{M},p)$ means that $\mathfrak{M}$ *believes that* p *(holds), at a sense order* $\leq n$. Of course, for $n \in \mathbb{N}_*$, we can use the same B^n in all the sentences

(2.8) $\mathrm{B}^n(\mathfrak{M}, \alpha_r)$ for $r = 0, \ldots, n - 1$ (see (γ) and the considerations following it in N7).

N3 Formation rules for the extensional analogue $\mathfrak{L}_\alpha$ of the interpreted modal sense language $\mathrm{S}\mathfrak{L}^\nu_\alpha$ deprived of nonlogical operators Both $\mathrm{S}\mathfrak{L}^\nu_\alpha$ and $\mathfrak{L}_\alpha$ are based on the *individual types* 1 to ν, the *propositional type* 0, and more generally on the type system τ_ν defined as the least set τ_ν for which the following hold:

(i) $\{0,\ldots,\nu\} \subseteq \tau_\nu$,
(ii) If $t_0,\ldots,t_n \in \tau_\nu$, then the $(n+1)$-tuple $\langle t_1,\ldots,t_n,t_0\rangle \in \tau_\nu$.

For $t_0 \neq 0$, $\langle t_1,\ldots,t_n,t_0\rangle$ is used as a function type, viz. the type of the functions that carry objects of the respective types t_1 to t_n into objects of type t_0. For $t_0 = 0$ the same $(n+1)$-tuple is used as a relation type, viz. the one for relations holding for n-tuples of objects of the respective types t_1 to t_n.

In order to conform with Carnap's notations, I set

$$(3.1) \quad (t_1,\ldots,t_n) = \langle t_1,\ldots,t_n,0\rangle; \ (t_1,\ldots,t_n;t_0) = \langle t_1,\ldots,t_n,t_0\rangle \text{ for } t_0 \neq 0.$$

The symbols of $\mathcal{L}_\alpha$ are the *variables* $v^\beta_{t,n}$ and *constants* $c^\beta_{t,\mu}$ of *order* β, *type* t, and *index* n or μ respectively, for $0 \le \beta < \alpha$, $t \in \tau_\nu$, $n \in \mathbb{N}_*$, and $0 < \mu < \beta + \omega$, where ω is the first transfinite ordinal;[4] furthermore left and right parentheses, the identity sign $=$, the connectives $\sim$ and $\supset$, the all sign $\forall$, the symbol $\imath$ for descriptions, and λ for λ-expressions. This last symbol, denoted by λ^P in e.g. Bressan [10], has to be regarded as primitive. (In [10] λ is used for λ-expressions defined in terms of $\imath$; here λ^D can be used for this metalinguistic purpose.)

The class E_t of the wfe's of $\mathcal{L}_\alpha$ having the type t is defined recursively for $t \in \tau_\nu$ by the rules (φ_{1-8}) below, where $0 \le \beta < \alpha$, $n \in \mathbb{N}_*$, and $t, t_0,\ldots,t_n \in \tau_\nu$; the recursion concerns the lengths of wfe's.

(φ_1) $c^\beta_{t\mu}$, $v^\beta_{tn} \in \mathrm{E}_t$ $(0 < \mu < \beta + \omega)$.
(φ_2) If $\Delta_i \in \mathrm{E}_{t_i}$ $(i = 1,\ldots,n)$ and $\Delta_0 \in \mathrm{E}_{\langle t_1,\ldots,t_n,t_0\rangle}$, then $\Delta_0(\Delta_1,\ldots,\Delta_n) \in \mathrm{E}_{t_0}$.
(φ_3) If $\Delta_1,\Delta_2 \in \mathrm{E}_{t_1}$, then $\Delta_1 = \Delta_2 \in \mathrm{E}_0$.
(φ_{4-7}) If $p,q \in \mathrm{E}_0$, then $\sim p$, $p \supset q$, $(\forall v^\beta_{t,n})\, p \in \mathrm{E}_0$ and $(\imath v^\beta_{t,n})\, p \in \mathrm{E}_t$.
(φ_8) If $\Delta_0 \in \mathrm{E}_{t_0}$ and x_1 to x_n are n (distinct) variables with $x_i \in \mathrm{E}_{t_i}$ $(i = 1,\ldots,n)$, then $(\lambda x_1,\ldots,x_n)\Delta_0 \in \mathrm{E}_{\langle t_1,\ldots,t_n,t_0\rangle}$.

Definition 3.1 I say that β is the *order* of the wfe Δ, $\beta = \Delta^{\mathrm{ord}}$, if β is the maximum of the orders of the variables and constants occurring in Δ.

For every class $\mathfrak{A}^\beta$ depending on the ordinal β, I set

$$(3.2) \quad \mathfrak{A}^{<\beta} = \bigcup_{\delta<\beta} \mathfrak{A}^\delta, \ \mathfrak{A}^{\le\beta} = \bigcup_{\delta\le\beta} \mathfrak{A}^\delta, \ \mathfrak{A}^{\beta\not<} = \mathfrak{A}^\beta - \mathfrak{A}^{<\beta}.$$

For $t \in \tau_\nu$ I also set

$$(3.3) \quad \mathrm{E}^\beta_t = \{\Delta \in \mathrm{E}_t \mid \Delta^{\mathrm{ord}} \le \beta\}; \ \mathrm{wfe}^\beta = \bigcup_{t\in\tau_\nu} \mathrm{E}^{\beta\not<}_t \ (\beta < \alpha),$$

so that the wfe^β's are the wfe's of order β $(\beta < \alpha)$.

Convention 3.1 By $x,y,z,x_1,\ldots$, by $p,q,r,r_1,\ldots$, and by $\Delta,\Delta_1,\ldots$ arbitrary variables, wff's (well-formed formulas), and wfe's of $\mathcal{L}_\alpha$ will be denoted respectively; $x^\beta, \ldots, p^\beta, \ldots$, and $\Delta^\beta, \Delta_1^\beta, \ldots$ will denote any wfe$^\beta$'s of the respective kinds above.

Convention 3.2 Every expression of $\mathcal{L}_\alpha$ used in the sequel is assumed to be well-formed, except where otherwise noted.

One can consider a theory $\mathfrak{J}$ *belonging* to $\mathcal{L}_\alpha$. From the semiotic point of view, i.e., as far as formation rules are concerned, this means that: (i) $\mathfrak{J}$'s symbols are those of $\mathcal{L}_\alpha$ except for some constants (perhaps none or all) and possibly some variables, e.g. those whose orders are outside some set Σ of ordinals ($<\alpha$), and (ii) $\mathfrak{J}$'s wfe's are those of $\mathcal{L}_\alpha$ formed with $\mathfrak{J}$'s symbols or even only a certain subset of these. I say that $\mathfrak{J}$ is *based* on $\mathcal{L}_\alpha$ if (i) as well as the first alternative in (ii) hold. Thus a theory based on $\mathcal{L}_\alpha$ belongs to $\mathcal{L}_\alpha$ and may coincide with $\mathcal{L}_\alpha$.

N4 On the nonexisting object a^ν and the truth value* F *in the semantics for $\mathcal{S}\mathcal{L}_\alpha^\nu$ or $\mathcal{L}_\alpha$ to $\mathcal{L}_\alpha^{\shortparallel}$. On the translation in $\mathcal{L}_\alpha$ of some general locutions. On Bealer's work [1]. Hints at a typeless version of $\mathcal{L}_\alpha$. A comparison with projective geometry In extensional logical theories including descriptions, such as Rosser's (see [23]), all descriptions that fail to fulfill their conditions of exact uniqueness can be proved to coincide. Then, following Frege, it is natural to consider a common denotatum for them: the *nonexisting object*. This entity has for objects the analogue of the role played by zero for numbers or by the empty set for sets. Thus, (i) *this object appears to be unique.*

Consider now the man m^* (or horse h^*) that jumps (or can jump) over 100 meters, formally

(4.1) $m^* =_{def} (\imath x).x \in \mathrm{M} \wedge j(x)$, $h^* =_{def} (\imath x).x \in \mathrm{H} \wedge j(x)$ ($j(x) \equiv_{def} x$ jumps over 100 meters).

Sometimes the property M^* (H^*) of being a nonexisting man (horse) is used in assertions such as $m^* \in \mathrm{M}^*$, i.e. m^* is an M^*, and $h^* \notin \mathrm{M}^*$ ($h^* \in \mathrm{H}^*$ and $m^* \notin \mathrm{H}^*$). In conformity with (i) and in the light of hyper-intensional languages such as $\mathcal{S}\mathcal{L}_\alpha$, it is natural (ii) *to regard predicates such as* M^* *and* H^* *as hyper-intensional.*

In the semantics for typed logical theories (see, e.g., [7] where ML^ν is presented) it is technically convenient to introduce (a representative of) the *nonexisting object $a^\nu = a_t^\nu$ of type t.* However, it is not necessary for a_t^ν to depend on t. Since the objects of type t (t') can be men (horses), the example above forces us to assume that $a_t^\nu = a_{t'}^\nu$ (but this is not strictly essential by the representative character of a_t^ν).

Recalling the extension of ML^ν to the typeless modal language ML^∞ (see [8]),

(E) *the semantics for the languages $\mathcal{S}\mathcal{L}_\alpha^\nu$ and $\mathcal{L}_\alpha$ to $\mathcal{L}_\alpha^{\shortparallel}$ are also viewed here as steps towards the constructions of typeless (and possibly orderless or modal) analogues of them.*

So, in spite of its above representative character, a_t^ν *has to be chosen independent of t.*

As well as a modal language considered in [9], $\mathcal{SL}_\alpha^\nu$ is also expected to deal with probability functions (as in [13]), whose domains involve events, hence propositions. Therefore (and by criteria of uniformity and generality) $\mathcal{SL}_\alpha^\nu$ is based on the type system τ_ν, symmetric with respect to the types 0 and 1 to ν; and the same holds for $\mathcal{L}_\alpha$ to $\mathcal{L}_\alpha^{\text{iv}}$.

In the semantics for $\mathcal{SL}_\nu^\alpha$, or e.g. $\mathcal{L}_\alpha$ to $\mathcal{L}_\alpha^{\text{iv}}$, the identities

(4.2.1) $\mathbf{F} = a^\nu \qquad (t \in \tau_\nu)$,
(4.2.2) $a^\nu = a_t^\nu \qquad (t \in \tau_\nu)$

hold. The second of them has been justified above. In connection with (4.2.1) let us first consider

Case C1 For uniformity reasons, e.g., one is willing to use any description $y =_{def} (\iota x)p$ where x is a propositional variable.

For the sake of simplicity, also assume that $x, y \in \mathrm{E}_0^0$ (see (3.3)). As a consequence, if for $i = 1$ to 7 p is the ith of the wff's

(4.3) $x,\ x \vee.\ q \wedge \sim q,\ x = q,\ x \equiv.\ q \vee \sim q,\ x =.\ q \wedge \sim q,\ r \wedge (x = r)\ .\vee.\ \sim r \wedge (x = \sim r),\ r \wedge (x = \sim r)\ .\vee.\ \sim r \wedge (x = r)$,

then no problem arises, since the condition of exact uniqueness is satisfied. Moreover, y's QE, $\bar{y}$, coincides with the one of the ith of the wff's

(4.4) $x,\ x,\ q,\ q \vee \sim q,\ q \wedge \sim q,\ q \vee \sim q,\ q \wedge \sim q$.

However, if p is, e.g., any of the first (last) two wff's

(4.5) $x \wedge \sim x,\ x \wedge q \wedge \sim q;\ x \vee \sim x,\ x \vee q \vee \sim q$,

then the existence (uniqueness) condition for the description y fails to hold, so that $\bar{y} = a_0^\nu$. Therefore, if a_0^ν is required not to be a truth value, then one is in effect concerned with at least a three-valued propositional logic. In spite of some interesting features of this logic, for reasons of simplicity it is important not to be compelled to use it. Hence it is better to assume that $a_0^\nu \in \{\mathbf{T},\mathbf{F}\}$, where $\mathbf{T}$ is the truth value true; yet at this time several people are inclined to choose the alternative that $a_0^\nu = \mathbf{F}$ (see the second part of this section).

Conclusion C1 In the case C_1 the compliance with two-valued propositional logic forces us to choose $a_0^\nu = \mathbf{F}$ and hence, whenever we prefer a_t^ν not to depend on t, to assume (4.2).

Case C2 One prefers not to use descriptions of type zero. In this case, complementary to C_1, the identities of (4.2) are still very convenient when the extension plan (E) is taken into account.

In fact, in a typeless version $\mathcal{L}_\alpha^\infty$ of $\mathcal{L}_\alpha$ any variable x ranges over individuals, sets (i.e., attributes), and propositions. Incidentally, this is in accordance with, e.g., the possibility of George's liking objects, properties, and facts. The aforementioned range of the variables of $\mathcal{L}_\alpha^\infty$ practically forces us to regard a_t^ν

as independent of t and as defined for $t = 0$, i.e. to accept (4.2.2). Then the acceptance of (4.2.1) is practically required, for the same reasons as were considered in case C1.[5]

The above motivations for accepting (4.2) may be sufficient for some people. However, some other people may require more detailed and technical reasons for preferring the alternative $a_0^\nu = \mathbf{F}$ to $a_0^\nu = \mathbf{T}$. Therefore the considerations below are added.

First note that, following Frege's views, in $\mathcal{S}\mathcal{L}_\alpha^\nu$ or any typed language the nonexisting object of type t ($\in \tau_\nu$) can be defined (metalinguistically, since a_t^* is not a constant of $\mathcal{S}\mathcal{L}_\alpha^\nu$) by

$$(4.6)\quad a_t^* =_{def} (\imath v_{t1}^0) v_{t1}^0 \neq v_{t1}^0 \qquad (t \in \tau_\nu)$$

and for every $\Delta \in \mathrm{E}_t$, $\Delta = a_t^*$ ($\Delta \neq a_t^*$) is equivalent to: Δ *fails to exist* (Δ *exists*). In particular, $\Delta = a_t^*$ holds for every description Δ that fails to meet its conditions of exact uniqueness.

Note that, e.g., Carnap regards **T** and **F** as the extensions of propositions, so that there are only two propositions. Furthermore, Russell seems to think that there is no difference between facts and true propositions—as Jon Barwise asserts on the basis of what is said on p. 223 in his work "Situations, facts, and true propositions," in preparation.

Russell's view can certainly be shared in connection with a propositional logic $\mathcal{P}\mathcal{L}_1$ that includes descriptions with propositional variables and is endowed with a two-valued interpretation. In it, for every proposition p, we can regard the description

$$(4.7)\quad y =_{def} (\imath x)(x = p \,.\!\wedge p) \qquad (p = q. \equiv p = q)$$

as denoting the fact that p (holds). The requirements (α) to (β) below must be satisfied, as Russell's view above suggests.

(α) the proposition p is false iff the fact y that p holds fails to exist, or
(α') the assertion 'p' is false iff the term y is nondenoting; and
(β) if p is true, then $y = p$ is also true.

Let us now show that *requirements* (α) *and* (β) *imply the validity of* (4.2.1) *for* $\mathcal{P}\mathcal{L}_1$.

Indeed, letting $\bar{p}$ ($\bar{y}$) be p's (y's) QE, we can render (i) $\bar{p} = \mathbf{T}$ true. Then, by (β), (ii) $\bar{y} = \bar{p}$; and by (α), $\bar{y} \neq a_0^\nu$. Hence $\mathbf{T} \neq a_0^\nu$ by (i) to (ii). Furthermore $a_0^\nu \in \{\mathbf{T},\mathbf{F}\}$ by $\mathcal{P}\mathcal{L}_1$'s definition. Hence $a_0^\nu = \mathbf{F}$.

The above proof of the identity (4.2.1) for $\mathcal{P}\mathcal{L}_1$ renders it natural to regard it as holding in connection with $\mathcal{L}_\alpha$ too.

It is also natural to try to deduce (4.2.1) for $\mathcal{L}_\alpha$ directly; furthermore, it is worthwhile to consider this problem explicitly, because this allows us to speak briefly of other interesting topics, such as certain views about properties—e.g. Bealer's. Therefore we now, only for the sake of brevity, regard English as supplemented with variables—how to avoid them being obvious—and we set

(4.8.1) $p \equiv_{def}$ George was born in Rome,
(4.8.2) $y =_{def}$ the fact that p,
(4.8.3) $z =_{def}$ the proposition (that) p.

Then, even if p is false, z certainly exists, unlike y. Therefore, remembering that propositions (and facts) are generally said to be very many, it is natural to identify z with p's sense. Hence, according to general criteria used, e.g., in [7] in connection with ML^{ν}, z can be defined to be the hyper-intensional singleton $\{p\}^h$ of p (see (4.9) below).[6] Of course, here a certain definition of *the proposition that* . . . is presupposed, of the same kind as Dedekind's definition of real numbers; in fact it identifies that intuitive notion with a certain set theoretical structure (which can be, and sometimes is replaced with, other such structures).

Note that the sense of p, or better $\{p\}^h$, is used above as an explicatum (in Carnap's sense) for the explicandum z. Therefore it is not strange that it may appear different from the intuitive notion of the proposition that p.

In connection with a second explicandum for z to be considered below (4.10), let us remember that the explicatum for a given explicandum $\mathcal{E}$ may have several (possible) versions. Among these, roughly speaking, the most useful and simple has to be chosen (see [16], p. 7). Of course this choice may depend on the situation and the purposes for which $\mathcal{E}$ is used. In particular (see [16], p. 26), within mechanics *linear momentum* and *kinetic energy* are both important explicata for the intuitive notion of *living force*.

The above considerations also show that when a field is studied (scientifically), then the senses in which a term related with it is used can change; and their number can increase. This is a way in which ambiguities may arise.

Note that the above first explicatum $\{p]^h$ for z is also different from p. Correspondingly, in accordance with Russell's views above, we can identify y with $\{p\}^h$ in case p holds; and we can use (α) otherwise.

The preceding considerations can be rendered more precise by: (i) using $\mathcal{L}_{\alpha}$, (ii) considering p (see (4.8.1)) as a wfe of $\mathcal{L}_{\alpha}$, and (iii) regarding p, y, and z as metalinguistic variables, in contrast with their use above. Then by (4.8.1) we can turn (4.8.2)–(4.8.3) into (4.9.1)–(4.9.2), respectively:

(4.9.1) $y =_{def} (\imath x)\,[x = [p]^h \wedge p]$,
(4.9.2) $z =_{def} \{p\}^h(\{\Delta\}^h =_{def} \{x | x \between \Delta\})$,

where

(4.10) $\Delta \between \Delta_1 \equiv_{def} (\forall G).\ G(\Delta) = G(\Delta_1) \qquad (G^{\mathrm{ord}} = 1 + \max\{\Delta^{\mathrm{ord}}, \Delta_1^{\mathrm{ord}}\})$.

Thus y and z are hyper-intensional properties of type (0) (instead of propositions) and the proof of (4.2.1) for $\mathcal{PL}$, cannot be repeated for $\mathcal{L}_{\alpha}$.

The analogue for properties of the above explicatum is useful in the following considerations. Within plane geometry let σ be the segment of end points A and $B(\neq A)$ and set $F(x) \equiv_{def} x$ belongs to σ's axis (which is the perpendicular to σ through σ's middle point) and $G(x) \equiv_{def} x$'s distances from A and B coincide. Then, by a well-known theorem, $F = G$ (and even $\Box(F = G)$). However, as Bealer in effect emphasizes, (generally) one says that

(γ) the properties F and G do not coincide.

Therefore, in connection with the explicata above, (γ) can be translated in $\mathcal{L}_\alpha$ (or $\mathcal{SL}_\alpha^\nu$) by $\{F\}^h \neq \{G\}^h$. Likewise, the translation in, e.g., $\mathcal{L}_\alpha$ of the assertion

(δ) F_1 to F_n are n (distinct) properties

can be

(4.11) $\{F_1,\ldots,F_n\}^{(h)} \in n$, where $\{F_1,\ldots,F_n\}^{(h)} =_{def} \{F_1\}^h \cup \ldots \cup \{F_n\}^h$.

Let us add that within scientific contexts it is natural to consider

(ϵ) the probability of the event $\mathcal{E}$ in the situation Σ ("$\mathcal{E}$","Σ" $\in \mathrm{E}_0$)

as an intensional function $\mathrm{pr}(\Sigma,\mathcal{E})$ of $\mathcal{E}$ (see [9], [13]). On the other hand, to construct for (ϵ) a modal analogue of the explicatum (4.11) for the intuitive notion (γ) – in which explicatum "$\ldots \in n$" is extensional – we can consider the extensional function Pr determined by the condition

(4.12) $\mathrm{pr}(\Sigma,\mathcal{E}) = \mathrm{Pr}(\Sigma,\{\mathcal{E}\}^i)$ $\qquad (\{\Delta\}^i =_{def} \{x \mid \Box(x = \Delta)\})$

and can use $\mathrm{Pr}(\Sigma,\{\mathcal{E}\}^i)$, instead of $\mathrm{pr}(\Sigma,\mathcal{E})$, as an explicatum for the intuitive notion (ϵ).

Note that, conversely, (4.12) can be used as a definition of pr in terms of Pr. This suggests that we consider the hyper-intensional analogue of (this converse) pr for every relation $R(\ldots,\sigma_1,\ldots,\sigma_n)$ that is extensional in the propositional or relational arguments σ_1 to σ_n:

(4.13) $R^{(h)}(\ldots,\xi_1,\ldots,\xi_n) \equiv_{def} R(\ldots,\{\xi_1\}^h,\ldots,\{\xi_n\}^h)$.

Then all assertions on $\{\xi_1\}^h$ to $\{\xi_n\}^h$ can be turned into assertions (directly) on ξ_1 to ξ_n. Thus the following explicata appear useful for the notions y, z, and $R(\ldots,\sigma_1,\ldots,\sigma_n)$ (see (4.8.2)–(4.8.3) and (4.13)):

(4.14.1) $y =_{def} (\imath p)(x = p \mathbin{.\wedge} p)$,
(4.14.2) $z =_{def} p$,
(4.14.3) $R^{(h)}$ or something equivalent.

For instance, we can translate (γ) (in $\mathcal{L}_\alpha$) into

(4.15) $\sim(F \mathrel{\Join} G)$ (see (4.10)).

That is, instead of = we can use synonymy; and (δ) can be translated directly into the wff (4.11), regarded as a condition on F_1 to F_n. Note that the use of the first explicata does not compel us to change the relation R involved in (δ) into $R^{(h)}$.

By (4.14.1), $y \in \mathrm{E}_0$. *Hence the above proof of* (4.2.1) *for* $\mathcal{PL}$, *also works for* $\mathcal{L}_\alpha$ *(or* $\mathcal{SL}_\alpha^\nu$*)*.

Conditions (4.2) concern a relation between assertions and terms, such as (4.8.2), that can be considered to be of type 0 according to the second way given above of formalizing, e.g., (γ). Hence it is natural to suggest the possibility of extending the considerations above to terms such as 'George's birth in Rome', which in some situations are equivalent and perhaps synonymous with terms such as (4.8.2) – remember Carnap's principle of tolerance.[7]

Bealer accords great importance to assertions such as (γ); he likes to translate, e.g., (γ) in formal languages with as few formal changes as possible as

$F = G$ (see [1] and (4.15)). Thus identity must in effect work, in his formal languages, the way synonymy does in $\mathcal{L}_\alpha$, unlike what happens for $\mathcal{L}_\alpha$, $\mathcal{SL}_\alpha^\nu$, ML$^\nu$, and most intensional languages including Carnap's (see e.g. [15]).

I strongly prefer the latter languages for the following reasons:

(i) there is no need of introducing (contingent) identity as an additional (possibly nonlogical) primitive;
(ii) the power of set theory is available directly (e.g., $F = \varnothing \equiv (\forall G)\, F \cap G = F$ is true), and
(iii) number theory is treatable at least as well as in [1], Chapter 6 (where = is used in effect as a synonymy relation).

If I have understood [1] sufficiently, an aim attained there (in Chapter 6) is that by a certain hyper-intensional explicatum NN (for *natural number)* all of Peano's axioms follow from laws of logic. Hence they are logically valid, including the 4th axiom, which is often regarded as a law on nature; and e.g. for the individuals of type 1 in a given interpretation of $\mathcal{L}_\alpha$ (or $\mathcal{SL}_\alpha^\nu$ or ML$^\nu$) this axiom is true iff the set D_1 of the above individuals is at least denumerable.

The above aim of [1] is also attained in, e.g., the set theory presented in Monk [20] which, as well as Bealer's logic, is typeless (and orderless).

Unlike Bealer I regard sets as properties, and hence set theory as a part of logic. Therefore I am not very interested in the above aim in itself—i.e., as considered in typeless logical languages.

However, Chapter 6 in [1] also has a positive value from my point of view, in that it suggests the considerations below on the natural numbers for the above individuals, in case $(\mathfrak{F})$ *D_1 is finite and $\nu = 1$*:

Assume $(\mathfrak{F})$. Then, since for $\alpha > 1$ $\mathcal{L}_\alpha$ has hyper-intensional properties, the properties of type (1) are denumerably many. Hence, briefly speaking, if we define the set $\mathbb{N}_{(1)}$ of natural numbers for objects of type (1) in the usual way, $\mathbb{N}_{(1)}$ renders all of Peano's axioms logically valid. Therefore, in particular the 4th axiom can be rendered a theorem in some suitable logical calculus valid in $\mathcal{L}_\alpha$. Even if it is left as an axiom, it is no longer an axiom of nature, i.e. on D_1.

Roughly speaking, if $n \in \mathbb{N}_{(t_1)}$ and $t_1 \in \tau_\nu$, then by "the set F of type (t_1) has n elements" or "the F's are n" it is natural to mean that there is a one-to-one correspondence between F and the above numbers $\leq n$—or, in case $t_1 = 1$, that $\{\{x\} | x \in F\} \in n$ if preferred.

In case $(\mathfrak{F})$, the ordinary extensional language $\mathcal{L}_1$ has only finitely many objects of type t for all $t \in \tau_\nu$. Hence no analogue $\mathbb{N}_t$ of the above explicatum $\mathbb{N}_{(1)}$ for the set of natural numbers exists for $\mathcal{L}_1$ (unlike what happens in a typeless theory of sets based on D_1).

In my opinion the above difference between Bealer's view and mine about sets is strongly connected with this: the primitive logical relation of identity used by Bealer [me] is in effect hyper-intensional identity, i.e. synonymy [contingent identity (which in connection with, e.g., attributes is meant, roughly speaking, as the contingent validity for the same hyper-intensional entities)].

I prefer to use the latter identity as a logical primitive, because it is simpler than the former; and in fact it is widely used. Furthermore, in modal logic $\Delta \bowtie \Delta'$ implies $\Box(\Delta = \Delta')$, i.e. transworld identity; and this identity is widely unknown in even some important cases. In addition the (natural) synonymy relation is not uniquely determined, as it already appears from Carnap [15]; moreover it is widely subjective.

Above we made some remarks concerning C2, where the extension (E) (below (4.1)) is taken into account; among them, identities (4.2) were in effect shown to hold, because $\mathcal{L}_\alpha^\infty$'s variables range over individuals, sets, and propositions. Now it is conversely shown that (4.2.1)–(4.2.2) practically allow us to give this range to those variables: there is a device compatible with (4.2)—see (D) below—that enables us to interpret, e.g., $\Delta \wedge p$ when Δ denotes something different from truth values.

(D) Roughly speaking, interpret $C_n(\Delta_1, \ldots, \Delta_n)$, where C_n is an n-ary connective, as follows. C_n holds for the n propositions: Δ_i is a true proposition $(i = 1, \ldots, n)$. Furthermore, interpret (possibly nonlogical) quantifiers similarly.[8]

The interpretation above is equivalent to the ordinary one, when it is used for the (ordinary) LPC (lower predicate calculus); and it also applies to the *extended* LPC, i.e., to the expressions obtained from LPC's wff's by replacing some wff's in them with terms. Furthermore it is easy to check that, if (only) $\sim$, $\supset$, and $(\forall x)$ for every variable x are used as primitive connectives or quantifiers and condition (D) is satisfied (see Note 8), then (D) holds for all connectives and logical operators of the extended LPC. In addition, all theorems and metatheorems of the ordinary LPC hold for the extended LPC.

In the contexts referred to above a term Δ has the role of a false assertion, and this complies perfectly with condition (4.2) when Δ denotes a^ν.

The use of (4.2) can induce, so to speak, the identification of the truth value false with the nonexisting object; and this may appear strange. However remember that, similarly, for certain technical reasons of uniformity and simplicity, in projective geometry (non-oriented) directions are said to be *points*; hence they are regarded as similar to ordinary points and one could say that, together with these, those directions constitute the *projective points* (instead of *the points*). Thus the new (actual) way of speaking—besides the one of thinking—contrasts with the old one.

To exhibit an analogue of *projective point* for the identifications (4.2), note that: (i) any truth value can naturally be regarded as a property (of propositions), (ii) following Frege and Russell the same holds for zero, and (iii) likewise the nonexisting object could be meant as the property of being nonexistent. Hence the use of (4.2) practically amounts to regarding **F** as

(η) [(η')] *the property of being either a false assertion (false proposition), or a nondenoting term (the nonexisting object),*

the disjunction being exclusive, if terms of type 0 are excluded, and inclusive in the general case, which was shown to be compatible with (4.2).

Thus the use of (4.2) does not require us to identify (or confuse) notions, viz. **F** with (η) or (η'); it only induces an ambiguity in the use of '**F**', similar to using 'point' sometimes in its ordinary sense and sometimes for 'projective point'.

Likewise, for certain technical reasons, ostensive QS's for $\mathcal{L}_\alpha$ ($\mathcal{S}\mathcal{L}_\alpha^\nu$) are identified with QE's (quasi-intensions) (see N6 and N5 in [10]). This may induce the identification of ostensive senses with extensions (intensions). But in fact the above identification at most induces us to use 'sense' in a special way, e.g. as 'technical sense' (see the end of N2 in [12]).

N5 Some preliminaries for the semantics of $\mathcal{L}_\alpha$ and designation rules for its ordinary extensional segment Let $\mathcal{L}_\alpha$ be given; and for $0 < \lambda \leq \alpha$ regard $\mathcal{L}_\lambda$ as the λ-th segment of $\mathcal{L}_\alpha$, i.e. the part of $\mathcal{L}_\alpha$ formed with wfe's of orders $<\lambda$. The least among these segments, $\mathcal{L}_1$, has to be regarded as an ordinary extensional (interpreted) language.

Remembering (4.2), it is natural to regard as *proper individual domains* for $\mathcal{L}_\alpha$ any sets $\mathfrak{D}_1$ to $\mathfrak{D}_\nu$ such that

(5.1) $\mathbf{F} \notin \mathfrak{D}_s$, $\mathfrak{D}_s \neq \varnothing \qquad (s = 1, \ldots, \nu) \quad$ ($\mathbf{F}$ is elementless).

Furthermore, let us set

(5.2) $\mathfrak{D}_0 = \{\mathbf{T}\}$, $D_j = \mathfrak{D}_j \cup \{\mathbf{F}\} \qquad (j = 0, \ldots, \nu)$.

Hence D_0 is the set of truth values and D_i is the set of the *possibly nonexisting individuals of type i* $(i = 1, \ldots, \nu)$.

The domain (counterdomain) of any function f is denoted by $\mathfrak{D}_f(\mathcal{C}\mathfrak{D}_f)$; and for any classes A and B we set

$$(5.3) \quad \begin{cases} A \to B \\ A \mapsto B \end{cases} =_{def} \{f \mid f \text{ is a function, } \mathfrak{D}_f \begin{cases} = A \\ \subseteq A \end{cases}, \mathcal{C}\mathfrak{D}_f \subseteq B\},$$

so that $A \mapsto B$ can be called the set of *partial functions* from A to B.

One wants to define gradually the class QE_t^β of the HQE's (hyper-quasi-extensions or ordered QE's) of (sense) orders $\leq\beta$ and type t, as well as the class QS_t^β of QS's of orders $\leq\beta$ and type t, for $0 < \beta < \alpha$ and $t \in \tau_\nu$. This will be done by means of a multiple definition based on a transfinite induction on $\beta(<\alpha)$ and recursion on $t(\in\tau_\nu)$, which also determines the HQE and QS designated by any wfe Δ, provided contextual definitions be absent (QS-designation rules for theories endowed with a definition system and based on a generalized (modal) version of ML^ν are dealt with in sections 17–20 of [5], p. 200).

For $\beta = 0$ we first determine QE_t^0 $(t \in \tau_\nu)$ recursively, in a rather usual extensional way:

(5.4) $\mathrm{QE}_t^0 = D_t \qquad (t = 0, \ldots, \nu)$

and

(5.5) $\mathrm{QE}^0_{\langle t_1, \ldots, t_n, t_0\rangle} = (\mathrm{QE}^0_{t_1} \times \ldots \times \mathrm{QE}^0_{t_n} \mapsto \mathrm{Q}\mathcal{E}_{t_0}) \cup \{\mathbf{F}\}$,

where $\mathrm{Q}\mathcal{E}_t^\beta$ is the class of *proper* QE_t^β's:

(5.6) $\mathrm{Q}\mathcal{E}_t^\beta = \mathrm{QE}_t^\beta - \{\mathbf{F}\}$, hence $\mathrm{QE}_t^\beta = \mathrm{Q}\mathcal{E}_t^\beta \cup \{\mathbf{F}\} \qquad (t \in \tau_\nu, \beta < \alpha)$.

Definitions (5.4)–(5.6) have been written so as to be helpful in stating the designation rules (h_{1-8}) for HQE's and those (ϵ_{1-8}) for QS's. First (h_{1-8}) are given for $\beta = 0$ (see below) while the same with (ϵ_{1-8}) is done in N6. This allows us to define QS_t^β for $\beta = 0$ and $t \in \tau_\nu$. In N7 QE_t^β is defined for $\beta = 1$ and $t \in \tau_\nu$, while the same for QS_t^β is done in N8. The process continues for any $\beta < \alpha$.

In spite of the ordinary extensional character of $\mathcal{L}_1$, the rules (h_{1-8}) are written explicitly for it in order to show, in a simple case, how **F** and definition (5.3) work in the semantics under consideration. For similar reasons it is convenient to say in advance that, e.g., the designata assigned by those rules to a_t^* (see (4.6)), $(\lambda x)\, x \neq x$, and $(\lambda x).x = a_1 \vee x = a_2 \vee x = a_3$ are **F**, $\varnothing$, and the constant function g with $\mathfrak{D}_g = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ and $\mathfrak{C}\mathfrak{D}_g = \{\mathbf{T}\}$ respectively, where $\bar{a}_i$ is the designatum of $a_i (i = 1,2,3)$. More generally, the QE designated by any relator will be a constant function g with $\mathfrak{C}\mathfrak{D}_g = \{\mathbf{T}\}$, so that g is determined by $\mathfrak{D}_g$.

For $0 < \lambda \leq \alpha$ let V^λ (I^λ) denote the set of the v-valuations (c-valuations) for $\mathcal{L}_\lambda$, to be determined below; in more detail, $\mathcal{V} \in V^\lambda$ $(\mathcal{I} \in I^\lambda)$ iff $\mathcal{V}$ $(\mathcal{I})$ is a function defined only on the variables (constants) of $\mathcal{L}_\lambda$, for which the first (second) of the relations

(5.7) $\mathcal{V}(v_{tn}^\beta) \in A_t^\beta$, $\mathcal{I}(c_{t\mu}^\beta) \in A_t^\beta$ with $A_t^0 = \mathrm{QE}_t^0$ $(t \in \tau_v,\ \beta < \lambda,\ n \in \mathbb{N}_*,\ 0 < \mu < \beta + \omega)$

holds, A_t^β being the set of entities assignable to the variables of order β $(<\alpha)$ and type t (see $(5.7)_3$ and (7.6)).

By (5.7) V^0 and I^0 are completely determined. Since the definition (7.6) of A_t^β for $\beta > 0$ differs relevantly from its analogue $(5.7)_3$ for $\beta = 0$, $\mathcal{L}_2$ appears to differ from $\mathcal{L}_1$ more than $\mathcal{L}_{\lambda+1}$ differs from $\mathcal{L}_\lambda$ $(1 < \lambda < \alpha)$. Furthermore, if λ is a limit ordinal (i.e., $\lambda = \cup\lambda$, according to Monk [20]) and $\mathcal{L}_\delta$ is known for $0 < \delta < \lambda$, then $\mathcal{L}_\lambda$ also is in effect known. In fact its HQE's (QS's) of type t form the class $\mathrm{QE}_t^{<\lambda}$ $(\mathrm{QS}_t^{<\lambda})$ $(t \in \tau_\nu)$.

It is useful to consider the c-valuations $\mathcal{I}_*^\lambda \in I^\lambda$ and $\mathcal{I}_*$ for which

(5.8) $\mathcal{I}_*^\lambda(c_{t\mu}^\beta) = \mathbf{F}$ $\quad (\beta < \lambda \leq \alpha,\ t \in \tau_\nu,\ 0 < \mu < \beta + \omega)$; $\qquad \mathcal{I}_* =_{def} \mathcal{I}_*^\alpha$.

For every wfe Δ of $\mathcal{L}_\lambda$ we want to define the *(hyper)-quasi-extensional designatum* $\mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ at any $\mathcal{I} \in I^\lambda$ and $\mathcal{V} \in V^\lambda$ $(0 < \lambda \leq \alpha)$. This definition must imply that for some t

(5.9) $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta) \Rightarrow \bar{\Delta} \in \mathrm{QE}_t^{<\lambda}$ $(\Delta \in \mathrm{E}_t^{<\lambda},\ \mathcal{I} \in \mathrm{I}^\lambda,\ \mathcal{V} \in \mathrm{V}^\lambda)$.

Hence, for $\lambda = 1$, (5.5) shows that, if Δ is any relational or functional expression $\Delta_0(\Delta_1, \ldots, \Delta_n)$, then $\bar{\Delta}_0 = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ is either a function, or (in case it fails to exist) **F**; and the same holds for $\lambda > 1$ (see (7.7)). Therefore it will appear useful to state, besides definition (5.3), the following convention:

(5.10) $$f(\xi_1, \ldots, \xi_n)^\dagger = \begin{cases} f(\xi_1, \ldots, \xi_n), \text{ if } f \text{ is a function and } \langle \xi_1, \ldots, \xi_n \rangle \in \mathfrak{D}f; \\ \mathbf{F}, \text{ otherwise.} \end{cases}$$

Furthermore, as is customary, in case $0 < \lambda \leq \alpha$ and $\mathcal{V}, \mathcal{V}' \in V^\lambda$ we write

(5.11) $$\mathcal{V}' = \mathcal{V}\begin{pmatrix} x_1 \ldots x_m \\ \xi_1 \ldots \xi_m \end{pmatrix} \text{ iff } \mathcal{V}'(x) = \begin{cases} \xi_i, \text{ for } x = x_i \ (i = 1, \ldots, m); \\ \mathcal{V}(x), \text{ otherwise.} \end{cases}$$

The rules (h_{1-8}) below define (5.9) for $\lambda = 1$ recursively, under assumptions (i), (ii), and (5.12)–(5.14) below.

(i) $n = m + 1$ and $\Delta_j \in \mathrm{E}_{t_j}^0$ with $t_j \in \tau_\nu$ $(j = 0, \ldots, n)$.
(ii) x_1 to x_m are m $(=n - 1)$ variables in $\mathrm{E}_{t_1}^0$ to $\mathrm{E}_{t_m}^0$ respectively (see (3.3)).

(5.12) $\bar{\Delta}_j = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta_j)$ $(j = 0, \ldots, n)$.

(5.13) $\bar{\Delta}'_n = \mathrm{des}_{\mathcal{I}\mathcal{V}'}(\Delta_n)$ where $\mathcal{V}' = \mathcal{V}\begin{pmatrix} x_1 \ldots x_m \\ \xi_1 \ldots \xi_m \end{pmatrix}$ $(n = m + 1)$.

(5.14) $f = \{\langle \xi_1, \ldots, \xi_n \rangle | \xi_i \in \mathrm{QE}^{\beta}_{t_i}$ $(i = 1, \ldots, m)$, $\mathbf{F} \neq \xi_n = \bar{\Delta}'_n\}$ (see (5.13)).

Rule	If Δ is	then $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ is
(h_1)	v^0_{tn} $(c^0_{t\mu})$,	$\mathcal{V}(v^0_{tn})$ $(\mathcal{I}(c^0_{t\mu}))$.
(h_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$\bar{\Delta}_0(\bar{\Delta}_1, \ldots, \bar{\Delta}_n)^{\dagger}$ (see (5.10)).
(h_3)	$\Delta_1 = \Delta_2$ $(t_1 = t_2)$,	**T** if $\bar{\Delta}_1 = \bar{\Delta}_2$; **F** otherwise.
(h_4)	$\sim\Delta_1$ $(t_1 = 0)$,	**F** (**T**) if $\bar{\Delta}_1$ is **T** (**F**).
(h_5)	$\Delta_1 \supset \Delta_2$ $(t_1 = t_2 = 0)$,	**T** if $\bar{\Delta}_1$ is **F** or $\bar{\Delta}_2$ is **T**; **F** otherwise.
(h_6)	$(\forall x_1)\Delta_2$ $(t_2 = 0)$,	**T** if $\bar{\Delta}'_2 = \mathbf{T}$ for all $\xi_1 \in \mathrm{QE}^0_{t_1}$ (see (5.13) for $n = 2$); **F** otherwise.
(h_7)	$(\imath x_1)\Delta_2$ $(t_2 = 0)$,	ξ_1, if ξ_1 is the unique element of $\mathrm{QE}^0_{t_1}$ such that $\bar{\Delta}'_2 = \mathbf{T}$ (see (5.13) for $n = 2$); **F** if such a unique element fails to exist.
(h_8)	$(\lambda x_1, \ldots, x_m)\Delta_n$,	f (see (5.13)-(5.14)).

By induction on the length of Δ_n we can prove that, for $\beta = 0$:

(5.15) $f \in \mathrm{Q}\mathcal{E}^{\beta}_{\langle t_1, \ldots, t_m, t_n \rangle}$ (see (5.14) and (5.5)-(5.6)).

Incidentally, every f satisfying (5.15) can have the form (5.14). In fact, for some choice of Δ_n and $\mathcal{V}$, Δ_n has the form $F(x_1, \ldots, x_m)$ where F is a variable, and $\mathcal{V}(F)$ is (or, for $\beta > 0$, has the QE) f.

N6 On the senses and* QS*'s of the wfe's of $\mathcal{L}_1$. Rules for assigning* QS*'s Now we can assign a sense, or better a QS, $\breve{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$, to any wfe of $\mathcal{L}_\lambda$, at any $\mathcal{I} \in I^\lambda$ and $\mathcal{V} \in V^\lambda$, for $\lambda = 1$.

The simplest among the senses of the (possible) expressions for a given object ξ is *ostensive*, i.e. to be grasped by direct inspection (in a wide sense) of something in the world (see [5], p. 184). In this way all extensions (including those for assertions) are put in a canonical one-to-one correspondence with ostensive senses. Therefore it is possible for us to conform with the following usage, which much simplifies our semantics:

(I) to identify the QS representing any ostensive sense s with the QE that represents the extension corresponding to s.

This simplifies our semantics—see, e.g., rule (h_1) below (8.5) ((ϵ_1) below (8.6))—by allowing us not to consider the aforementioned one-to-one correspondence.

Let us now consider an interpreted theory $\mathfrak{I}$ in which '1', '2', ... are all regarded as primitive terms (in contrast to the usual definitions $2 = 1 + 1$, $3 = 2 + 1, \ldots$). Then the sense of '3' can be identified with 3 and is regarded as differing from those of '6/2' and '$\lg_2 8$'. These last senses are obviously more complex.

It is rather natural to represent the senses of '3', '6/2' or 'quot(6,2)', and '3 = 6/2' or '=(3,6/2)' by the following objects (QS's) respectively:

(6.1) $3, \langle /, 6, 3 \rangle, \langle =, 3, \langle /, 6, 3 \rangle \rangle$.

Regarding $\mathcal{L}_\lambda$ as a theory belonging to $\mathcal{L}_\alpha$ all of whose constants are primitive, we want to determine the QS $\breve{\Delta} = \text{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ of any wfe Δ of $\mathcal{L}_\lambda$ at any $\mathcal{I} \in I^\lambda$ and any $\mathcal{V} \in V^\lambda$ $(0 < \lambda \leq \alpha)$. The rules (ϵ_{1-8}) of QS-designation used below to this end are similar to and simpler than their analogues for $\mathcal{S}\mathcal{L}^\nu_\alpha$.

In this section they are given only for $\lambda = 1$, assuming conditions (i) and (ii) in N5 and definitions (6.2)–(6.3) below.

(6.2.1) $\breve{\Delta}'_n = \text{sens}_{\mathcal{I}\mathcal{V}'}(\Delta_n)$ for $\mathcal{V}' = \mathcal{V}\begin{pmatrix} x_1 \ldots x_m \\ \xi_1 \ldots \xi_m \end{pmatrix}$,

(6.2.2) $\breve{\Delta}_j = \text{sens}_{\mathcal{I}\mathcal{V}}(\Delta_j)\,(j = 0, \ldots, n)$

and (compare with (5.14))

(6.3) $g =_{def} \{\langle \xi_1, \ldots, \xi_m, \xi_n \rangle \mid \xi_i \in \text{QE}_{t_i}\ (i = 1, \ldots, m),\ \xi_n = \breve{\Delta}'_n\}$ (see (6.2.1)).

Rule	If Δ is	then $\breve{\Delta} = \text{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ is
(ϵ_1)	$v^0_{tn}\ (c^0_{t\mu})$,	$\mathcal{V}(v^0_{tn})\ (\mathcal{I}(c^0_{t\mu}))$.
(ϵ_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$\langle 0, \breve{\Delta}_0, \breve{\Delta}_1, \ldots, \breve{\Delta}_n \rangle$.
(ϵ_3)	$\Delta_1 = \Delta_2\ (t_1 = t_2)$,	$\langle =, \breve{\Delta}_1, \breve{\Delta}_2 \rangle$.
(ϵ_{4-5})	$\sim\Delta_1$ or $\Delta_1 \supset \Delta_2\ (t_1 = t_2 = 0)$,	$\langle \sim, \breve{\Delta}_1 \rangle$ or $\langle \supset, \breve{\Delta}_1, \breve{\Delta}_2 \rangle$ respectively.
(ϵ_{6-7})	$(\forall x_1)\Delta_2$ or $(\imath x_1)\Delta_2 (t_2 = 0)$,	$\langle \forall, g \rangle$ or $\langle \imath, g \rangle$ respectively (see (6.2)–(6.3) for $n = 2$).
(ϵ_8)	$(\lambda x_1, \ldots, x_m)\Delta_n$,	$\langle \lambda, g \rangle$ (see (6.3)).

The occurrence of 0 in $\breve{\Delta}$ for every Δ treated by (ϵ_2) will appear reasonable and natural only when $\mathcal{L}_1$ is effectively considered as a segment of $\mathcal{L}_\lambda$ $(1 < \lambda \leq \alpha)$.

For $t \in \tau_\nu$ the QS^0_t's, i.e. the QS's of order zero and type t, can be defined as follows (see (5.8)):

(6.4) $\text{QS}^0_t = \{\text{sens}_{\mathcal{I}^1_*\mathcal{V}}(\Delta) \mid \Delta \in \text{E}^0_t,\ \mathcal{V} \in V^1, \text{ and } \Delta \text{ is constant-free}\}$.

The analogue for $\text{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ of condition (5.9) on $\text{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ can be easily checked for $\lambda = 1$ (and any $\mathcal{I} \in I^1$). Furthermore, the following theorem can now be proved for $\lambda = 1$.

Theorem 6.1, λ *If Δ and Δ' are (constant-free) wfe's of $\mathcal{L}_\lambda$ while $\mathcal{I}, \mathcal{I}' \in I^\lambda$ and $\mathcal{V}, \mathcal{V}' \in V^\lambda$, then*

(6.5) $\text{sens}_{\mathcal{I}\mathcal{V}}(\Delta) = \text{sens}_{\mathcal{I}'\mathcal{V}'}(\Delta') \Rightarrow \text{des}_{\mathcal{I}\mathcal{V}}(\Delta) = \text{des}_{\mathcal{I}'\mathcal{V}'}(\Delta')$.[9]

Hence for $\beta < \lambda = 1$, every $\sigma \in \text{QS}^\beta$ determines *its* QE, say σ^{E}.

Definition 6.1, λ For $\sigma \in \text{QS}^{<\lambda}$ we say that ξ is its QE (and denote it by σ^{E}) in case $\xi = \text{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ and $\sigma = \text{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$, for some (constant-free) wfe$^{<\lambda}$ Δ and some $\mathcal{V} \in V^\lambda$ (and $\mathcal{I} \in I^\lambda$).

Definitions 6.2 (6.3) If δ is the least ordinal for which $\xi \in \text{QE}^\delta$ $(\sigma \in \text{QS}^\delta)$, then we say that δ is the *order* of the QE ξ (QS σ) briefly $\delta = \xi^{\text{ord}}$ $(\delta = \sigma^{\text{ord}})$.

Note that Definitions (6.2)–(6.3) do not conflict with one another in spite of the QE's being special QS's, because if $\sigma \in \text{QE}(\subset\text{QS})$ then the order of σ as a QS equals its order as a QE.

N7 Intuitive considerations on the wfe's of $\mathcal{L}_2$. Examples of effectively hyperintensional functions and relations of arbitrary types. On belief sentences, and hints at the semantics of $\mathcal{L}_\lambda$ $(2 < \lambda \leq \alpha)$ Now we can evaluate the wfe's of $\mathcal{L}_2$. Their orders are 0 and 1. For instance, we can represent the belief relation expressed by B^1 (see (2.7)) with a QE of order 1, say $\xi = \overline{B^1}$ as follows. Since that relation exists, ξ is a constant function of range $\mathcal{C}\mathfrak{D}_\xi = \{T\}$; and $\mathfrak{D}_\xi$ can be characterized as follows:

(7.1) $\mathfrak{D}_{\overline{B^1}} = \{\langle\rho,\sigma\rangle | \rho^E$ is a person η and σ is the QS $\breve{p}$ of an assertion p of $\mathcal{L}_1$ believed by $\eta\}$.

Then $B^1(\mathfrak{M}, p)$ is true iff $\langle\breve{\mathfrak{M}}, \breve{p}\rangle \in \mathfrak{D}_{\overline{B^1}}$. Assuming persons to be objects of type 1, we have thus constructed $\overline{B^1}$ as an object of type $(1,0) = \langle 1,0,0\rangle$. This object is new, i.e. outside the semantics of $\mathcal{L}_1$, and it is natural to regard it as of order 1. Let us now show that

(A) *for any $t \in \tau_\nu$ a new object (or* QE*) of type t can be constructed iff $t \notin \{0, \ldots, \nu\}$; however*
(B) *new senses or* QS's *of any type t can be constructed.*

In fact, for $t = (t_1, \ldots, t_n) \in \tau_\nu$ consider $R \in E^1_{(t_1,\ldots,t_n)}$ and n variables x_1 to x_n of the respective types t_1 to t_n. Then, setting

(7.2) $S =_{def} (\lambda x_1, \ldots, x_n) B^1[\mathfrak{M}, R(x_1, \ldots, x_n)]$,

S expresses a new object (or QE) of type t (and order 1), for a suitable choice of $\mathfrak{M}$'s QS-designatum $\breve{\mathfrak{M}}$.

Now add the assumptions that: (i) x_0 to x_n are $n+1$ variables, (ii) $0 \neq t_0 \in \tau_\nu$, and (iii) $x_0, a, b \in E_{t_0}$. Then, setting

(7.3) $f =_{def} (\lambda x_1, \ldots, x_n)(\imath x_0).\ x_0 = a \wedge S(x_1, \ldots, x_n) \vee x_0 = b \wedge \sim S(x_1, \ldots, x_n)$,

f expresses a new object of type $(t_1, \ldots, t_n; t_0)$ for a suitable choice of $\breve{\mathfrak{M}}$.

Of course, the QS's of the wfe's $S, f, S(x_1, \ldots, x_n)$, and $f(x_1, \ldots, x_n)$ are also new. This occurs in particular for $t_0 \in \{1, \ldots, \nu\}$; but in this case the QE's of $S(x_1, \ldots, x_n)$ and $f(x_1, \ldots, x_n)$ are not new (in accordance with (A)). Likewise no new (proper) object can be denoted by any sentence of $\mathcal{L}_2$; however, $B^1(\mathfrak{M}, p)$ e.g. has a new propositional sense.

Thus (A) and (B) have been shown to hold for $\mathcal{L}_2$. Incidentally, they can be shown to hold for $\mathcal{L}_\lambda$ in a similar way (by considering B^n or B^β for $\beta < \lambda$) $(0 < \lambda \leq \alpha)$.

At this point it is natural to consider the directions (α) and (β) below for evaluating any functional or relational expression $\Delta_0(\Delta_1, \ldots, \Delta_n)$ of $\mathcal{L}_2$, of order 1.

(α) Use the QS $\breve{\Delta}_i$ of the argument Δ_i, if $(\breve{\Delta}_i)^{ord} < \Delta_0^{ord}$ – i.e. if $\breve{\Delta}_i$ belongs to $\mathcal{L}_1$'s semantics, while Δ_0 is outside $\mathcal{L}_1$ $(i = 1, \ldots, n)$.
(β) Use the HQE (hyper-quasi-extension) $\bar{\Delta}_i$ of Δ_i otherwise $(i = 1, \ldots, n)$.

In fact (α) is justified by the above considerations on belief sentences and by a uniformity criterion; e.g. by this criterion alone the QS ρ is used in (7.1), in spite of its QE $\xi = \rho^E$ being sufficient. Furthermore, (β) is practically imposed by (i) the ordinary (nonintensional) semantics of $\mathcal{L}_1$ (already constructed), in that e.g. the evaluation rules (h_{1-8}) in N5 do not involve QS's, and by (ii) similar features of $\mathcal{L}_{\beta+1}$ for $1 \leq \beta < \alpha$. (ii) refers to the fact that, for $1 \leq \beta < \alpha$, it is natural to construct $\mathcal{L}_{\beta+1}$ from $\mathcal{L}_\beta$ just as $\mathcal{L}_2$ is constructed from $\mathcal{L}_1$, so that many QE's not expressible in $\mathcal{L}_\delta (0 < \delta \leq \beta)$ have an expression Δ_1 in $\mathcal{L}_{\beta+1}$—and hence $\beta = \bar{\Delta}_1^{\mathrm{ord}} (\leq \breve{\Delta}_1^{\mathrm{ord}})$. Then it is natural to treat Δ_1 in $\mathcal{L}_{\beta+1}$ like designators are treated in $\mathcal{L}_1$, and hence to determine the QE of any wfe in $\mathcal{L}_{\beta+1}$, of the form $\Delta_0(\Delta_1)$, by using only Δ_1's QE (and disregarding $\breve{\Delta}_1$).

For instance, in case $\beta + 1 = \lambda = 2$, Δ_1 can be B^1 and Δ_0 can express the property of being an effectively hyper-intensional relation; furthermore, if Δ_1 is a complex expression equivalent to B^1 ($\bar{\Delta}_1 = \overline{B^1}$), then it is natural to require that, in $\mathcal{L}_\lambda$, $\Delta_0 (\Delta_1)$ should be equivalent to $\Delta_0 (B^1)$—which practically implies the above conclusion that (β) is justified.

Note that direction (α) has the alternative

(α') use $\breve{\Delta}_i$ whenever $(\breve{\Delta}_i)^{\mathrm{ord}} < (\bar{\Delta}_0)^{\mathrm{ord}}$ $(i = 1, \ldots, n)$.

It can be supported by the same arguments considered for (α). I prefer (α') to (α), because it is only according to (α') that the QE of $\Delta_0 (\Delta_1, \ldots, \Delta_n)$ depends on what Δ_1 to Δ_n express, disregarding the orders of these arguments, and this is very useful (i) to render a nice axiom system valid and (ii) to construct an orderless sense language.

Note that one may want the following condition on belief sentences to hold:

(γ) $B^m(\mathfrak{M}, p)$ is false when p expresses a proposition of sense order β with $m \leq \beta < \alpha$ (see the considerations following (2.6)).

This can be rendered compatible with directions (β) and either (α) or (α') by stipulating that $B^m(\mathfrak{M}, p)$ is false when p's QS, $\breve{p}$, is ostensive. Incidentally, this appears reasonable in that truth and falsity are generally regarded in everyday life as properties of sentences or propositions, but never as propositions.

However, for some special purposes, one might be interested in the possibility of rendering condition (γ) compatible with the condition

(δ) $B^m(\mathfrak{M}, p)$ is true (false) when $\breve{p}$ is **T** (**F**).

This requires B^m to be sensitive to senses of order m, which can be carried out only by identifying B^m with a constant of order $>m$.

Incidentally I decided to define the QS^β's after having stated the rules of QS-designation—see e.g. (6.4) or (8.7)—(and by using these), because only in connection with the first five rules among (ϵ_{1-8}) in N5 or N8 can some simple recursive clauses for a direct definition of QS_t^β ($t \in \tau_\nu$) be written (disregarding rules (ϵ_{1-8})):[10]

($\mathcal{E}_1$) $A_t^\beta \subseteq QS_t^\beta$.
($\mathcal{E}_2$) If $\sigma_j \in QS_{t_j}^\beta$ where $t_j \in \tau_\nu$ $(j = 0, \ldots, n)$, then $\langle \beta, \sigma_0, \ldots, \sigma_n \rangle \in QS_{t_0}^\beta$.
($\mathcal{E}_{3\text{-}4}$) If $\sigma_1, \sigma_2 \in QS_0^\beta$, then $\langle \sim, \sigma_1 \rangle$, $\langle \supset, \sigma_1, \sigma_2 \rangle \in QS_0^\beta$.
($\mathcal{E}_5$) If $\sigma_1, \sigma_2 \in QS_t^\beta$, then $\langle =, \sigma_1, \sigma_2 \rangle \in QS_0^\beta$.

Clauses (ϵ_{6-8}) in N6 (N8) involve the function g defined by (6.3) ((8.3.2)) using $\mathrm{sens}_{\mathfrak{I}\mathfrak{V}'}$. Due to the very special nature of this function, to define it independently of the rules for QS-designation (though certainly possible) appears very complex; therefore the chosen way of defining QS's seems to me strongly preferable, in connection with the aims of the present work.

As far as the choice of A_t^β for $0 < \beta < \alpha$ is concerned (see the considerations following (5.7)), note that it is useful to render the following wff true:

(7.4) $F = G \equiv (\forall x).F(x) = G(x)$ for $F, G \in \mathrm{E}^\beta_{(t)}$ and $x \in \mathrm{E}_t^{\beta \not<}$ $(t \in \tau_\nu)$,

at least when $F^{\mathrm{ord}} = G^{\mathrm{ord}}$. For $\beta = 0$ (7.4) is valid by rules (h_{1-8}) in N5. For $\beta_1 = 1$ $G(x)$ can be equivalent to $\mathrm{B}^1(\mathfrak{M}, x)$ (see (2.7)). Then it is obvious that we have to assign to x every $\sigma \in \mathrm{QS}_t^0$ (and hence every $\xi \in \mathrm{QE}_t^0$).

Of course, by (α) or (α') and (β) the new QE's of type t have also to be assignable to x in (7.4). At this point it is natural to accept for $\beta = 1$ the multiple recursive definition of QE_t^β, A_t^β, and $\mathrm{Q}\mathcal{E}_t^\beta$ $(t \in \tau_\nu)$ afforded by clauses (5.6) and (7.5)–(7.7).

(7.5) $\mathrm{QE}_t^\beta = D_t$ $\quad (t = 0, \ldots, \nu)$ (see (5.2)),
(7.6) $A_t^\beta = \mathrm{QE}_t^\beta \cup \mathrm{QS}_t^{<\beta}$ $\quad$ (see (3.2)) $(t \in \tau_\nu)$

and, for $t_0, \ldots, t_n \in \tau_\nu$,

(7.7) $\mathrm{QE}^\beta_{\langle t_1, \ldots, t_n, t_0 \rangle} = [A^\beta_{t_1} \times \ldots \times A^\beta_{t_n} \mapsto \mathrm{Q}\mathcal{E}^\beta_{t_0}] \cup \{\mathbf{F}\}$.

Note that, for $\beta = 0$, (7.6) implies $(5.7)_3$, so that (7.5)–(7.7) are equivalent to (5.4)–(5.5). Hence the definition above can be accepted for $\beta = 0$. It will be accepted also for every $\beta < \alpha$, because the considerations above based on (7.4), (α), and (β), can be repeated *mutantis mutandis* for every $\beta (< \alpha)$ that has the form $\delta + 1$; it suffices to consider B^β instead of B^1 (see N2 for $\delta < \omega$).

In case $\beta = \cup\beta$, e.g. $\beta = \omega$, we can also construct new QE's of relational and functional types, in spite of each QE used for this having already been used singularly to construct such a new QE of some order $\delta < \beta$.[11] Furthermore, the considerations based on (7.4), (α), and (β) can also easily be adapted to support the acceptance of (7.5)–(7.7) and (5.6) for every $\beta = \cup\beta < \alpha$.

***N8 Rules of* QE- *and* QS-*designation for* $\mathcal{L}_2$** Objects (5.6) and (7.5)–(7.7) are now determined for $\beta = 0, 1$; hence the set V^λ (I^λ) of λ-v-valuations (λ-c-valuations), i.e. v-(c-)valuations for $\mathcal{L}_\lambda$, is determined for $\lambda = 1,2$.

Now we want, first, to evaluate any wfe Δ of $\mathcal{L}_\lambda$, i.e. to determine $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ at any $\mathcal{I} \in I^\lambda$ and $\mathcal{V} \in V^\lambda$, for $\lambda = 2$; then the QS $\breve{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ will be determined for these Δ, $\mathcal{I}$, and $\mathcal{V}$. We do this, in the present section, according to the directions (α) and (β) in N7; and in N9 the present section will be extended from $\mathcal{L}_2$ to $\mathcal{L}_\lambda$ $(0 < \lambda \le \alpha)$. Then, in N11, the directions (α') and (β) in N7 will be considered (mainly for $\mathcal{L}_2$), and the resulting sets of rules will be compared with the rules (h_{1-8}) and (ϵ_{1-8}) in N9.

In accord with (α) and (β) let us consider conditions (i) to (iv) below, for $\lambda = 2$.

(i) Δ and Δ_0 to Δ_n are wfe's having the respective types t, t_0 to t_n, and the respective orders δ, δ_0 to δ_n; furthermore $t_0 = \langle t_1, \ldots, t_n, t \rangle$, $m + 1 = n \in \mathbb{N}_*$, and $0 < \mu < \lambda + \omega$.
(ii) x_1 to x_m are m variables and $x_i \in \mathrm{E}_{t_i}^{\delta_i \not<}$ (see (3.2)) $(i = 1, \ldots, m)$.

(iii) The orders β, δ, and δ_0 to δ_n are $<\lambda$ $(\leq\alpha)$ (hence $d_f \leq \delta_g < \lambda$, see (8.5) below), $\mathcal{I} \in I^\lambda$, and $\mathcal{V} \in V^\lambda$.
(iv) One uses the definitions

(8.1.1) $\bar{\Delta}_j = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta_j)$ $(j = 0, \ldots, n)$,

(8.1.2) $\bar{\Delta}'_n = \mathrm{des}_{\mathcal{I}\mathcal{V}'}(\Delta_n)$ with $\mathcal{V}' = \mathcal{V}\begin{pmatrix} x_1 \ldots x_m \\ \xi_1 \ldots \xi_m \end{pmatrix}$,

and

$$(8.2)\quad \hat{\Delta}_i = \begin{cases} \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta_i), \text{ if } (\bar{\Delta}_i)^{\mathrm{ord}} < (\Delta_0)^{\mathrm{ord}} \\ \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta_i), \text{ otherwise} \end{cases} \qquad (i = 1, \ldots, n)$$

and the first of the definitions

$$(8.3)\quad \begin{cases} f \\ g \end{cases} =_{def} \{\langle \xi_1, \ldots, \xi_n \rangle | \, \xi_n = \begin{cases} \mathrm{des}_{\mathcal{I}\mathcal{V}'}(\Delta_n) \neq \mathbf{F} \\ \mathrm{sens}_{\mathcal{I}\mathcal{V}'}(\Delta_n) \end{cases}, \ \xi_i \in A_{t_i}^{\delta_i} \ (i = 1, \ldots, n-1)\} \qquad \text{(see (5.11))}$$

which, as can be checked (for $\lambda = 2$) on the basis of rules (h_{1-8}) and (ϵ_{1-8}) below, imply that

$$(8.4)\quad \begin{cases} f \\ g \end{cases} \in (A_{t_1}^{\delta_1} \times \ldots \times A_{t_m}^{\delta_m} \mapsto \begin{cases} \mathrm{Q}\mathcal{E}_{t_n}^{d_f} \\ \mathrm{QS}_{t_n}^{\delta_g} \end{cases}), \text{ with } d_f \leq \delta_g < \lambda,$$

where[12]

$$(8.5)\quad \begin{cases} d_f \\ \delta_g \end{cases} =_{def} \sup\{ \begin{cases} f(\xi)^{\mathrm{ord}} \\ g(\xi)^{\mathrm{ord}} \end{cases} \Big| \, \xi \in \begin{cases} \mathcal{D}_f \\ A_{t_1}^{\delta_1} \times \ldots \times A_{t_m}^{\delta_m} \end{cases} \}, \ (\xi = \langle \xi_1, \ldots, \xi_m \rangle).$$

($\mathfrak{A}$) For $\lambda = 2$, $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ is defined recursively by rules (h_{1-8}) below, regarded to hold for all entities that satisfy conditions (i) to (iv):

Rule	If Δ is	then $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ is
(h_1)	v_{tn}^β $(c_{t\mu}^\beta)$,	σ^{E}, where $\sigma = \mathcal{V}(v_{tn}^\beta)$ $(\sigma = \mathcal{I}(c_{t\mu}^\beta))$, if $\sigma \in \mathrm{QS}_t^{<\beta}$; and σ otherwise (see Theorem 6.1λ, or (8.10) below).
(h_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$\bar{\Delta}_0$ $(\hat{\Delta}_1, \ldots, \hat{\Delta}_n)^\dagger$ (see (8.2) and (5.10)).
(h_3)	$\Delta_1 = \Delta_2$ $(t_1 = t_2)$,	**T**, if $\bar{\Delta}_1 = \bar{\Delta}_2$; **F** otherwise.
(h_4)	$\sim\Delta_1$ $(t_1 = 0)$,	**T** (**F**), if $\bar{\Delta}_1$ is **F** (**T**).
(h_5)	$\Delta_1 \supset \Delta_2$ $(t_1 = t_2 = 0)$,	**T**, if $\bar{\Delta}_1 = \mathbf{F}$ or $\bar{\Delta}_2 = \mathbf{T}$; **F** otherwise.
(h_6)	$(\forall x_1)\Delta_2$ $(t_2 = 0)$,	**T**, if $\bar{\Delta}'_2 = \mathbf{T}$ for all $\xi_1 \in A_{t_1}^{\delta_1}$ (see (8.1.2) for $n = 2$); **F** otherwise.
(h_7)	$(\iota x_1)\Delta_2$ $(t_2 = 0)$,	η if η is the unique element of $\mathrm{QE}_{t_1}^{\delta_1}$ such that, for some $\xi_1 \in A_{t_1}^{\delta_1}$, $\bar{\Delta}'_2 = \mathbf{T}$ (see (8.1.2) for $n = 2$) and either $\xi_1^{\mathrm{ord}} = \beta$ and $\eta = \xi_1$, or $\xi_1^{\mathrm{ord}} < \beta$ and $\eta = \xi_1^{\mathrm{E}}$ (see (8.10)); **F** otherwise.
(h_8)	$(\lambda x_1, \ldots, x_m)\Delta_n$,	f (see $(8.3)_1$).

($\mathcal{B}$) For $\lambda = 2$, $\breve{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ is defined recursively by rules (ϵ_{1-8}) below, regarded to hold for all entities that satisfy (i) to (iii), $(8.3)_2$, and

(8.6) $\breve{\Delta}_j = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta_j)\,(j = 0, \ldots, n)$, $\breve{\Delta}'_n = \mathrm{sens}_{\mathcal{I}\mathcal{V}'}(\Delta_n)$ with $\mathcal{V}' = \mathcal{V}\begin{pmatrix} x_1 \ldots x_m \\ \xi_1 \ldots \xi_m \end{pmatrix}$.

Rule	If Δ *is*	then $\breve{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ is
(ϵ_1)	v^{β}_{tn} $(c^{\beta}_{t\mu})$,	$\mathcal{V}(v^{\beta}_{tn})$ $(\mathcal{I}(c^{\beta}_{t\mu}))$.
(ϵ_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$\langle(\Delta_0)^{\mathrm{ord}}, \breve{\Delta}_0, \breve{\Delta}_1, \ldots, \breve{\Delta}_n\rangle$ (see Definition 3.1).
(ϵ_3)	$\Delta_1 = \Delta_2$ $(t_1 = t_2)$,	$\langle =, \breve{\Delta}_1, \breve{\Delta}_2\rangle$.
(ϵ_{4-7})	$\sim\Delta_2$, $\Delta_2 \supset \Delta_3$, $(\forall x_1)\Delta_2$, $(\imath x_1)\Delta_2$ $(t_2 = t_3 = 0)$,	$\langle\sim, \breve{\Delta}_2\rangle$, $\langle\supset, \breve{\Delta}_2, \breve{\Delta}_3\rangle$, $\langle\forall, g\rangle$, or $\langle\imath, g\rangle$ respectively, where $(8.3)_2$ holds for $n = 2$.
(ϵ_8)	$(\lambda x_1, \ldots, x_m)\Delta_n$,	$\langle\lambda, g\rangle$ (see $(8.3)_2$).

The class QS^{β}_t of the QS's of type t and orders $\leq\beta$ can now be defined for $\beta = 1$:

(8.7) $\mathrm{QS}^{\beta}_t =_{def} \{\mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta) \mid \mathcal{V} \in V^{\beta+1}, \mathcal{I} = \mathcal{I}^{\beta+1}_{*}, \Delta \in \mathrm{E}^{\beta}_t, \Delta \text{ is constant-free}\}$ (see (5.8)).

By the considerations adduced following (7.7), the above rules (h_{1-8}) and (ϵ_{1-8}) induce the corresponding rules in N5 and N6 for $\mathcal{L}_1$.

For $\lambda = 1,2$ it is not difficult to check the following:

Theorem 8.1 *Assume that* $0 < \delta < \lambda \leq \alpha$. *Then theses* (a) *to* (c) *below hold.*
(a) *The restrictions* $\mathcal{V}'^{\delta}$ $(\mathcal{I}'^{\delta})$ *of the valuations* $\mathcal{V} \in V^{\lambda}$ $(\mathcal{I} \in I^{\lambda})$ *to the variables (constants) of orders* $<\delta$ *are the* δ*-v-valuations (*δ*-c-valuations), i.e. they form the set* V^{δ} (I^{δ}).
(b) *If* Δ *is a* $\mathrm{wfe}^{<\delta}$, $\mathcal{V} \in V^{\lambda}$, $\mathcal{I} \in I^{\lambda}$, $\mathcal{V}' \in V^{\delta}$, $\mathcal{I}' \in I^{\delta}$, *and* $\mathcal{V}$ $(\mathcal{I})$ *agrees with* $\mathcal{V}'$ $(\mathcal{I}')$ *on the variables (constants) that occur in* Δ, *then*

(8.8) $\mathrm{des}_{\mathcal{I}'\mathcal{V}'}(\Delta) = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$, $\mathrm{sens}_{\mathcal{I}'\mathcal{V}'}(\Delta) = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$.

(c) *If* $\mathcal{V}_r \in \mathcal{V}^{\lambda}$, $\mathcal{I}_r \in I^{\lambda}$, *and* Δ_r *is a* $\mathrm{wfe}^{<\lambda}$ $(r = 1,2)$, *then*

(8.9) $\mathrm{sens}_{\mathcal{I}_1\mathcal{V}_1}(\Delta_1) = \mathrm{sens}_{\mathcal{I}_2\mathcal{V}_2}(\Delta_2) \Rightarrow \mathrm{des}_{\mathcal{I}_1\mathcal{V}_1}(\Delta_1) = \mathrm{des}_{\mathcal{I}_2\mathcal{V}_2}(\Delta_2)$.

Definition 8.1 A valuation $\mathcal{V} \in V^{\lambda}$ $(\mathcal{I} \in I^{\lambda})$ is said to be *ostensive* if it assigns every variable (constant) an ostensive QS, i.e. a QE.

By Theorem 8.1(c), for every $\sigma \in \mathrm{QS}^{<\lambda}$ we can define σ^{E} by requiring that *for some constant-free* $\mathrm{wfe}^{<\lambda}$ Δ *and some (ostensive)* $\mathcal{V} \in V^{\lambda}$ *we should have*

(8.10) $\sigma^{\mathrm{E}} = \mathrm{des}_{\mathcal{I}_*\mathcal{V}}(\Delta)$, $\sigma = \mathrm{sens}_{\mathcal{I}_*\mathcal{V}}(\Delta)$ (see (5.8)).

Note that Theorem 8.1(c) and (8.10) are extensions of Theorem 6.1,λ and Definition 6.1,λ, respectively (practically considered only for $\lambda = 1$); furthermore, to prove Theorem 8.1(c), it is convenient to note that the same class QS^{β}_t could be defined by the identity obtained from (8.7) by either adding to "$\mathcal{V} \in V^{\beta+1}$" the assertion "$\mathcal{V}$ is ostensive", or replacing "$\mathcal{I} = \mathcal{I}_*$" with "$\mathcal{I} \in I^{\beta+1}$" and crossing out "$\Delta$ is constant-free".

N9 A general recursive definition of the main semantical notions for the extensional sense language $\mathfrak{L}_\alpha$ The rules (h_{1-8}) and (ϵ_{1-8}) given in N8 in full generality but considered there only for $\lambda = 2$ induce the corresponding rules for $\mathfrak{L}_1$ (see the considerations following (8.7)). Now we want to consider them, together with some definitions—such as Definition 6.1.λ, which was used only for $\lambda \leq 2$—in full generality, i.e. for $\beta < \lambda \leq \alpha$. In more detail, we want to define, for $\beta < \lambda \leq \alpha$:

(a) the class QE_t^β (QS_t^β) of the HQE's (QS's) designatable by (constant-free) wfe's in E_t^β, and the classes A_t^β and $\mathrm{Q}\mathcal{E}_t^\beta$ (see (7.6) and (5.6)) ($t \in \tau_\nu$);

(b) the class V^λ (I^λ) of λ-v-valuations (λ-c-valuations (see (5.7));

(c) the HQE σ^{E} of any $\sigma \in \mathrm{QS}^{<\lambda}$, to be denoted by $I_\lambda(\sigma)$; and

(d) ((e)) the HQE $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ (the QS $\breve{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$) designated by any wfe$^{<\lambda}$ Δ at any $\mathcal{I} \in I^\lambda$ and $\mathcal{V} \in V^\lambda$.

We do this by possibly transfinite induction on λ and the length of Δ and by recursion on t. Therefore fix a value ($\leq\alpha$) for λ (>0), and assume that the objects in (a) to (e) are known for the smaller values of λ.

First we consider the case $0 < \lambda = \cup\lambda$ (λ is a limit ordinal). Then all objects above are in effect known for the actual value of λ too, except V^λ, I^λ, and $I_\lambda(\cdot)$. These can be defined by

$$(9.1)\quad \begin{Bmatrix} V^\lambda \\ I^\lambda \end{Bmatrix} = \Big\{\text{function} \begin{Bmatrix} \mathcal{V} \\ \mathcal{I} \end{Bmatrix} \Big| \begin{Bmatrix} \mathfrak{D}_{\mathcal{V}} \\ \mathfrak{D}_{\mathcal{I}} \end{Bmatrix} \text{ is the set of the } \begin{Bmatrix} \text{variables} \\ \text{constants} \end{Bmatrix} \text{ of } \mathfrak{L}_\alpha \text{ and}$$
$$\begin{Bmatrix} (5.7)_1 \\ (5.7)_2 \end{Bmatrix} \text{ holds}\Big\},\ I_\lambda = \bigcup_{\mu<\lambda} I_\mu \ (\text{i.e. } I_\lambda(\sigma) = I_\mu(\sigma) \text{ for } \sigma^{\mathrm{ord}} < \mu < \lambda).$$

In the remaining case it is useful to assume that $\lambda = \beta + 1$ and to order the objects in (a) to (e) as follows:

(1) QE_t^β, A_t^β, and $\mathrm{Q}\mathcal{E}_t^\beta$ ($t \in \tau_\nu$), (2) V^λ and I^λ, (3) (f (see $(8.3)_1$) and) des^λ, i.e. $\bar{\Delta} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ for $\mathcal{V} \in V^\lambda$, $\mathcal{I} \in I^\lambda$, and $\Delta \in \mathrm{E}_t^\beta$ ($t \in \tau_\nu$), (4) (g and) sens^λ i.e. $\breve{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ for $\mathcal{I}$, $\mathcal{V}$, and Δ as in (3) (see $(8.3)_2$), (5) QS_t^β ($t \in \tau_\nu$) see (8.7)), and (6) σ^{E} (see (8.10)), i.e. $I_\lambda(\sigma)$ for $\sigma \in \mathrm{QS}_t^\beta$ ($t \in \tau_\nu$).

Having assumed that $\lambda = \beta + 1$, we determine the objects (1) to (6) in the written order by simultaneous recursion. In more detail, we determine: (1) by using (5.6) and (7.5)-(7.7), as well as recursion on $t \in \tau_\nu$; (2) by (9.1); (3) by rules (h_{1-8}) in N8 and $(8.3)_1$, using recursion on Δ's length l_Δ and assumptions (i) to (iv) in N8; it is sufficient to do this only for $\Delta^{\mathrm{ord}} = \beta$, provided we state (9.2.1) below—see Theorem 8.1(a) for the definition of e.g. $\mathcal{I}'^\delta$

$$(9.2.1)\quad \mathrm{des}^\lambda_{\mathcal{I}\mathcal{V}}(\Delta) = \mathrm{des}^\delta_{\mathcal{I}'^\delta\mathcal{V}'^\delta}(\Delta)$$
$$(9.2.2)\quad \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta) = \mathrm{sens}^\delta_{\mathcal{I}'^\delta\mathcal{V}'^\delta}(\Delta), \text{ for } \delta = \Delta^{\mathrm{ord}} + 1 \leq \beta.$$

Then we can determine the objects (4) likewise by $(8.3)_2$, (8.6), and rules (ϵ_{1-8}) in N8, using induction on l_Δ ($<\infty$) and assumptions (i) to (iii) in N8; it is sufficient to do this only for $\Delta^{\mathrm{ord}} = \beta$, provided we state (9.2.2). Objects (5) and (6) are determined by (8.7) and (8.10) respectively.

Now the validity of Theorem 8.1 can be checked for $0 < \lambda \le \alpha$. It can also be checked that, besides (5.6), we have[13]:

$$(9.3)\quad \begin{cases} \mathrm{QE}_t^\beta = \mathrm{QE}_t^0, & \mathrm{Q}\mathcal{E}_t^\beta = \mathrm{Q}\mathcal{E}_t^0 \quad (t = 0,\ldots,\nu;\ \beta < \alpha), \\ \mathrm{QE}_t^\delta \subset \mathrm{QE}_t^\beta, & \mathrm{Q}\mathcal{E}_t^\delta \subset \mathrm{Q}\mathcal{E}_t^\beta \quad (t \in \tau_\nu - \{0,\ldots,\nu\};\ \delta < \beta < \alpha), \\ \mathrm{QS}_t^\delta \subset \mathrm{QS}_t^\beta, & (t \in \tau_\nu,\ \delta < \beta < \alpha). \end{cases}$$

Furthermore, for $\delta < \beta < \alpha$ and $t \neq t'$

$$(9.4)\quad \mathrm{QE}_t^\delta \wedge \mathrm{QE}_{t'}^\beta = \begin{cases} \mathbf{F} \\ \{\mathbf{F},\varnothing\} \end{cases} \text{and } \mathrm{Q}\mathcal{E}_t^\delta \wedge \mathrm{Q}\mathcal{E}_{t'}^\beta = \begin{cases} \varnothing, \text{ for } \{t,t'\} \cap \{0,\ldots,\nu\} \neq \varnothing, \\ \{\varnothing\}, \text{ otherwise.} \end{cases}$$

Definition 9.1 Under condition (4.4) we say that

(9.5) $\mathbf{\mathfrak{I}} = \langle \mathfrak{D}_1,\ldots,\mathfrak{D}_\nu,\mathfrak{I}\rangle$, where $\mathfrak{I} \in I^\alpha$

is an *interpretation* of $\mathcal{L}_\alpha$ relative to the proper individual domains $\mathfrak{D}_1$ to $\mathfrak{D}_\nu$, and that $\mathcal{V}$ is an $\mathbf{\mathfrak{I}}$-valuation in case $\mathcal{V} \in V^\alpha$.

N10 Discussion of the semantics for $\mathcal{L}_\alpha$ I want now to discuss some features of the preceding semantics for $\mathcal{L}_\alpha$, and in particular to show that, within it, the attribution of a property to an object depends on the order of the predicate expressing the property. The same can be said of the application of an n-ary function to n objects.

As in N6, let us consider '1','2',... as primitive constants of a theory $\mathfrak{T}$ based on (or coinciding with) $\mathcal{L}_\alpha$. By using '3' autonomously, let us set within $\mathcal{L}_2$:

(10.1.1) $P^0(x^0) \equiv_{def} x^0 = 3,$
(10.1.2) $P^1(x^1) \equiv_{def} x^1 = 3,$
(10.1.3) $Q^1(x^1) \equiv_{def} (\forall G^1).G^1(x^1) \equiv G^1(3),$

so that, incidentally, $Q^1(\Delta^0)$ can be translated as "Δ^0 is synonymous with 3". Now remember that the extension $\bar{\bar{F}}$ of a predicate F, considered in the version most used in the extensional case, is the set of objects of which F can be correctly predicated; thus $\bar{\bar{F}}$ is the domain $\mathfrak{D}_{\bar{F}}$ of the QE $\bar{F}$ designated by F in $\mathcal{L}_\alpha$. Using these notions we can write:

(10.2.1) $\bar{\bar{P}}^0 = \{3\},$
(10.2.2) $\bar{\bar{P}}^1 = \{\sigma \in \mathrm{QS} \mid \sigma^{\mathrm{E}} = 3\},$
(10.2.3) $\bar{\bar{Q}}^1 = \{3\}.$

By rules ($h_{2,6}$) in N8 the wff's:

(10.3.1) $(\forall x^1).P^0(x^1) \equiv P^1(x^1),$
(10.3.2) $P^0(5-2),$
(10.3.3) $\sim Q^1(5-2)$

are true. Incidentally, (10.2)–(10.3) still hold in case the replacements $x^1 \to x^\beta$, $P^1 \to P^\beta$, $Q^1 \to Q^\beta$, and $G^1 \to G^\beta$ are performed on (10.1)–(10.3) $(0 < \beta < \alpha)$.

The truth of (10.3.2)–(10.3.3) and (10.2.1) and (10.2.3), show that:

(i) certain predicates of different orders, e.g. P^0 and Q^1, fail to be correctly applicable to the same entity, e.g. $5-2$, in spite of the identity $\bar{\bar{P}}^0 = \bar{\bar{P}}^1$ of their (set theoretical) designata.

On the other hand, by (10.2.1)–(10.2.2) and the truth of (10.3.1), kept intact by the replacement $x^1 \to x^\beta$,

(ii) certain predicates of different orders, e.g. P^0 and P^1, have different (set theoretical) designata, $\bar{\bar{P}}^0 \neq \bar{\bar{P}}^1$, and in spite of this they are applicable to the same entities (see (10.3.1)).

Assertions (i) and (ii) on $\mathfrak{L}_2$ have obvious analogues holding for $\mathfrak{L}_\lambda$, induced by the above replacements $P^1 \to P^\beta$ and $x^1 \to x^\beta$.

Now recall that by the rules (h_{1-8}) in N8 the wff

(10.4) $p = q \equiv . \, p \equiv q$ (see Convention 3.1)

is logically valid, and that, if $a^\nu = \varnothing$ and $F^{\mathrm{ord}} = G^{\mathrm{ord}}$, the same holds for (7.4), which is the conjunction of the wff's

(10.5.1) $F = G \supset (\forall x).F(x) = G(x)$,
(10.5.2) $(\forall x)\,[F(x) = G(x)]\,. \supset F = G$, where $F^{\mathrm{ord}} = G^{\mathrm{ord}}$.

In general $a^\nu \neq \varnothing$ (see (13.1)). Now we remark that:

(iii) both (10.5.1) and (10.5.2) fail to be logically valid in $\mathfrak{L}_\alpha$ when $F^{\mathrm{ord}} \neq G^{\mathrm{ord}}$, even if $x^{\mathrm{ord}} \geq \max\{F^{\mathrm{ord}}, G^{\mathrm{ord}}\}$.

Indeed (10.5.1) ((10.5.2)) is false when F and G are replaced by P^0 and Q^1 (and P^1) respectively. This proves (iii) for $F^{\mathrm{ord}} = 0$ and $G^{\mathrm{ord}} = 1$. Quite analogous examples can be constructed for arbitrary different choices ($<\alpha$) of F^{ord} and G^{ord}.

The considerations above, i.e. (i) to (iii), simply show that:

(iv) in $\mathfrak{L}_\alpha$ the predication of a (set theoretical) attribute depends on the order of the predicate that designates it.

This fact, in accordance with direction (α) in N7, is not strange in itself. However, by its consequence (iii), it is natural to try and change $\mathfrak{L}_\alpha$'s semantics in order to render (10.5.1)–(10.5.2) valid in more general cases. At this point it is natural to try and follow the alternative direction (α') considered in N7.

N11 The interpreted language $\mathfrak{L}'_\alpha$ obtained from $\mathfrak{L}_\alpha$ by a first semantical change. A corresponding change in the underlying ontology Let us accept direction (α') in N7. Then it is important to determine $(\bar{\Delta}_0)^{\mathrm{ord}}$, hence $\bar{\Delta}_0$, suitably. Since we call $\mathfrak{L}'_\alpha$ the interpreted language being constructed according to (α'), let us now denote by $\bar{\Delta}'$ ($\breve{\Delta}'$) the QE (QS) of any designator Δ in the modified semantics, unless no confusion arises.

In applying (α') we may wish to use Δ_0 like P^0 or Q^1 in N10. Suppose we want to regard the wff's (10.1) and (10.3) as still true. By (10.2.1) and (10.2.3) $\bar{P}^0 = \bar{Q}^1$, so that $\bar{Q}^1$ seems to be of order 0. Hence it is practically necessary to regard $\bar{\Delta}'_0$ in the above two cases to be $\langle \bar{P}^0, 0\rangle$ or $\langle \bar{Q}^1, 1\rangle$ respectively. Thus we can use Δ_0 as P^0 or Q^1, no matter what Δ_0^{ord} is, provided Δ_0^{ord} is ≥ 0 or ≥ 1 respectively. Something quite similar happens with functions. To accept this point of view implies that

(i) in $\mathcal{L}'_\alpha$ an attribute or function is considered as determined only if, besides its set theoretical part, the rule for applying it is also given.

Thus the ontologies underlying $\mathcal{L}_\alpha$ and $\mathcal{L}'_\alpha$ differ in general and fundamental ways. This has an analogue in mathematics: analysts and topologists use different notions of functions, as is explicitly remarked in [2], Chapter 1, §2. Briefly, analysts regard n-ary functions as special $(n+1)$-ary relations, i.e. special sets of $(n+1)$-tuples, while topologists regard them as determined by such a set f of $(n+1)$-tuples, associated with two other sets A and B. These are called the *domain* and *counterdomain* respectively, but they can fail to coincide with the analysts' $\mathfrak{D}_f$ and $\mathcal{C}\mathfrak{D}_f$ respectively (however $\mathfrak{D}_f \subseteq A$ and $\mathcal{C}\mathfrak{D}_f \subseteq B$). Obviously the new ontology is to the old one as the one of topologists is to the ontology of analysts.

The technical changes sufficient to carry out this first semantical change can be reduced to the following four.

(1) Replace clause (7.7) in the recursive definition (5.6) $\cup$ (7.5)–(7.7) of QE_t^β, A_t^β, and $\mathrm{Q}\mathcal{E}_t^\beta$ with the clause:

(11.1) $\mathrm{QE}^\beta_{\langle t_1,\ldots,t_n,t_0\rangle} = \{\xi \mid$ either $\xi \in \mathrm{QE}^{<\beta}_{\langle t_1,\ldots,t_n,t_0\rangle}$, or $\xi = \mathbf{F}$, or else $\xi = \langle f,\beta\rangle$ for some $f \in (\mathrm{A}^\beta_{t_1} \times \ldots \times \mathrm{A}^\beta_{t_n} \mapsto \mathrm{Q}\mathcal{E}^\beta_{t_0})$ with $f \neq \varnothing$ for $\beta > 0\}$ $(t_0,\ldots,t_n \in \tau_\nu)$.

Note that the alternative $\xi = \mathbf{F}$ in (11.1) could be cancelled (only) for $0 < \beta < \alpha$. Furthermore, note that the empty n-ary attribute is trivially applied (in $\mathcal{L}_\alpha$) in the same way by all wfe's; therefore the clause "with $f \neq \varnothing$ for $\beta > 0$" has been inserted in (11.1), in implementing (i) technically.

Lastly, remember that for the sake of simplicity, for $t = \langle t_1,\ldots,t_n,t_0\rangle \in \tau_\nu$ one may wish to identify the nonexisting object of type t with the empty object of the same type (a relation or a function), i.e. to render e.g. $a_t^* = (\lambda x_1, \ldots, x_n) x_1 \neq x_1$ $(x_1 \in \mathrm{E}_{t_1})$ true. In fact, in this case the hyper-intensionality principle has the form (1.1), instead of the form (13.1) which is (mathematically acceptable but) more complex. For $\mathcal{L}_\alpha$ the above identification can be obtained by setting (a) $\mathbf{F} = \varnothing$. The replacement of (a) with (b) $\mathbf{F} = \langle \varnothing, 0\rangle$ achieves the same goal for $\mathcal{L}'_\alpha$.

(2) Replace the definition (8.2) of the object $\hat{\Delta}_i$ depending on $\mathcal{I}$ and $\mathcal{V}$ with:

$$\text{(11.2)}\quad \hat{\Delta}_i = \begin{cases} \check{\Delta}_i, & \text{if } (\check{\Delta}_i)^{\mathrm{ord}} < (\bar{\Delta}_0)^{\mathrm{ord}}, \\ \bar{\Delta}_i, & \text{otherwise.} \end{cases} \qquad (i = 1,\ldots,n).$$

Of course, rule (h'_2) consequently differs from (h_2) only in that—by writing e.g. $(c)_1$ for a_1 when $c = \langle a_1, a_2\rangle$—it reads:

Rule (h'_2) If Δ is $\Delta_0(\Delta_1,\ldots,\Delta_n)$, then $\bar{\Delta} (= \mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta))$ is $\mathbf{F}$ for $\bar{\Delta}_0 = \mathbf{F}$, and otherwise $\bar{\Delta}$ is $(\bar{\Delta}_0)_1(\hat{\Delta}_1,\ldots,\hat{\Delta}_n)^\dagger$ (see (11.2) and (5.10)).

(3) Change rule (h_8) in N8 into the following:

Rule (h'_8) If Δ is $(\lambda x_1,\ldots,x_m)\Delta_n$, then $\bar{\Delta} = \langle f, d_f\rangle$ (see $(8.3)_1$ and $(8.5)_1$).

(4) Simplify rule (ϵ_2) in N8 by crossing out $(\Delta_0)^{\text{ord}}$:

Rule ($\grave{\epsilon}_2$) If Δ is $\Delta_0(\Delta_1, \ldots, \Delta_n)$, then $\breve{\Delta} = \langle \breve{\Delta}_0, \ldots, \breve{\Delta}_n \rangle$.

In the sequel I may denote any rule or semantical object for $\grave{\mathcal{L}}_\alpha$ by the same symbol of its counterpart for $\mathcal{L}_\alpha$, endowed with a grave accent. When $\grave{\mathcal{L}}_\alpha$ will have been turned into the interpreted language $\grave{\grave{\mathcal{L}}}_\alpha$ by a second semantical change—see N12—two such accents will be used for the corresponding purpose connected with $\grave{\grave{\mathcal{L}}}_\alpha$, as long as no confusion arises. For instance I may write that $\bar{\Delta}\grave{} = \grave{\text{des}}_{\mathcal{I}\mathcal{V}}(\Delta)$ and $\breve{\Delta}\grave{} = \grave{\text{sens}}_{\mathcal{I}\mathcal{V}}(\Delta)$ are determined by rules ($\grave{h}_{1-8}$) and ($\grave{\epsilon}_{1-8}$), for $\mathcal{I} \in I^{\grave{}\lambda}$, $\mathcal{V} \in V^{\grave{}\lambda}$, and $\Delta \in E^{<\lambda}$. However, des $(\bar{\Delta}'_n)$ was preferred to des$\grave{}$ $((\grave{\bar{\Delta}}_n)')$ in Rule ($\grave{h}_2$) (($\grave{h}_8$)).

By changes (1) to (4) above, ($\grave{h}_r$) and ($\grave{\epsilon}_r$) equal (h_r) and (ϵ_r) respectively (up to notations) for $r \in \{1\} \cup \{3, \ldots, 8\}$; ($\grave{h}_2$) differs from ($h_2$) only in that it is based on definition (11.2) instead of (8.2) and it involves $(\Delta_0)_1$ instead of Δ_0. The obvious analogues for $\grave{\mathcal{L}}_\alpha$ of definitions (8.7) and (8.10) can be used.

In case we want to compare the versions of QE_t for $\mathcal{L}_\alpha$ and $\grave{\mathcal{L}}_\alpha$, we denote them by QE_t and $\grave{\text{QE}}_t$ respectively. We assert, e.g.,

(11.3) $\quad \grave{\text{QE}}_t^{\beta} = \text{QE}_t^{\beta}$, only for $t \in \{0, \ldots, \nu\}$; $\grave{\text{QS}}_t^{\beta} \neq \text{QS}_t^{\beta}$, $\forall t \in \tau_\nu$ $(0 \leq \beta < \alpha)$.

The analogue of Theorem 8.1 for $\grave{\mathcal{L}}_\alpha$ can be proved.

Note that *in* $\grave{\mathcal{L}}_\alpha$ (10.5.1) *is valid for arbitrary orders of F, G, and x.* More generally, *the wff*

(11.4) $\quad f = g \supset (\forall x_1, \ldots, x_m).f(x_1, \ldots, x_m) = g(x_1, \ldots, x_m)$

is valid, where $f, g \in E_{\langle t_1, \ldots, t_m, t_0 \rangle}$ *and* $x_i \in E_{t_i}$ $(i = 1, \ldots, m)$.

A general version of the instantiation axiom (1.2) is also valid in $\grave{\mathcal{L}}_\alpha$ (see N14).

Lastly, we remark that *the converse of* (11.4) *cannot be a valid axiom scheme in* $\grave{\mathcal{L}}_\alpha$. This can easily be seen by considering its special case (10.5.2) for $F^{\text{ord}} < G^{\text{ord}} \leq x^{\text{ord}} < \alpha$.

In fact, since by (10.2.1)-(10.2.2) $\bar{P}^0 \neq \bar{P}^1$, we have that $\overline{P^0}\grave{} = \langle \bar{P}^0, 0 \rangle \neq \langle \bar{P}^1, 1 \rangle = \overline{P^1}\grave{}$. However, by these definitions of $\overline{P^0}\grave{}$ and $\overline{P^1}\grave{}$ and by rules ($\grave{h}_2$) and ($\grave{h}_6$) (=(h_6) in N8), the wff (10.3.1) is also true in $\grave{\mathcal{L}}_\alpha$, while (since $\overline{P^0}\grave{} \neq \overline{P^1}\grave{}$) the wff $P^0 = P^1$ is false in $\grave{\mathcal{L}}_\alpha$; hence (10.5.2) is also false in $\grave{\mathcal{L}}_\alpha$.

N12 The language $\grave{\grave{\mathcal{L}}}_\alpha$ obtained from $\grave{\mathcal{L}}_\alpha$ by a second semantical change. Ontological considerations. QE$^\beta$'s for $\grave{\grave{\mathcal{L}}}_\alpha$ in terms of the QE$^{<\beta}$'s and QS$^{<\beta}$'s It is natural to try and render the converse of (11.4) true; this is substantially equivalent to regarding the HQE's of f and g as identical whenever they render the consequent of (11.4) true. This strengthening of identity induces a corresponding strengthening of the synonymy relation. Thus, the second semantical change which is being chosen appears similar to an extension to $\mathcal{S}\mathcal{L}_\alpha^\nu$ of some among the changes of the basic semantics for $\mathfrak{M}\mathcal{L}^\nu$ that are connected with a strengthening of only synonymy, and were considered in [5] (for $\alpha = 1$).

It is convenient to note that such a strengthening is not new; it occurred during the mathematical analysis of basic notions. E.g. when the intuitive notions of rational, real, and complex numbers were analysed, strictly speaking they were redefined within set theory by identifying them with certain mutually disjoint classes $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ respectively. A certain part $\mathbb{R}^{(\mathbb{Q})}$ of $\mathbb{R}$ is formed by the so-called *rational real numbers*, and certain parts $\mathbb{C}^{(\mathbb{R})}$ and $\mathbb{C}^{(\mathbb{Q})}$ of $\mathbb{C}$ are formed by the so-called *real* or *rational complex numbers* respectively.

By using an overbar, $\bar{a}$, to express the object denoted by a wfe a, assume that

$$(12.1)\quad \bar{a}, \bar{b}, \bar{c} \in \mathbb{Q} \text{ and e.g. } \overline{a^{\mathbb{R}}}\ (\overline{a^{\mathbb{C}}}) \text{ is the counterpart of } \bar{a} \text{ in } \mathbb{R}(\mathbb{C}).$$

Then the mutual equivalence of the relations

$$(12.2)\quad \bar{a} + \bar{b} = \bar{c},\ \overline{a^{\mathbb{R}}} + \overline{b^{\mathbb{R}}} = \overline{c^{\mathbb{R}}}, \text{ and } \overline{a^{\mathbb{C}}} + \overline{b^{\mathbb{C}}} = \overline{c^{\mathbb{C}}}$$

is well known. Furthermore, people often make the identifications $\bar{a} = \overline{a^{\mathbb{R}}} = \overline{a^{\mathbb{C}}}$. These identities induce the synonymy relations

$$(12.3)\quad a \asymp a^{\mathbb{R}} \asymp a^{\mathbb{C}} \text{ (and } a + b \asymp a^{\mathbb{R}} + b^{\mathbb{R}} \asymp a^{\mathbb{C}} + b^{\mathbb{C}})$$

between terms, and the following ones between wff's[14]:

$$(12.4)\quad (a + b = c) \asymp (a^{\mathbb{R}} + b^{\mathbb{R}} = c^{\mathbb{R}}) \asymp (a^{\mathbb{C}} + b^{\mathbb{C}} = c^{\mathbb{C}}).$$

As well as many changes in a hyper-intensional semantics, connected with the basic synonymy notion – see section 17 in [5], p. 200 – our second semantical change

(1) allows us to treat synonymy relations that are more similar to those used in practice, at least in certain interesting situations, and
(2) it allows us to write a more efficient (practically more powerful) corresponding axiom system.

Now, in order to define $\mathcal{L}''_\alpha$ technically, we consider the λ-segment $\mathcal{L}''_\lambda$ of $\mathcal{L}''_\alpha$ $(0 < \lambda \leq \alpha)$, use (a part of) the semantics for $\mathcal{L}'_\alpha$, and define simultaneously the $\mathcal{L}''_\alpha$-versions of the objects (1) to (6) mentioned following (9.1) by transfinite induction on λ $(0 < \lambda \leq \alpha)$ and recursion on $t \in \tau_\nu$ or the length l_Δ of the wfe Δ being considered.

Recalling N11, as starting clause we use that

(i) we have

$$(12.5.1)\quad \mathrm{QE}_t^\beta = \mathrm{QE}_t^{\prime\beta} \text{ if either } \beta = 0 \text{ or } t \in \{0, \ldots, \nu\},$$
$$(12.5.2)\quad \mathrm{QS}_t^0 = \mathrm{QS}_t^{\prime 0}\ (t \in \tau_\nu),$$

where (as well as often in the sequel) it is convenient to write QE_t^β (QS_t^β) for $\mathrm{QE}_t^{\prime\prime\beta}$ $(\mathrm{QS}_t^{\prime\prime\beta})$, and

(ii) the versions of all objects (1) to (6) (see below (9.1)) for the extensional segments $\mathcal{L}'_1$ and $\mathcal{L}''_1$ of $\mathcal{L}'_\alpha$ and $\mathcal{L}''_\alpha$ respectively coincide.

Now fix λ $(0 < \lambda \leq \alpha)$ and suppose those objects to have been defined for $\mathcal{L}''_\vartheta$ $(0 < \vartheta < \lambda)$. In case λ is a limit ordinal, determine V^λ, I^λ, and $I_\lambda(\cdot)$ by (9.1). The others among the versions for $\mathcal{L}''_\lambda$ of the objects (1) to (6) are in effect known.

In case λ is a successor ordinal, set $\lambda = \beta + 1$.

Since the subcase $\beta = 0$ need not be considered because of (12.5.1), we first assume that $\beta = 1$. Furthermore we remember the case (10.1.1)–(10.1.2) by which (10.2.1)–(10.2.2), (10.3.1), and $\overline{P^{0}}' \neq \overline{P^{1}}'$ hold. In it, when f is P^0 and g is P^1, the converse of (11.4) (for $n = 1$, $t = (t_1)$ where t is P^1's type, and $\beta = 1$) is false; therefore we want to construct a sense language $\mathcal{L}''_\alpha$ in which $\overline{P^{0}}'$ and $\overline{P^{1}}'$ are identified and the analogue holds for any $n \geq 1$, $\beta \geq 1$, and $t \in \tau_\nu - \{0, \ldots, \nu\}$.

Technically this identification can be carried out by

(α) regarding $\xi =_{def} \overline{P^{1}}'$ ($\in \mathrm{QE}_t^{1\not<}$) as a pre-QE of order 1 and type $t = (t_1)$

and by

(β) stipulating that $\overline{P^{1}}' \notin \mathrm{QE}_t^1$ in $\mathcal{L}''_\alpha$ because of the truth of the wff (10.3.1) in $\mathcal{L}'_\alpha$, which shows that $\overline{P^{1}}'$ represents the same extension as $\overline{P^{0}}'$, and hence that it is useless.

Note that the above truth in $\mathcal{L}'_\alpha$ involves QE-designation in $\mathcal{L}'_\alpha$ and that, in a first (tentative technical) approach, $\overline{P^{1}}'$ could be regarded as the *pre*-QE *designated by* P^1 in $\mathcal{L}''_\alpha$. However, for $\beta > 0$ the pre-QE designation function cannot coincide with $\mathrm{des}'^{\beta+1}$ (see below (9.1) and above (11.3)) (unlike what happens for $\beta = 0$). This adds to the complexity of this approach.

Furthermore, on the one hand, for $\mathcal{L}_{\beta+1}$ and $\mathcal{L}'_{\beta+1}$, QE_t^β is defined, together with some other notions, by a multiple recursion on $t \in \tau_\nu$, which does not involve the functions des and sens for $\mathcal{L}_{\beta+1}$ and $\mathcal{L}'_{\beta+1}$ respectively—i.e. $\mathrm{des}^{\beta+1}$ and $\mathrm{sens}^{\beta+1}$. Instead, on the other hand, in the approach above a similar function (of pre-QE designation) appears to be involved in the corresponding recursion on $t \in \tau_\nu$; and this also contributes to the complexity of the approach being considered.

Lastly, note that within a multiple recursion on $t \in \tau_\nu$ the stipulation mentioned in (β) is carried out by simply failing to include $\overline{P^{1}}'$ in QE_t^1.

Therefore we look for a second approach to perform identifications of the above type; and remembering that (10.3.1) is true because of (10.2.1)–(10.2.2), we begin by replacing the above identification based on (α) and (β) with statement (β') below, for case $\delta = 0$ and $\beta = 1$.

(β') We fail to include the case PQE_t^β $\zeta = \langle \zeta_1, \beta \rangle = \overline{P^{1}}'$ in QE_t^β with $t = (t_1)$ because there is some $\eta \in \mathrm{QE}_t^{\delta\not<}$, and precisely $\eta = \langle \eta_1, \delta \rangle = \overline{P^{0}}'$, that satisfies the following condition—which directly involves $\mathfrak{D}_{\eta_1}$ ($= \overline{\overline{P^0}}$) and $\mathfrak{D}_{\zeta_1}$ ($= \overline{\overline{P^1}}$):

$$(12.6)\quad \mathfrak{D}_{\zeta_1} = \mathfrak{D}_{\beta,\eta_1} =_{def} \{\sigma \in \mathrm{A}_{t_1}^\beta \mid \sigma^{\mathrm{E}\delta} \in \mathfrak{D}_{\eta_1}\} = \{\sigma \in \mathrm{QS}_{t_1}^{<\beta} \cup \mathrm{QE}_{t_1}^\delta \mid \sigma^{\mathrm{E}\delta} \in \mathfrak{D}_{\eta_1}\}$$
$$(\delta = \eta^{\mathrm{ord}} \leq \beta),$$

where (referring to QS's and QE's for $\mathcal{L}''_\alpha$ (see (12.5)))

$$(12.7)\quad \sigma^{\mathrm{E}\delta} =_{def} I_{\beta\delta}(\sigma) =_{def} \begin{cases} \sigma^{\mathrm{E}} \\ \sigma \end{cases} \text{if } \sigma \begin{matrix} \notin \\ \in \end{matrix} \mathrm{QS}_{t_1}^{<\delta}, \text{ for } \sigma \in \mathrm{A}_{t_1}^\beta,$$

so that $\sigma^{\mathrm{E}\delta} = \sigma = \sigma^{\mathrm{E}}$ for $\sigma \in \mathrm{QE}_{t_1}^\delta$.

In order to see that (β') is reasonable, first note that, for $\xi \in \mathrm{QE}^{\delta\not<}$ (i) $\xi^{[\beta]} =_{def} \{\sigma \in \mathrm{QS}^{<\beta} \mid \sigma^{\mathrm{E}} = \xi\}$ is the set of the QS's in $\mathcal{L}''_{\beta+1}$ that correspond to the QE ξ of $\mathcal{L}''_{\delta+1}$ and (ii) $\xi \in \xi^{[\beta]}$. Furthermore, in the examples (10.2.1)–(10.2.2) (iii) $\delta = 0$ so that $\sigma \notin \mathrm{QS}_{t_1}^{<\delta} = \varnothing$ for any σ, whence $\mathfrak{D}_{1,\eta_1} =$

$\{\sigma \in \mathrm{QS}^0_{t_1} | \sigma^{\mathrm{E}} \in \mathfrak{D}_{\eta_1}\}$. Then, by (10.2.1)-(10.2.2), $\mathfrak{D}_{\eta_1} = \{3\}$ so that $(12.6)_1$ yields $\mathfrak{D}_{\zeta_1} = \overline{\overline{P^1}}$.

Thus in the case of the example, where $P^1 \in \mathrm{E}_{\langle t_1,\ldots,t_m,t_0\rangle}$ with $m = 1$ and $t_0 = 0$, the approach being constructed works. That it works in general will be proved rigorously by Theorem 13.1 (and Theorem 13.2). Below one aims at showing, at least in part, that this approach has natural features, and at rendering these more easily understandable.

Now, from an intuitive point of view, assume that (a) $\beta > \delta > 0$, (b) we know $\mathfrak{L}_\beta$, and (c) we have defined pre-QE^β_t's, i.e. the set PQE^β_t, properly. Then we can consider, first, an $\eta = \langle \eta_1, \delta\rangle \in \mathrm{QE}^{\delta \not<}_t$ with $t = (t_1)$, $\mathfrak{D}_{\eta_1} \subseteq \mathrm{QS}^{<\delta}_{t_1}$, and e.g. $\delta = 1$, and second, a $\zeta = \langle \zeta_1, \beta\rangle \in \mathrm{PQE}^\beta_t$ that satisfies condition (12.6). Then $D_{\zeta_1} = \mathfrak{D}_{\beta,\eta_1} = \{\sigma \in \mathrm{QS}^{<\beta}_t | \sigma \in \mathfrak{D}_{\eta_1}\} = \mathfrak{D}_{\eta_1}$ and hence $\zeta = \langle \eta_1, \beta\rangle$ $(\neq \eta)$. Therefore, roughly speaking, if we had that $\zeta \in \mathrm{QE}^\beta_t$, then the converse of (11.4) would be false for $\bar{f} = \eta$ and $\bar{g} = \zeta$. Thus the lower part of definition (12.7) is also essential in order to extend (β') to case (a) correctly, e.g. for $t = (t_1)$.

Looking forward to extending the present considerations to e.g. 1-ary functions, let us note that even in the second case above—where $t = (t_1)$—

(12.8) $\quad \zeta_1(\sigma) = \eta_1(\sigma^{\mathrm{E}\delta}) \qquad \forall \sigma \in \mathfrak{D}_{\zeta_1}\ (= \mathfrak{D}_{\beta,\eta_1}).$

Now let us define $\mathfrak{L}^{\nu}_{\lambda}$ rigorously in the case $1 < \beta + 1 = \lambda \leq \alpha$. To obtain the analogue for $\mathfrak{L}^{\nu}_{\lambda}$ of the definition of the objects (1) following (9.1), we first add to them the set $\mathrm{PQE}^\beta_\vartheta$ of the *pre*-QE's of order β and type $\vartheta \in \tau_\nu - \{0,\ldots,\nu\}$, the β-mate $m^\beta(\xi)$ of any $\xi \in \mathrm{QE}^{<\beta}_\vartheta$, σ's δ-equivalent $\sigma^{\mathrm{E}\delta} = I_{\beta\delta}(\sigma)$, and the set $\mathrm{LQE}(\zeta)$ of the lower-order QE's (of type ϑ) corresponding to any $\zeta \in \mathrm{PQE}^\beta_\vartheta$. Second, we define the resulting objects

(12.9) $\quad \mathrm{QE}^\beta_t,\ \mathrm{Q}\mathcal{E}^\beta_t,\ \mathrm{A}^\beta_t,\ \mathrm{PQE}^\beta_\vartheta,\ I_{\beta\delta}(\cdot)(\delta < \beta),\ \xi \vdash m^\beta(\xi),\ \zeta \vdash \mathrm{LQE}_\vartheta(\zeta)$

by recursion on $t \in \tau_\nu$ and $\vartheta \in \tau_\nu - \{0,\ldots,\nu\}$ in terms of the analogues for $\mathfrak{L}^{\nu}_{\beta}$ of both these objects and the objects (2) to (6) mentioned following (9.1). We do this by means of the initial clause (12.5.1) for $t \in \{0,\ldots,\nu\}$ and the recursive clauses (5.6), (7.6), and (12.11)-(12.14) below for all t, t_0 to t_m, ϑ, and δ, with $t \in \tau_\nu$ and

(12.10) $\quad \vartheta = \langle t_1,\ldots,t_m,t_0\rangle \in \tau_\nu,\ 0 \leq \delta < \beta < \alpha.$

Incidentally, Definition (12.12) (i.e. (12.12a)-(12.12c)) is equivalent to Definition 12.1 below.

(12.11) $\quad \mathrm{PQE}^\beta_\vartheta =_{def} \{f \in (\mathrm{A}^\beta_{t_1} \times \ldots \times \mathrm{A}^\beta_{t_m} \mapsto \mathrm{Q}\mathcal{E}^\beta_{t_0}) | f \neq \varnothing\} \times \{\beta\}.$

For all $\eta = \langle \eta_1, \delta\rangle \in \mathrm{QE}^\delta_\vartheta$:

(12.12a) $\quad m^\beta(\eta) =_{def} \langle \zeta_1, \beta\rangle \in \mathrm{PQE}^\beta_\vartheta$

where (remembering (12.7) and that $\mathfrak{D}_{\eta_1} \subseteq \mathrm{A}^\delta_{t_1} \times \ldots \times \mathrm{A}^\delta_{t_m}$) ζ_1 is the n-ary function with

(12.12b) $\quad \mathfrak{D}_{\zeta_1} = \{\sigma \in \mathrm{A}^\beta_{t_1} \times \ldots \times \mathrm{A}^\beta_{t_m} | \sigma^{\mathrm{E}\delta} \in \mathfrak{D}_{\eta_1}\} \qquad (\sigma^{\mathrm{E}\delta} = \langle \sigma_1^{\mathrm{E}\delta},\ldots,\sigma_m^{\mathrm{E}\delta}\rangle)$

and

(12.12c) $\quad \zeta_1(\sigma) = \eta_1\ (\sigma^{\mathrm{E}\delta}) \qquad \forall \sigma \in \mathfrak{D}_{\zeta_1}\ (\subseteq \mathrm{A}^\beta_{t_1} \times \ldots \times \mathrm{A}^\beta_{t_m}).$

Furthermore

(12.13) $\mathrm{LQE}(\zeta) =_{def} \{\eta \in \mathrm{QE}_{\vartheta}^{\lesssim\beta} \mid \zeta = m^{\beta}(\eta)\}$ $(\forall \zeta \in \mathrm{QE}_{\vartheta}^{\beta})$

(see property (γ) of $\zeta \vdash \mathrm{LQE}(\zeta)$ below). Finally

(12.14) $\mathrm{QE}_{\vartheta}^{\beta} =_{def} \mathrm{QE}_{\vartheta}^{\lesssim\beta} \cup \{\zeta \in \mathrm{PQE}_{\vartheta}^{\beta} \mid \mathrm{LQE}(\zeta) = \varnothing\}$.

Note that by (12.7) and $(12.12\mathrm{b})_2$, Definition (12.12) is equivalent to

Definition 12.1 For any $\eta = \langle \eta_1, \delta\rangle \in \mathrm{QE}_{\vartheta}^{\delta}$ where (12.10) holds, we say that $\zeta = m^{\beta}(\eta)$ in case $\zeta = \langle \zeta_1, \beta\rangle$ $(\in \mathrm{PQE}_{\vartheta}^{\beta})$ where ζ_1 is the n-ary function such that:

(i) $\langle \sigma_1, \ldots, \sigma_m\rangle \in \mathfrak{D}_{\zeta_1}$ iff, for some $\langle \rho_1, \ldots, \rho_m\rangle \in \mathfrak{D}_{\eta_1}$, $(\mathfrak{A}_i)$ either $\rho_i \in \mathrm{QS}_{t_i}^{<\delta} \cup \mathrm{QE}_{t_i}^{\delta}$ and $\sigma_i = \rho_i$, or $\rho_i \in \mathrm{QE}_{t_i}^{\delta}$ and $\sigma_i \in \mathrm{QS}_{t_i}^{<\beta}$ with $\sigma_i \neq \sigma_i^{\mathrm{E}\delta} = \rho_i$ $(i = 1, \ldots, m)$ and

(ii) the alternatives $(\mathfrak{A}_1)$ to $(\mathfrak{A}_m)$ in (i) imply (12.12c).

Now we can determine the objects (2) (following (9.1)) for $\mathfrak{L}_{\lambda}''$ just as those for $\mathfrak{L}_{\lambda}$ were determined. The objects (3) for $\mathfrak{L}_{\lambda}''$ —i.e. the function f and the QE^{β} $\mathrm{des}_{\mathcal{I}\mathcal{V}}(\Delta)$ for any $\mathcal{I} \in I^{\lambda}$, $\mathcal{V} \in V^{\lambda}$, and $\Delta \in \mathrm{wfe}^{\beta}$—can be constructed by means of $(8.3)_1$ and rules (h_{1-8}''), where (h_r'') coincides with rule (h_r) in N8 for $r \in \{1,3,4,\ldots,7\}$ and with rule (h_2') in N11 for $r = 2$. Furthermore we have that

(h_8'') if Δ is $(\lambda x_1, \ldots, x_m)\Delta_n$, then $\bar{\Delta}$ is $\langle f, d_f\rangle$ $(\in \mathrm{PQE}_{\vartheta}^{d_f}$ for $\vartheta = \langle t_1, \ldots, t_m, t_0\rangle)$ in case $\mathrm{LQE}(\langle f, d_f\rangle)$ is $\varnothing$; and the (unique) $\eta \in \mathrm{LQE}(\langle f, d_f\rangle)$ otherwise—see (12.11), (12.13), (8.3)-(8.5), and assertion (γ) below.

At this point let us determine (for $\mathfrak{L}_{\alpha}''$) the objects (4) (in N9), i.e. g and $\check{\Delta} = \mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$, by using induction on $\mathrm{l}_{\Delta}(< \omega)$, assumptions (i) to (iii) in N8, $(8.3)_2$, (8.6), and rules (ϵ_{1-8}''), where (ϵ_i'') is rule (ϵ_i) in N8 for $i \in \{1,3,4,\ldots,8\}$ and rule (ϵ_2') in N11 for $i = 2$.

Now we can define QS_t^{β} $(t \in \tau_\nu)$—i.e. the objects (5)—by (8.7) again. Theorem 8.1 also holds for $\mathfrak{L}_{\lambda}''$, so that one can define $I_{\lambda}(\cdot)$—i.e. σ^{E} for $\sigma \in \mathrm{QS}_t^{\beta}$ $(t \in \tau_\nu,\ \beta < \lambda)$ or object (6)—by (8.10). Thus objects (1) to (6) (following (9.1)) have been defined for $\mathfrak{L}_{\lambda}''$ $(0 < \beta + 1 = \lambda < \alpha)$. Furthermore, by determining $\mathfrak{L}_{\lambda}''$ for any limit ordinal λ as was done for $\mathfrak{L}_{\alpha}$ in N9, we define $\mathfrak{L}_{\alpha}''$ completely.

Note that relations (9.2)-(9.4) continue to hold; furthermore it is natural to use the obvious analogue for $\mathfrak{L}_{\alpha}''$ of Definition 8.1.

Incidentally, let us prove that:

(γ) *for* $\zeta \in \mathrm{PQE}_{\vartheta}^{\beta}$, $\mathrm{LQE}(\zeta)$ *is at most a singleton.*

Indeed assume that (i) $m^{\beta}(\xi) = \zeta = m^{\beta}(\eta)$ and (ii) $\xi, \eta \in \mathrm{QE}_{\vartheta}^{\lesssim\beta}$. Then the case $\xi^{\mathrm{ord}} = \eta^{\mathrm{ord}}$ clearly implies $\xi = \eta$, by (12.12). The opposite case cannot occur. In fact assume e.g. that (iii) $\xi^{\mathrm{ord}} < \eta^{\mathrm{ord}} = \delta < \beta$. Then (12.11) and (12.12) imply that $\eta = m^{\delta}(\xi)$, whence $\mathrm{LQE}(\eta) \neq \varnothing$ by (12.13). Then (12.14) yields $\eta \notin \mathrm{QE}_{\vartheta}^{\lesssim\beta}$, in contrast with (ii). Thus both inequality (iii) and its converse are impossible.

N13 The validity in $\mathfrak{L}_{\alpha}''$ of a general version of the hyper-intensionality axiom. A maximality property for the* QE*'s of $\mathfrak{L}_{\alpha}''$ The hyper-intensionality axiom has the form (1.1) if, as it happens very often, one uses the assumption mentioned in note 2, which in the semantics for $\mathcal{L}_\alpha$ (for $\mathcal{L}'_\alpha$ and $\mathcal{L}''_\alpha$) is equivalent to the condition $\mathbf{F} = \varnothing$ ($\mathbf{F} = \langle\varnothing,0\rangle$) (see the remarks following (11.3)). If this condition is not assumed, (1.1) must be replaced by (13.1) below. While this and (1.1) are valid in $\mathcal{L}_\alpha$ or $\mathcal{L}'_\alpha$ only under certain restrictions on orders—see NN 10,11—the following theorem on (13.1) holds for $\mathcal{L}''_\alpha$. Furthermore so does its analogue for (11.1) in case the QE $\mathbf{F}$ is identified with $\langle\varnothing,0\rangle$; and its proof is quite similar to the one of Theorem 13.1.

Theorem 13.1 *In $\mathcal{L}''_\alpha$ the wff*[15]

$$(13.1)\quad f=g \vee \{f,g\}=\{a^*,\varnothing_\vartheta\}. \equiv (\forall x_1,\dots,x_m).f(x_1,\dots,x_m)=g(x_1,\dots,x_m)$$

is true at $\mathcal{V} \in V^\alpha$ and $\mathcal{I} \in I^\alpha$ whenever (i) $f,g,\varnothing_\vartheta \in \mathrm{E}_\vartheta$ *and* $a^* = a^*_\vartheta(\in \mathrm{E}_\vartheta)$ *(see* (4.6)*) with $\vartheta = \langle t_1,\dots,t_m,t_0\rangle \in \tau_\nu$,* (ii) *$x_1$ to x_m are m variables of the respective types t_1 to t_m,* (iii) $\delta_i =_{def} x_i^{\mathrm{ord}} \geq \max(f^{\mathrm{ord}},g^{\mathrm{ord}})$, *and* (iv) *$\varnothing_\vartheta$ denotes (at $\mathcal{I}$ and $\mathcal{V}$) the empty attribute or function of type ϑ (see Definition* (13.15) *below).*

Proof: Let L (R) be the left (right) hand side of the formal equivalence (13.1). Then the $\mathcal{L}''_\alpha$-designatum $\bar{\mathrm{R}} = \mathrm{des}_{\mathcal{I}\mathcal{V}}(\mathrm{R})$, where $\mathcal{I} \in I^\alpha$ and $\mathcal{V} \in V^\alpha$, depends on the designata $\overline{f(\ )}'$ and $\overline{g(\ )}'$ of $f(x_1,\dots,x_m)$ and $g(x_1,\dots,x_m)$ respectively at $\mathcal{I}$ and $\mathcal{V}'$ (see (5.11)) for all ξ $(= \langle\xi_1,\dots,\xi_m\rangle) \in \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m}$.

In order to evaluate these designata, let $\hat{x}_i^f$ ($\hat{x}_i^g$) be the object $\hat{\Delta}$ defined by (11.2) in case Δ_i, $\mathcal{V}$, and Δ_0 are replaced by x_i, $\mathcal{V}'$, and f (g) respectively ($i = 1,\dots,m$). We have that:

$$(13.2)\quad \delta =_{def} \bar{f}^{\mathrm{ord}} \leq f^{\mathrm{ord}} \leq \delta_i =_{def} x_i^{\mathrm{ord}},\ \beta =_{def} \bar{g}^{\mathrm{ord}} \leq x_i^{\mathrm{ord}}\ (< \alpha)\ (i = 1,\dots,m).$$

Furthermore, e.g. $\bar{f} =_{def} \mathrm{des}_{\mathcal{I}\mathcal{V}}(f)$ is either $\mathbf{F}$ or a couple $\langle \bar{f}_1,\delta\rangle$, where $\bar{f}_1$ $(=_{def}(\bar{f})_1)$ is an m-ary function. We have that:

$$(13.3)\quad \mathfrak{D}_{\bar{f}_1} \subseteq \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m} \text{ if } \bar{f} \neq \mathbf{F} \quad (\mathfrak{D}_{\bar{g}_1} \subseteq \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m} \text{ if } \bar{g} \neq \mathbf{F}).$$

Now assume that $\bar{f} \neq \mathbf{F}$ and $\xi \in \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m}$, so that $\bar{f} \neq \langle\varnothing,0\rangle$. Then $\xi_i \in \mathrm{QS}_{t_i}^{<\delta_i} \cup \mathrm{QE}_{t_i}^{\delta_i \not<}$ ($i = 1,\dots,m$); and since $\delta \leq \delta_i = x_i^{\mathrm{ord}}$, for some $\mathcal{V}' \in V^\alpha$ (5.11) holds. Hence by (11.2) $\hat{x}_i^f = \xi_i$ ($\hat{x}_i^f = \xi_i^{\mathrm{E}}$) for $\xi_i^{\mathrm{ord}} < \delta$ ($\xi_i^{\mathrm{ord}} \geq \delta$); and the analogue holds for $\hat{x}_i^g$ because $\beta \leq x_i^{\mathrm{ord}}$ ($i = 1,\dots,m$). Then by (12.7)

$$(13.4)\quad \hat{x}^f = \langle \hat{x}_1^f,\dots,\hat{x}_m^f\rangle = \xi^{\mathrm{E}\delta} \qquad (\hat{x}^g = \xi^{\mathrm{E}\beta}).$$

By (13.3), (α_f) *if $\bar{f} \neq \mathbf{F}$, when ξ ranges over* $\mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m}$, *$\hat{x}^f$ ranges over a set containing $\mathfrak{D}_{\bar{f}_1}$ (and $\hat{x}^f = \xi$ for $\xi \in \mathfrak{D}_{\bar{f}_1}$).* Likewise, ($\alpha_g$) holds.

Furthermore, by (13.4)

$$(13.5)\quad \delta = \beta \Rightarrow (\hat{x}^f = \hat{x}^g \text{ for all } \xi \in \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m}).$$

Step 1: The $\supset$-part L $\supset$ R *of* (13.1) *holds.*

Assume the truth of L (at $\mathcal{I}$ and $\mathcal{V}$). Then either $\{\bar{f},\bar{g}\} = \{\mathbf{F},\langle\varnothing,0\rangle\}$ or $\bar{f} = \bar{g}$. In the first alternative, by (h''_2) (which is (h'_2) in N11) R is true because $\overline{f(\ )}' = \mathbf{F} = \overline{g(\ )}'$ for all $\xi \in \mathrm{A}_{t_1}^{\delta_1} \times \dots \times \mathrm{A}_{t_m}^{\delta_m}$. The other alternative implies both $\mathfrak{D}_{\bar{f}_1} = \mathfrak{D}_{\bar{g}_1}$ and $\delta = \beta$, so that $(13.5)_1$ implies $(13.5)_2$. Then by assertions (α_f) and (α_g), and by rule (h''_2), R is true again. Thus Step 1 holds.

Step 2: The part R $\supset$ L *of* (13.1) *also holds.*

Indeed let R $\supset$ L be false, as a hypothesis for a *reductio ad absurdum*. Then at some $\mathcal{I} \in I^\alpha$ and $\mathcal{V} \in V^\alpha$, R is true and L is false. By this falsity

(13.6) $\mathcal{S} =_{def} \{\mathbf{F}, \langle\varnothing,0\rangle\} \neq \{\bar{f},\bar{g}\}, \bar{f} \neq \bar{g}$.

Then at most one of the QE's $\bar{f}$ and $\bar{g}$ is inside the set $\mathcal{S}$. Furthermore f and g have symmetric roles in Theorem 13.1. Hence it suffices to consider the case $\bar{f} \notin \mathcal{S}$; in full detail:

(13.7) $\bar{g} \neq \bar{f} \neq \mathbf{F}, \bar{f}_1 = (\bar{f})_1 \neq \varnothing$ (hence $\bar{f} \notin \mathcal{S}$).

In the subcase $\bar{g} = \mathbf{F}$ or $\bar{g}_1 = (\bar{g})_1 = \varnothing$, for some $\xi \in \mathfrak{D}_{\bar{f}_1}$, $\overline{f(\ )}' \neq \mathbf{F} = \overline{g(\ )}'$. This contrasts with R's truth, deduced above. Thus Step 2 holds in this subcase ($\bar{g} \in \mathcal{S}$).

In the remaining subcase we have that:

(13.8) $\bar{f} \neq \bar{g}$, $\mathfrak{D}_{\bar{f}_1} \neq \varnothing \neq \mathfrak{D}_{\bar{g}_1}$ $\quad$ ($\{\bar{f},\bar{g}\} \cap \mathcal{S} = \varnothing$).

The additional assumptions $\delta = \beta$—i.e. $(\bar{f})_2 = (\bar{g})_2$—yields $\bar{f}_1 \neq \bar{g}_1$ and, by (13.5), assertion $(13.5)_2$ too. If we had $\mathfrak{D}_{\bar{f}_1} \neq \mathfrak{D}_{\bar{g}_1}$, then for some $\sigma \in (\mathfrak{D}_{\bar{f}_1} - \mathfrak{D}_{\bar{g}_1}) \cup (\mathfrak{D}_{\bar{g}_1} - \mathfrak{D}_{\bar{f}_1})$ only one of the QE's $\overline{f(\ )}'$ and $\overline{g(\ )}'$ would be $\mathbf{F}$ by rule ($h\frac{1}{2}$) in N11 and by (α_f) and (α_g). Thus R would be false, which is absurd. Hence $\mathfrak{D}_{\bar{f}_1} = \mathfrak{D}_{\bar{g}_1}$.

However, $\bar{f}_1 \neq \bar{g}_1$ and $(13.5)_2$ hold. Then $\bar{f}_1(\hat{x}^f) \neq \bar{g}_1(\hat{x}^g)$ for some $\xi \in \mathfrak{D}_{\bar{f}_1}$. Hence $\overline{f(\ )}' \neq \overline{g(\ )}'$ by rule ($h\frac{1}{2}$) in N11. Thus R would be false again. Therefore, in the aforementioned remaining subcase we must have $\delta \neq \beta$ and $(13.8)_{2-3}$. Hence it is not restrictive to assume that

(13.9) $\delta < \beta$ $\quad$ (and $\mathfrak{D}_{\bar{f}_1} \neq \varnothing \neq \mathfrak{D}_{\bar{g}_1}$).

Now choose arbitrarily

(13.10) $\xi \in \mathrm{A}_{t_1}^{\delta_1} \times \ldots \times \mathrm{A}_{t_m}^{\delta_m}$ (see (13.2) and (iii) following (13.1)).

Then, by R's truth, rule ($h\frac{1}{2}$) in N11, and (13.4)

(13.11) $\bar{f}_1(\xi^{\mathrm{E}\delta})^\dagger = \bar{f}_1(\hat{x}^f)^\dagger = \overline{f(\ )}' = \overline{g(\ )}' = \bar{g}_1(\hat{x}^g)^\dagger = \bar{g}_1(\xi^{\mathrm{E}\beta})^\dagger$.

Let us also assume (iv) $\sigma \in \mathfrak{D}_{\bar{g}_1}$, so that $(13.3)_2$ yields (v) $\xi^{\mathrm{E}\beta} = \sigma$ for some $\xi \in \mathrm{A}_{t_1}^{\delta_1} \times \ldots \times \mathrm{A}_{t_m}^{\delta_m}$. For any such ξ, (13.11) holds with (vi) $\bar{f}_1(\xi^{\mathrm{E}\delta}) = \bar{g}_1(\xi^{\mathrm{E}\beta}) \neq \mathbf{F}$. Then, by rule ($h\frac{1}{2}$) in N11, (vii) $\rho =_{def} \xi^{\mathrm{E}\delta} \in \mathfrak{D}_{\bar{f}_1}$. We can now show that:

(β) *for* $i = 1$ *to* m, *either* (viii) $\rho_i \in \mathrm{QS}_{t_i}^{<\delta} - \mathrm{QE}_{t_1}^{\delta}$ *and* (ix) $\rho_i = \xi = \sigma_i$, *or (otherwise)* (x) $\rho_i \in \mathrm{QE}_{t_i}^{\delta}$ *and both* (xi) $\xi_i^{\mathrm{E}\delta} = \rho_i$ *and* (xii) *either* $\sigma_i = \rho_i$ *or* $\sigma_i = \xi_i$ ($\neq \rho_i$ *for* $\rho_i^{\mathrm{ord}} < \xi_i^{\mathrm{ord}} < \beta$).

In fact, first (v), (vii), (viii), (13.9), and (12.7) yield (ix). Second, (vii) implies (xi) in any case. Third, assume the falsity of (viii) and let (xii) be false as a hypothesis for a *reductio ad absurdum*. Then (a) $\xi_i \neq \sigma_i \neq \rho_i$. Hence by (v) and (vii), (b) $\xi_i^{\mathrm{E}\beta} = \sigma_i \neq \rho_i = \xi^{\mathrm{E}\delta}$. By $(a)_1$, $(b)_1$, and (12.7), (c) $\xi_i^{\mathrm{E}} = \sigma_i \in \mathrm{QE}_{t_i}^{\beta\not<}$. Then $\xi_i^{\mathrm{ord}} \geq \sigma_i^{\mathrm{ord}} = \beta > \delta \geq \rho_i^{\mathrm{ord}}$. Hence $\xi_i \neq \rho_i$, so that by (xi) and (12.7), $\xi_i^{\mathrm{E}} = \rho_i \in \mathrm{QE}_{t_i}^{\delta\not<}$, which contradicts (c) and $(b)_2$. Thus (on the third place) the falsity of (viii) implies (xii). Therefore (β) holds.

Since (iv) yields (β) and, by (2.7), (β) yields $\sigma^{E\delta} = \rho$, we conclude that

(γ) *if* (iv) $\sigma \in \mathfrak{D}_{\bar{g}_1}$, *then* (xiii) $\sigma^{E\delta} = \rho$ *for some* $\rho \in \mathfrak{D}_{\bar{f}_1}$; *hence*

(13.12) $\mathfrak{D}_{\bar{g}_1} \subseteq \mathfrak{D}_{\beta,\bar{f}_1} =_{def} \{\sigma \in A^{\beta}_{t_1} \times \ldots \times A^{\beta}_{t_m} | \sigma^{E\delta} \in \mathfrak{D}_{\bar{f}_1}\}$ $\qquad (\delta = \bar{f}^{\text{ord}})$.

Now conversely assume (xiv) $\sigma \in \mathfrak{D}_{\beta,\bar{f}_1}$, i.e. (xiii), and (xv) $\sigma \in A^{\beta}_{t_1} \times \ldots \times A^{\beta}_{t_m}$. Furthermore set $\xi = \sigma$, hence (xvi) $\xi^{E\beta} = \sigma^{E\beta} = \sigma$. By (xv) and (13.2)—so that $x_i^{\text{ord}} \geq \beta$ for $i = 1$ to m—some $\mathcal{V}' \in V^{\alpha}$ renders (5.11) true. Then (13.4) holds (see below (13.3)). Furthermore, on the hypothesis that R $\supset$ L is false (made below Step 2) (13.11) holds. Therefore (xvi) and (xiii) (whence $\sigma^{E\delta} \in \mathfrak{D}_{\bar{f}_1}$) imply that $\bar{f}_1(\sigma^{E\delta}) = \bar{g}_1(\sigma^{E\beta}) = \bar{g}_1(\sigma) \neq \mathbf{F}$; hence (xvii) $\sigma \in \mathfrak{D}_{\bar{g}_1}$. Thus (xiv) entails (xvii), and we conclude that $\mathfrak{D}_{\beta,\bar{f}_1} \subseteq \mathfrak{D}_{\bar{g}_1}$. Hence, by $(13.12)_1$, (xviii) $\mathfrak{D}_{\bar{g}_1} = \mathfrak{D}_{\beta,\bar{f}_1}$.

Since (13.10) implies (13.11), by (α_f) and (α_g) (xv) yields $\bar{g}_1(\sigma) = \bar{f}_1(\sigma^{E\delta})$. Hence, by (xviii) and (12.12), $\bar{g}$ is the β-mate $m^{\beta}(\bar{f})$ of $\bar{f}(\in \text{QE}^{\delta\not<}_{\vartheta})$. Then, by (12.13), (xix) LQE$(\bar{g}) \neq \varnothing$.

However, since $\bar{g} \in \text{QE}^{\beta\not<}_{\vartheta}$ (so that $\bar{g} \notin \text{QE}^{<\beta}_{\vartheta}$), by (12.14) $\bar{g} \in \text{PQE}^{\beta}_{\vartheta}$ and LQE$(\bar{g}) = \varnothing$, which contradicts (xix). This absurd consequence of the assumed falsity of R $\supset$ L yields the truth of R $\supset$ L. Thus Step 2 also holds. This completes the proof of Theorem 13.1

Let us now show that the version of $\text{QE}^{\beta}_{\vartheta}$ defined for $\mathcal{L}''_{\alpha}$ in the above recursive step (12.14) is the maximal version of $\text{QE}^{\beta}_{\vartheta}$ that renders the wff (13.1) true according to rule (h''_2), and that satisfies the natural condition $\text{QE}^{\beta}_{\vartheta} \subseteq \text{QE}^{<\beta}_{\vartheta} \cup \text{PQE}^{\beta}_{\vartheta}$.

Thus it is clear that our $\text{QE}^{\beta}_{\vartheta}$'s for $\mathcal{L}''_{\alpha}$ fail to be some sorts of (proper) *general* $\text{QE}^{\beta}_{\vartheta}$'s, i.e. some $\text{QE}^{\beta}_{\vartheta}$'s for the so-called *general models* which in some logics are used to prove completeness theorems.

Theorem 13.2 *Fix $\beta < \alpha$ and $\vartheta = \langle t_1, \ldots, t_m, t_0\rangle \in \tau_\nu$; furthermore assume that*

(13.13) $\text{QE}^{\beta}_{\vartheta} \subseteq \text{QE}^{*\beta}_{\vartheta} \subseteq \text{QE}^{<\beta}_{\vartheta} \cup \text{PQE}^{\beta}_{\vartheta} \qquad (\text{QE}^{\beta}_{\vartheta} = \text{QE}''^{\beta}_{\vartheta})$

and that $\text{QE}^{*\beta}_{\vartheta}$, *regarded as the set of the* QE's *(for $\mathcal{L}''_{\alpha}$) of orders $\leq\beta$ and type ϑ, is compatible with Theorem 13.1—as well as the set* $\text{QE}^{\beta}_{\vartheta}$ *defined by clause* (12.14). *Then* (i) $\text{QE}^{*\beta}_{\vartheta} = \text{QE}^{\beta}_{\vartheta}$.

Proof: Let (i) be false as a hypothesis for a *reductio ad absurdum*. Then by (13.13) and (12.14) there is some $\zeta \in \text{QE}^{*\beta}_{\vartheta}$ with LQE$(\zeta) \neq \varnothing$. Hence by (12.13), for some $\eta \in \text{QE}^{<\beta}_{\vartheta}$, $\zeta = m^{\beta}(\eta)$ $(\eta^{\text{ord}} < \zeta^{\text{ord}} = \beta)$. At this point it is easy to show the falsity of R $\supset$ L for $\bar{f} = \eta$ and $\bar{g} = \zeta$ where L (R) is the left (right) hand side of the wff (13.1). Thus Theorem 13.1 is violated. Therefore (i) must hold.

It is easy to check the validity in $\mathcal{L}''_{\alpha}$ of the wff

(13.14) $\Delta = a^*_{\vartheta} \supset (\forall x_1, \ldots, x_m).\Delta(x_1, \ldots, x_m) = a^*_{t_0}$ (see (4.6) and $(1.1)_2$)

where $\vartheta = \langle t_1, \ldots, t_m, t_0\rangle \in \tau_\nu$ and $\Delta \in E_{\vartheta}$.

Hence if Δ is a nondenoting functor (attribute) of $\mathcal{L}''_{\alpha}$, then $\Delta(\Delta_1, \ldots, \Delta_n)$ is a nondenoting (false) wfe.

Consider the metalinguistic definition

(13.15) $\varnothing_\vartheta =_{def} (\imath f).f \neq a_\vartheta^* \wedge (\forall y_1,\ldots,y_m).f(y_1,\ldots,y_m) = a_{t_0}^*$ $(\vartheta = \langle t_1,\ldots,t_m,t_0\rangle)$

where y_r is $v_{t_r,r}^0$ $(r = 1,\ldots,m)$. And remark that in the *ordinary (refined) version of* $\mathcal{L}_\alpha''$, I mean the one where $\mathbf{F} = (\neq) \langle\varnothing,0\rangle$ is assumed, the first (second) of the wff's

(13.16) $\varnothing_\vartheta = a_\vartheta^*, \varnothing_\vartheta \neq a_\vartheta^*$ $(\vartheta = \langle t_1,\ldots,t_m,t_0\rangle \in \tau_\nu)$

is valid. Furthermore, both this wff and (13.1) can be used as axioms for a logical calculus based on $\mathcal{L}_\alpha''$.

N14 The Validity in $\mathcal{L}_\alpha'$ and $\mathcal{L}_\alpha''$ of a general version of the instantiation axiom Consider the instantiation axiom scheme—which includes (1.2)

(14.1) $(\forall x)F(x). \supset F(\Delta)$, Δ being free for x in $F(x)$ (see Convention 3.2)

where either (i) $\delta =_{def} x^{\text{ord}} \geq F(\Delta)^{\text{ord}} = 0$, or (ii) $\delta \geq \delta_\Delta =_{def} \Delta^{\text{ord}}$ and Δ's length l_Δ equals 1, or else (iii) $\delta > \delta_\Delta$.

The assertion in N1 that embodies (1.2) is included in the following:

Theorem 14.1 *Axiom scheme* (14.1) *is valid in both* $\mathcal{L}_\alpha'$ *and* $\mathcal{L}_\alpha''$.

Proof: Assume (a) $\text{des}_{\mathcal{I}\mathcal{V}}((\forall x)F(x)) = \mathbf{T}$ and (i). Then (in any of the languages $\mathcal{L}_\alpha'$ and $\mathcal{L}_\alpha''$) for some $\mathcal{V}' \in V^\alpha$

(14.2) $\overline{\mathbf{F}(x)}' =_{def} \text{des}_{\mathcal{I}\mathcal{V}'}[F(x)] = \mathbf{T}$ for $\xi_1 = \bar{\Delta} =_{def} \text{des}_{\mathcal{I}\mathcal{V}}(\Delta)$

(see (5.11) with $m = 1$). Furthermore $F(\Delta)$ belongs to the ordinary extensional segment $\mathcal{L}_1$ of $\mathcal{L}_\alpha$. Hence the truth of $F(\Delta)$ (at $\mathcal{I}$ and $\mathcal{V}$) can be deduced in a usual way. Thus (14.1) is (logically) valid in case (i).

Now replace assumption (i) with either (ii) or (iii); and set (b) $\xi_1 = \breve{\Delta} =_{def} \text{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$. Then the $\mathcal{V}' \in V^\alpha$ given by (5.11) for $m = 1$ exists. Thus, in obvious notations, (c) $\breve{x}' = \xi_1 = \breve{\Delta}$ and hence (d) $\bar{x}' = \bar{\Delta}$. Furthermore (a) implies (e) $\overline{F(x)}' = \mathbf{T}$ again.

Briefly speaking, let $\phi(x)$ and $\phi(\Delta)$ be any two corresponding sub-wfe's of $F(x)$ and $F(\Delta)$ respectively. By (c) and (d), it is a matter of routine to deduce (f) $\phi(x)^{\breve{}\,\prime} = \phi(\Delta)^{\breve{}}$ and hence (g) $\overline{\phi(x)}' = \overline{\phi(\Delta)}$, by induction on the length of $\phi(x)$. When $\phi(x)$ is $F(x)$, (g) and (f) yield that $\overline{F(\Delta)} = \mathbf{T}$. Thus (14.1) is also valid in cases (ii) and (iii).

It is easy to check the validity (in, e.g., $\mathcal{L}_\alpha''$) of any wfe

(14.3) $(\exists y)y = \Delta$ with y not occurring free in Δ,

where Δ's type t_Δ is in $\{0,\ldots,\nu\}$—see $(9.3)_1$—or $t_\Delta \in \tau_\nu$ and $y^{\text{ord}} \geq \Delta^{\text{ord}}$. Hence these wff's have to be (axioms or) theorems of any (strong) logical calculus $\mathcal{LC}_\alpha''$ based on $\mathcal{L}_\alpha''$.

Now consider any wff

(14.4) $(\forall x)(\exists y)(y = x \wedge G(x)) \supset G(\Delta)$ with Δ free for x in $G(x)$,

where (α) y fails to occur in $x = x \wedge G(\Delta)$, (β) $y^{\text{ord}} = \Delta^{\text{ord}}$ (or $t_\Delta \in \{0,\ldots,\nu\}$), and either (ii) or (iii) (below (14.1)) holds.

Note that, in effect, in e.g. (14.4), $(\forall x)(\exists y)(y = x \wedge \ldots)$ acts as a restricted operator that can roughly be translated into "for all entities x of type t_Δ, whose sense belongs to $\mathcal{L}''_{\delta_\Delta+1}, \ldots$". Therefore wff (14.4) (roughly) asserts that $G(\Delta)$ occurs whenever $G(x)$ holds for all the above entities x. Furthermore any (some) of these is the $\mathrm{sens}_{\mathcal{I}\mathcal{V}}(\Delta)$ for some (an arbitrarily prefixed) δ, $\mathcal{I}$, and $\mathcal{V}$. Hence the antecedent of (14.4) appears neither too weak nor too strong; thus (14.4) appears as a certainly satisfactory version of the instantiation principle (for $\mathcal{L}'_\alpha$ or $\mathcal{L}''_\alpha$), in case (iii).

By the above considerations on (14.3), if (h) $G(x)$ and $G(\Delta)$ are $F(x)$ and $F(\Delta)$, respectively, then for $t_\Delta \in (\notin)\{0, \ldots, \nu\}$ the antecedent of (14.1) appears equivalent to (stronger than) the one of (14.4). In more detail (in, e.g., $\mathcal{L}''_\alpha$) the former antecedent roughly says, in case (iii): $F(x)$ holds for all entities whose sense is expressible in $\mathcal{L}''_{\delta_\Delta+1}$ or whose extension is designatable in $\mathcal{L}''_{\delta_\Delta+2}$. Therefore, if case (iii) holds and $t_\Delta \notin \{0, \ldots, \nu\}$, the antecedent of (14.1) appears superabundant; and (on the assumption (h)) (14.4) appears as a logical consequence of (14.1). In addition, in case (i) or (ii), (14.1) is certainly satisfactory (like (14.4) in case (iii)). Therefore—see Theorem 14.2 below—(14.1) can be the instantiation axiom of, e.g., a satisfactory lower predicate calculus LPC''_α based on $\mathcal{L}''_\alpha$.

Clearly LPC''_α is expected to contain any ordinary axiom system for the propositional calculus (obviously valid in $\mathcal{L}''_\alpha$, $\mathcal{L}'_\alpha$, and $\mathcal{L}_\alpha$) and, e.g., the following (certainly valid) axiom (scheme).

(14.5) $(\forall x)(p \supset q) \supset .\, p \supset (\forall x)q$ *with x not occurring free in p.*

The inference rules for LPC''_α can also be assumed to be *modus ponens* and *multiple generalization of axioms*.

Theorem 14.2 *Any wff* (14.4) *is a theorem of* LPC''_α.

Proof: Briefly speaking, identify, e.g., $F(\Delta)$ with $(\exists y)[y = \Delta \wedge G(\Delta)]$, so that (iii) holds and (14.1) is an axiom. Furthermore, assume $(\forall x)F(x)$ formally—i.e. in a deduction within LPC''_α. Then (14.1) yields $F(\Delta)$, which by (α) (below (14.4)) is equivalent to $(\exists y)\, y = \Delta \,.\wedge\, G(\Delta)$ according to any lower predicate calculus. Thus $G(\Delta)$ can be deduced.

Remark. *For* $x^{\mathrm{ord}} = \beta =_{def} F(\Delta)^{\mathrm{ord}} > 0$, *wff* (14.1) *can be false* (*in* $\mathcal{L}'_\alpha$ *or* $\mathcal{L}''_\alpha$).

The following example shows this for $\beta = 1$. Briefly speaking, assume that—see convention 3.1—$Q^1 \in \mathrm{E}_{(1)}$, $f^0 \in \mathrm{E}_{(1;1)}$, P^0 and Δ are in E^1_1 as well as x^1 and y^1, and that the above wfe's Q^1 to Δ are closed. Furthermore let $\mathcal{I} \in \mathrm{I}^\alpha$ be such that, for all $\mathcal{V} \in V^\alpha$ (in obvious notations) (a) $\check{P} = \bar{P} = \overline{f^0(\Delta)}$ (hence $\check{P}$ is ostensive), (b) $\check{P}^{\mathrm{ord}} = 0$, (c) $\check{\Delta}^{\mathrm{ord}} = 1$, and (d) $\overline{\mathrm{Q}^1(y^1)} = \mathbf{T}$ *iff* $\check{y}^1 = \bar{P}$ (which determines $\overline{\mathrm{Q}^1}$). Then, for all $\mathcal{V} \in V^\alpha, f^0(x^1)\check{}$ is $\neq \check{P}$ and has the order zero because $x^1 \in \mathrm{E}_1$ and $(9.3)_1$ holds. Hence, by (d) and rule $(\mathrm{h}''_2) = (\mathrm{h}'_2)$ in N11, (e) $\overline{\mathrm{Q}^1(f^0(x^1))} = \mathbf{T}$ *for all* $\mathcal{V} \in V^\alpha$.

Now we identify x with x^1 and $F(x)$ ($F(\Delta)$) with $\sim\mathrm{Q}^1[f^0(x)]$ ($\sim\mathrm{Q}^1[f^0(\Delta)]$). Then, by (e), the antecedent of (14.1) is (closed and) true. Instead the consequent $F(\Delta)$ is false. Indeed, by (c), $f^0(\Delta)\check{}^{\,\mathrm{ord}} = 1$; hence, briefly, the predicate $\overline{\mathrm{Q}^1}$ cannot be sensitive to $f^0(\Delta)\check{}$ and, by rule $(\mathrm{h}''_2) = (\mathrm{h}'_2)$ in N11, it perceives $\overline{f^0(\Delta)}$

($=\bar{P}$ by (a)). Thus $\overline{\mathrm{Q}^1[f^0(\Delta)]} = \overline{\mathrm{Q}^1(P)} = \mathbf{T}$ by (d). Hence $F(\Delta)$ is false and the same holds for (4.1).

NOTES

1. In [6] Bressan uses an unusual extensional language in which, e.g., the (primitive) matrix of the ordinary language *P is the position of the mass point M at the instant* ϑ, briefly $P = \mathrm{pos}^*(M, \vartheta)$, is replaced by the matrix *P is... instant* ϑ, *in the* Γ*-case* (*or possible world*) γ, briefly $P = \mathrm{pos}(M, \vartheta, \gamma)$. Note that the number of the argument places of the functions pos* and pos are different. The foundations of mechanics according to Mach and Painlevé can be based on such a language, in spite of physically possible phenomena, and hence modalities, being essential for that purpose.

   The above substitution of $\mathrm{pos}^*(M, \vartheta)$ with $\mathrm{pos}(M, \vartheta, \gamma)$ has to be performed at the intuitive level when the language of [6] is used, while no such analogue is required when the modal language ML^ν (see [7]) is chosen as logical basis. Hence the formalization carried out in [6] appears partial. Incidentally, ML^ν can deal with very general modal contexts, for which the substitutions required at the intuitive level on the basis of [6] would be much more complex than the above one.

   By means of $\mathcal{SL}_\alpha^\nu$ a strong formalization is performed, so that this theory differs much from, e.g., Quine's work in [22]. In constructing $\mathcal{SL}_\alpha^\nu$ powerful results were preferred to simplicity.

2. For the sake of simplicity it is assumed here that the nonexisting relation (function) coincides with the empty set, and hence with the empty relation (function).

3. Definition (2.4) is criticized because it may happen that (i) p happens to be true, (ii) $\mathfrak{M}$ believes this, but (iii) he reaches this belief by an incorrect procedure, and (iv) if $\mathfrak{M}$ became aware of this incorrectness, he would change that belief. Thus definition (2.4) may appear not to be adequate in this case.

   However, now we are considering the notion of knowing from a rigorous point of view; and this is closer to science than to everyday life. Thus, in accordance with some ideas of Carnap (see [16], pp. 3–7) the rather vague everyday notion of knowing is an explicandum to be replaced with possibly several exact notions, called by him explicata, which need not be very similar to the explicandum, but are expected to fulfill other requirements, possibly including simplicity. I think that definition (2.4) affords us one of these explicata, which is useful in many situations—including the one described by (i) to (iv)—and is particularly simple.

4. In order to comply with the treatment of $\mathcal{SL}_\alpha^\nu$ in [10], we ought to write $0 < \mu < \alpha + \omega$. However, only the condition $0 < \mu < \beta + \omega$ (which practically fails to affect that treatment) renders, e.g., the λ-th segment $\mathcal{L}_\lambda$ of $\mathcal{L}_\alpha$ a language of the same type as $\mathcal{L}_\alpha$ $(0 < \lambda \leq \alpha)$.

5. The examples considered for Case C1 and based on (4.5) or (4.8)–(4.10) can also be formulated within $\mathcal{L}_\alpha^\infty$. This can be done directly if a variable x^0 of order zero belongs to $\mathcal{L}_\alpha^\infty$, or indirectly—i.e. by restricting x suitably—if every variable x of $\mathcal{L}_\alpha^\infty$ is orderless.

6. For instance, the intension of an entity Δ can be defined within ML^ν itself as *the intensional singleton* $\{\Delta\}^i$ of Δ, where $\{a_1, \ldots, a_n\}^i =_{def} (\lambda x)\Box x = a_1 \vee \ldots \vee \Box x = a_n$ (see Definition 18.13 in [7], p. 68).

7. By definitions (4.8)–(4.10), under suitable circumstances, the term "George's birth in Rome", say Δ, has the same sense—or more precisely the same ambiguity-sense (considered for 'dormouse' in N2)—as the term y. Thus, by the considerations above (involving (4.7)–(4.13)), in the same circumstances the nonexistence of George's birth in Rome is equivalent to p's falsity.

   In that case Δ can be reasonably translated in $\mathcal{L}_\alpha$ by a wfe in E_0 or $E_{(0)}$; e.g. by y (see (4.8.1) and (4.7), or (4.9.1)). Of course Δ has a complex meaning; in other circumstances Δ may express the process of George's birth (supposed to occur in Rome) with its starting and ending instants determined according to the criteria of some particular physician. Then Δ's translation in $\mathcal{L}_\alpha$ may perhaps be similar to (4.7) or (4.9.1), but certainly more complex and possibly of a type $\neq 0$. Since the ostensivization symbol $\mathfrak{O}$ introduced in Bressan [12] is expected to simplify such translations, some proposals for them are planned to be given in future works.

8. According to device (D) one has to interpret e.g. (D_1) $\Delta_1 \supset \Delta_2$ as *if* $\tilde{\Delta}_1$, *then* $\tilde{\Delta}_2$, where $\tilde{\Delta}$ stands for Δ *is a true proposition*, (D_2) $(\forall x)\Delta$ as *for every* x $\tilde{\Delta}$, and (D_3) $(\imath x)\Delta$ as *the* x *such that* $\tilde{\Delta}$. Furthermore, more generally, every possibly nonlogical operator $(\Omega x_1, \ldots, x)$ to be applied to assertions can also be applied to terms by interpreting $(\Omega x_1, \ldots, x_n)\Delta$ as $(\Omega x_1, \ldots, x_n)\tilde{\Delta}$.

9. Theorem 6.1,λ for $0 < \lambda \leq \alpha$ is in effect a particular case of Theorem 6.1 asserted and proved in [12]. More precisely, (6.5) is in effect implication $(6.1)_3$ in [12].

10. The recursion hinted at by means of clauses ($\mathcal{E}_{1-5}$) concerns, not the types of the QS$^\beta$'s, but their levels (which roughly speaking are the levels of the simplest wfe's capable of denoting them).

11. Consider e.g. a property-QE whose domain contains exactly one QS of order δ for every $\delta < \beta$, a limit ordinal. Its order is β.

12. If ξ is an n-tuple, its ith component is denoted by $\xi_i \quad (i = 1, \ldots, n) : \xi = \langle \xi_1, \ldots, \xi_n \rangle$.

13. Relations (9.3) are in effect stated and proved in Bressan [11]—see note 7.

14. In presenting the theory of $\mathbb{R}$, any mathematician $\mathfrak{M}$ carefully distinguishes $\mathbb{R}^{(\mathbb{Q})}$ from $\mathbb{Q}$; and he defines the set $\mathbb{IR}$ of irrational numbers to be $\mathbb{R} - \mathbb{R}^{(\mathbb{Q})}$. Later, generally for economy of expressions (and thoughts), especially in speaking with physicists, $\mathfrak{M}$ identifies $\mathbb{Q}$ with $\mathbb{R}^{(\mathbb{Q})}$; and practically he says, e.g; "if x is a rational number and y is irrational, then $x + y$ is irrational". Here $\mathfrak{M}$ in effect forgets or disregards his rigorous definitions of $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{IR}$; these definitions have now at most the roles of intuitive characterizations of the notions $\mathbb{Q}$ to $\mathbb{IR}$, which thus are used as primitives of an axiomatic theory.

    Since the difference between $\mathbb{Q}$ and $\mathbb{R}^{(\mathbb{Q})}$ is generally disregarded by physicists (completely), especially in talking with them it is natural for $\mathfrak{M}$ to use the synonymy relations (12.3)–(12.4). Furthermore, the sense in which $\mathfrak{M}$ uses e.g. $\mathbb{Q}$ in this situation can reasonably be regarded as ostensive according to [5], pp. 184–186.

    Something similar is practically required to solve the so-called paradox of analysis:

    (a) P = H (Phosphorous = Hesperous), B(G,P $\neq$ H) (George believes that P $\neq$ H).

    In fact (a) can be solved in $\mathcal{L}_\alpha$ by regarding P and H as constants of orders >0, that are primitives of a theory $\mathfrak{T}$ based on $\mathcal{L}_\alpha$. In $\mathcal{L}_\alpha - \mathfrak{T}$, P (H) can be intuitively characterized by means of a definition (such as those considered in [5], p. 178) as the most brilliant morning (evening) star. Then $(a)_{1,2}$ can be postulated in $\mathfrak{T}$, in that they can be true in $\mathcal{L}_\alpha$'s semantics.

15. Set $S_0 = (\imath F)(\forall G \subseteq \mathbb{R}).F \cup G = G$ and $S_1 = (\imath F)(\forall G \subseteq \mathbb{R}).F \cup G = [0,1]$. Then mathematicians (generally) assert that S_0 exists, S_0 is $\varnothing$, S_1 is not $\varnothing$, and S_1 does not exist—i.e. $a^* \neq S_0 = \varnothing \neq S_1 = a^*$. These assertions contrast with (11.1) and are compatible with (13.1).

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