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The Nonaxiomatizability of $L(Q^2_{\aleph_1})$ by Finitely Many Schemata

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Abstract Under set-theoretic hypotheses, it is proved by Magidor and Malitz that logic with the Magidor–Malitz quantifier in the $\aleph_1$-interpretation is recursively axiomatizable. It is shown here, under no additional set-theoretic hypotheses, that this logic cannot be axiomatized by finitely many schemata.

Magidor and Malitz [2] introduced the n-variable-binding quantifiers Q^n. The language $L(Q^n)$ is formed by adding Q^n to first-order predicate logic. For an infinite cardinal κ, $Q^n x_1 x_2 \ldots x_n \varphi$ may be assigned the so-called κ-interpretation in a structure $\mathfrak{M}$, wherein $Q^n x_1 \ldots x_n \varphi$ is satisfied if there exists an $A \subseteq \mathfrak{M}$ of power κ that is homogeneous for φ, i.e., for any $a_1, \ldots, a_n \in A$, $\varphi(a_1, \ldots, a_n)$ holds in $\mathfrak{M}$. Among many other results Magidor and Malitz establish, under the set-theoretic axiom $\diamondsuit_{\aleph_1}$, a completeness theorem for $L(Q^n)$ in the $\aleph_1$-interpretation (hereafter $L(Q^n_{\aleph_1})$). Unfortunately, the complete axiom system for $L(Q^n_{\aleph_1})$ exhibited in [2] lacks the simplicity of, e.g., Keisler's set of axioms for $L(Q^1_{\aleph_1})$ (cf. [1]).

This paper, a sequel to [3], demonstrates that this failure of simplicity is not without reason. It will be shown here, without additional set-theoretic hypotheses, that $L(Q^2_{\aleph_1})$ cannot be axiomatized by finitely many schemata. Even more strongly, we prove:

Theorem 1 *No collection of axiom schemata of bounded quantifier depth suffices to axiomatize* $L(Q^2_{\aleph_1})$.

This result is due to the first author (Shelah). He communicated it via notes to the second author (Steinhorn), who then prepared this paper. The proof of

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Theorem 1 follows roughly the same plan as that of Theorem 3.1 in [3]. For any possible bound k on quantifier depth of axiom schemas for $L(Q^2_{\aleph_1})$, a model $\mathfrak{M}$ will be constructed, whose $L(Q,Q^2)$-theory under a nonstandard interpretation of Q and Q^2 does not have a standard model (one of power $\aleph_1$ with the $\aleph_1$-interpretation for Q and Q^2), but in which every valid $L(Q_{\aleph_1},Q^2_{\aleph_1})$-schema of quantifier depth less than k holds.

For the convenience of the reader, we include below the definition of an axiom schema from [3]. It will not be used until the very end of the paper. For examples, consult [3].

Definition 1 Let $\mathcal{L}$ be a logic and let $R_1(v_1^1,\ldots,v_{m_1}^1),\ldots,R_n(v_1^n,\ldots,v_{m_n}^n)$ be relation variables. A *schema* is an $\mathcal{L}(R_1,\ldots,R_n)$-formula $\Phi(R_1,\ldots,R_n)$ in which each of the variables v_j^i for $i \le n$ and $j \le m_i$ is bound to a quantifier of $\mathcal{L}$. The *quantifier depth* of the schema is $|\{v_j^i : i \le n \wedge j \le m_i\}|$.

Now we embark upon the sequence of definitions and lemmas leading to the proof of Theorem 1.

Definition 2 Let $k \geq 3$ and $n > k$ (for definiteness, let $n = k^2$). A *k-degenerate* structure $\mathfrak{M} = \langle M \cup \omega, P,<,R,F,G\rangle$ is an L-structure, where the nonlogical symbols of L are a unary predicate symbol P, a binary relation symbol $<$, a binary relation symbol R, an n-place function symbol F, and a k-place function symbol G, satisfying:

(a) $M \cap \omega = \varnothing$, $P(\mathfrak{M}) = \omega$
(b) $<$ linearly orders M
(c) R is a relation on $M \times M$
(d) $F: M^n \to P(\mathfrak{M})$, $G: M^k \to P(\mathfrak{M})$
(e) there is no sequence $\langle a_i : 1 \le i \le 2n\rangle$ from M such that $a_1 < a_2 < \ldots < a_{2n}$, $R(a_i,a_j)$ holds for all $i \ne j$, $F(a_1,\ldots,a_n) = F(a_{n+1},\ldots,a_{2n})$, and $\langle a_i : 1 \le i \le n\rangle \equiv \langle a_j : n+1 \le j \le 2n\rangle$ (mod G) (i.e., for any $1 \le i_1 < i_2 < \ldots < i_k \le n$, $G(a_{i_1},\ldots,a_{i_k}) = G(a_{n+i_1},\ldots,a_{n+i_k})$).

Lemma 1
(i) *Passage to substructures preserves k-degeneracy*
(ii) *If* $\langle \mathfrak{M}_\beta : \beta < \alpha\rangle$ *is an increasing chain of k-degenerate structures, then* $\bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is k-degenerate*
(iii) *If* $\mathfrak{M}_i$, $i = 0,1,2$, *are k-degenerate structures such that* $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathfrak{M}_0$, *then* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *can be amalgamated over* $\mathfrak{M}_0$ *into a k-degenerate structure* $\mathfrak{M}$ *with universe* $\mathfrak{M}_1 \cup \mathfrak{M}_2$ *without introducing any new equalities.*

Proof: The first two assertions are evident, so it remains to verify (iii). For this statement, simply shuffle together the linear orders on M_1 and M_2 and stipulate that $\neg R(a,b)$ holds whenever $a \in M_1 \backslash M_0$ and $b \in M_2 \backslash M_0$, or vice-versa.

The reader should observe that more subtle ways of amalgamation in (iii) may be possible if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy additional conditions (e.g., if $\omega \backslash (F''(M_1^n) \cup F''(M_2^n))$ is infinite and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are countable). We will have to avail ourselves of this additional freedom in what follows.

Let $\mathfrak{D}$ be the set of all quantifier-free L-types with parameters from ω that are realized in k-degenerate models. We then have:

Lemma 2 *Let $p \in \mathfrak{D}$. Then there is a finite $p_0 \subseteq p$ so that, if $\mathfrak{M}$ is k-degenerate, $\mathfrak{M} \models p_0 \to p$. Moreover, $|\mathfrak{D}| = \aleph_0$.*

Proof: Given $p(x_1,\ldots,x_m)$, let p_0 be the set of those formulas in p specifying whether or not $P(x_i)$ holds (and if it does, the element in ω equal to x_i), the complete $<$ and R diagram for pairs (x_i,x_j) for which $\neg P(x_i) \wedge \neg P(x_j)$ holds, and the value in ω of F and G for n-tuples and k-tuples from $\{x_1,\ldots,x_m\}$, respectively. Obviously, p_0 is as desired. To see that $\mathfrak{D}$ is countable, notice that there are but countably many possibilities for p_0 as above.

Lemma 3 *There exists a countable existentially closed k-degenerate structure $\mathfrak{M}^*$, which is unique up to isomorphism over $P(\mathfrak{M}^*) = \omega$.*

Proof: Using amalgamation and preservation under unions of increasing chains, the proof of the existence of $\mathfrak{M}^*$ is routine. Uniqueness follows via a back-and-forth argument, using the first statement of Lemma 2 above to ensure that the construction can be continued.

Lemma 4 *The structure $\mathfrak{M}^*$ is $\mathfrak{D}$-homogeneous and* $\mathrm{Th}_L(\mathfrak{M}^*)$ *admits elimination of quantifiers.*

Proof: The proof of the uniqueness of $\mathfrak{M}^*$ yields, *mutatis mutandis*, $\mathfrak{D}$-homogeneity. Let us now prove the quantifier elimination.

Let $\varphi(y_1,\ldots,y_m) \equiv \exists x\, \psi(x,y_1,\ldots,y_m)$ be given, where $\psi(x,y_1,\ldots,y_m)$ is quantifier-free. We show that $\varphi(y_1,\ldots,y_m)$ is equivalent, relative to $\mathrm{Th}_L(\mathfrak{M}^*)$, to a quantifier-free formula in at most the variables $y_1,\ldots,y_m$. If $\varphi(y_1,\ldots,y_m)$ is not satisfiable in any k-degenerate structure, then clearly $\mathrm{Th}_L(\mathfrak{M}^*) \models \varphi(y_1,\ldots,y_m) \leftrightarrow y_1 \neq y_1$. Thus we assume that $\varphi(y_1,\ldots,y_m)$ is satisfiable in k-degenerate structures.

Let $\mathcal{P} = \&\mathrm{p}(\mathrm{y}_1,\ldots,\mathrm{y}_\mathrm{m})\colon \mathrm{p}(\mathrm{y}_1,\ldots,\mathrm{y}_\mathrm{m}) \in \mathfrak{D} \wedge (\exists \mathfrak{M})(\exists a_1,\ldots,a_m \in \mathfrak{M})$ [M is k-degenerate $\wedge\ \mathfrak{M} \models p(a_1,\ldots,a_m) \wedge \exists x\, \psi(x,a_1,\ldots,a_m)$]$\}$. Next, let $\mathcal{P}_0$ be the collection of all finite $p_0 \subseteq p \in \mathcal{P}$ as guaranteed by Lemma 2. For $p_0 \in \mathcal{P}_0$, let p_1 be obtained by "deleting" the parameters from ω in p_0, e.g., replace $F(y_1,\ldots,y_m) = r$, $G(y_2,\ldots,y_{k+1}) = s$, where $r,s \in \omega$ by $F(y_1,\ldots,y_n)$ $\left\{\begin{smallmatrix}=\\ \neq\end{smallmatrix}\right\} G(y_2,\ldots,y_{k+1})$ according to $r \left\{\begin{smallmatrix}=\\ \neq\end{smallmatrix}\right\} s$. Each such p_1 is finite, and it also is clear that $\mathcal{P}_1$, the set of all such p_1, is finite.

We now claim that

$$\mathrm{Th}(\mathfrak{M}^*) \models \exists x\, \psi(x,y_1,\ldots,y_m) \leftrightarrow \bigvee_{p_1 \in \mathcal{P}_1} (\wedge p_1).$$

The implication from left to right needs no argument, so we turn our attention to the reverse implication. Suppose that $a_1,\ldots,a_m \in \mathfrak{M}^*$ and for some $p_1 \in \mathcal{P}_1$, $\mathfrak{M}^* \models \wedge p_1(a_1,\ldots,a_m)$. Moreover, let $\mathfrak{M}$ be k-degenerate, $c,b_1,\ldots,b_m \in \mathfrak{M}$, and $\mathfrak{M} \models \psi(c,b_1,\ldots,b_m) \wedge\ \wedge p_1(b_1,\ldots,b_m)$. The definition of $\mathcal{P}_1$ assures us of the existence of $\mathfrak{M}$. Since both $(a_1,\ldots,a_m)$ and $(b_1,\ldots,b_m)$ satisfy p_1, we can use $\mathfrak{M} \restriction \omega \cup \{c,b_1,\ldots,b_m\}$ (by permuting ω) to define a k-degenerate structure N with universe $\omega \cup \{a_1,\ldots,a_m,d\}$ — where d is a new element not in $\mathfrak{M}^*$ — so that

$$N \models \psi(d,a_1,\ldots,a_m)$$

and, for every quantifier-free formula $\theta(y_1,\ldots,y_m)$ with parameters from ω

$$N \models \theta(a_1,\ldots,a_m) \text{ iff } \mathfrak{M}^* \models \theta(a_1,\ldots,a_m).$$

Then, by amalgamating N and $\mathfrak{M}^*$ over $\mathfrak{M}^* \upharpoonright \omega \cup \{a_1,\ldots,a_m\}$, since $\mathfrak{M}^*$ is existentially closed, we find that $\mathfrak{M}^* \models \exists x\, \psi(x,a_1,\ldots,a_m)$, as desired.

Definition 3 *The Nonstandard Interpretation of* Q *and* Q^2 *in* $\mathfrak{M}^*$

(a) Let $\mathrm{Q}x\, \psi(x,y_1,\ldots,y_m)$ be an $\mathrm{L}(\mathrm{Q},\mathrm{Q}^2)$-formula and $a_1,\ldots,a_m \in \mathfrak{M}^*$. Then,

$$\mathfrak{M}^* \models \mathrm{Q}x\, \psi(x,a_1,\ldots,a_m)$$
$$\text{iff } (\forall c \in \mathfrak{M}^*\backslash\omega)(\exists b \in \mathfrak{M}^*)\mathfrak{M}^* \models b > c \wedge \psi(b,a_1,\ldots,a_m).$$

(b) Let $\mathrm{Q}^2xy\, \psi(x,y,z_1,\ldots,z_m)$ be an $\mathrm{L}(\mathrm{Q},\mathrm{Q}^2)$-formula and $a_1,\ldots,a_m \in \mathfrak{M}^*$. Then,

$$\mathfrak{M}^* \models \mathrm{Q}^2xy\, \psi(x,y,a_1,\ldots,a_m) \text{ iff}$$
$$(\exists \{b_i : i < \omega\} \subseteq \mathfrak{M}^*\backslash P(\mathfrak{M}^*))$$
$$[i < j \rightarrow b_i < b_j \wedge (\forall i \neq j)\mathfrak{M}^* \models \psi(b_i,b_j,a_1,\ldots,a_m) \wedge \langle b_i : i < \omega\rangle$$

is an unbounded indiscernible sequence in $\mathfrak{M}^*$ for quantifier-free formulas with parameters in $\omega \cup \{a_1,\ldots,a_m\}$ having fewer than k free variables.]

Lemma 5 *Let* Q *and* Q^2 *be interpreted in* $\mathfrak{M}^*$ *as in Definition 3.*
(a) *Let* $\psi(x,y_1,\ldots,y_m)$ *be quantifier-free. Then, for any* $\bar{a} \in \mathfrak{M}^*$,

$$\mathfrak{M}^* \models \mathrm{Q}x\, \psi(x,\bar{a}) \leftrightarrow \exists x\left[\psi(x,\bar{a}) \wedge \neg P(x) \wedge \bigwedge_{1\leq i\leq m} (\neg P(a_i) \rightarrow x > a_i)\right].$$

(b) *Let* $\psi(u,v,y_1,\ldots,y_m)$ *be quantifier-free. Then, for any* $\bar{a} \in \mathfrak{M}^*$,

$$\mathfrak{M}^* \models \mathrm{Q}^2uv\, \psi(u,v,\bar{a}) \leftrightarrow \exists u \exists v\Big[\psi(u,v,\bar{a}) \wedge \neg P(u) \wedge \neg P(v) \wedge u \neq v$$
$$\wedge \bigwedge_{1\leq i\leq m} (\neg P(y_i) \rightarrow u > y_i \wedge v > y_i)$$
$$\wedge \text{ “}u \text{ and } v \text{ realize same quantifier-free type over } \{\bar{a}\} \cup \omega\text{”}\Big].$$

Moreover, for any finite $C \subset \omega$, *there is a witnessing sequence* $\langle b_i : i \in \omega\rangle$ *for* $\psi(u,v,\bar{a})$ *contained in* $\mathfrak{M}^*$ *so that* $G''(\{b_i : i \in \omega\}) \cap C = \varnothing$.

Since the right-hand side of the equivalences in (a) and (b) in Lemma 5 are first-order, it follows immediately from Lemmas 4 and 5 that

Corollary 1 $\mathrm{Th}_{\mathrm{L}(\mathrm{Q},\mathrm{Q}^2)}(\mathfrak{M}^*)$ *admits elimination of quantifiers.*

Proof of Lemma 5(a): Before carrying out the proof, we observe that the interpretation of Q in $\mathfrak{M}^*$ is first-order, so we merely wish to obtain a sharper definition than the original one. To prove the equivalence, we first notice that the direction from left to right is clear, so we argue only for the reverse implication.

Let us abbreviate the right-hand side of the equivalence by $\theta(\bar{a})$. Suppose now that

$$\mathfrak{M}^* \models \theta(\bar{a}).$$

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be copies of $\mathfrak{M}^*$ so that

$$\mathfrak{M}_1 \cap \mathfrak{M}_2 = \omega \cup \{\bar{a}\}$$

and let $c \in \mathfrak{M}_2$ be such that

$$\mathfrak{M}_2 \models \psi(c,\bar{a}) \wedge \neg P(c) \wedge \bigwedge_{1 \le i \le m} (\neg P(a_i) \rightarrow x > a_i).$$

Now, amalgamate $\mathfrak{M}_1$ and $\mathfrak{M}_2$ over $\omega \cup \{\bar{a}\}$ in such a way that for all $d \in \mathfrak{M}_1 \backslash \omega$, $c > d$. Then, since $\psi(x,\bar{a})$ is quantifier-free and $\mathfrak{M}_1$ is existentially closed, it immediately follows that $\mathfrak{M}_1 \models Qx\, \psi(x,\bar{a})$, and we are done.

Proof of Lemma 5(b): Again, only the right to left direction demands proof. Without loss of generality, we may suppose that $\psi(u,v,y_1,\ldots,y_m)$ implies a complete quantifier-free type. It clearly suffices to construct a countable existentially closed structure $\mathfrak{N}$ containing $\omega \cup \{\bar{a}\}$ such that $\mathfrak{N} \models Q^2uv\, \psi(u,v,\bar{a})$, since $\mathfrak{N}$ will be isomorphic to $\mathfrak{M}^*$ over $\omega \cup \{\bar{a}\}$ by Lemma 4.

Let d_1 and d_2 satisfy the right-hand side of the equivalence in (b) and let $C \subset \omega$ be a finite set as in the last clause in (b). Let $\mathfrak{N}$ be a structure with universe $\omega \cup [\bar{a}\} \cup \{b_i : i \in \omega\}$ so that

(i) $\{\bar{a}\}$ has the same quantifier-free type over ω in $\mathfrak{N}$ as in $\mathfrak{M}^*$
(ii) $b_i < b_j$ if $i < j$ and $b_i > d$ for all $i \in \omega$ and $d \in C \cup \{\bar{a}\}$
(iii) for all $i < j$, (b_i, b_j) satisfies the same quantifier-free type over $\omega \cup \{\bar{a}\}$ in $\mathfrak{N}$ as (d_1, d_2) does in $\mathfrak{M}^*$
(iv) all $(k-1)$-tuples in $\{b_i : i \in \omega\}$ satisfy some fixed quantifier-free type over $\omega \cup \{\bar{a}\}$ in $\mathfrak{D}$
(v) n-tuples in $\mathfrak{N}$ for which F is not defined by (i)–(iv) are assigned distinct values of F different from any values of F determined by (i)–(iv)
(vi) k-tuples in $\mathfrak{N}$ for which G is not defined by (i)–(iv) are assigned distinct values of G different from any values of F determined by (i)–(iv), and $G''(\{b_i : i \in \omega\}) \cap C = \varnothing$.

Observe that (vi) does not conflict with (iii) or (iv) since we have assumed that $k \ge 3$, and that clauses (i)–(iv) determine only finitely many values of F and G, which makes it possible to satisfy (v) and (vi). Also, we can satisfy the second part of (ii) because the right-hand side of the equivalence in (b) is satisfiable in $\mathfrak{M}^*$.

Now we show that the structure $\mathfrak{N}$ is k-degenerate. Suppose for a contradiction that

$$e_1 < e_2 < \ldots < e_{2n}$$

is a sequence of elements from $\mathfrak{N} \backslash \omega$ violating k-degeneracy. First, it is clear that $e_i \in \{b_i : i \in \omega\}$ for some $i \le 2n$. Let i_0 be the least one with this property. Also, since the construction only fixes the $k-1$ type of a sequence from $\{b_i : i \in \omega\}$, it follows from (v) and (vi) that $i_0 > 2n - k$. However, for $r < k$, we see that

(iv) forces the quantifier-free type of an r-tuple from $\{b_i : i \in \omega\}$ to be in $\mathfrak{D}$, and so it is not possible that $\bar{e}$ violates k-degeneracy.

We next extend $\mathfrak{N}$ to a countable existentially complete k-degenerate structure $\mathfrak{N}_1$, and let $\mathfrak{N}_2$ be the initial segment of $\mathfrak{N}_1$ cofinal with $\{b_i : i \in \omega\}$. It is a simple matter to verify that $\mathfrak{N}_2$ is existentially complete. Since $\mathfrak{N}_2$ is isomorphic to $\mathfrak{M}^*$ over $\omega \cup \{\bar{a}\}$, we conclude that

$$\mathfrak{M}^* \models Q^2uv\ \psi(u,v,\bar{a}).$$

To show that all true $L(Q_{\aleph_1}, Q^2_{\aleph_1})$-schemes of quantifier depth less than k hold in $\mathfrak{M}^*$, we have to define a version of $\mathfrak{M}^*$ in which F is only a partial function.

Definition 4 Let $\bar{c} = \langle c_1, \ldots, c_m \rangle \in (\mathfrak{M}^*)^m$. Then $\mathfrak{M}^*_{\bar{c}}$ is the L-structure in which the interpretation of all nonlogical symbols but F is as in $\mathfrak{M}^*$, and $F(a_1, \ldots, a_n)$ is defined (and equal to its value in $\mathfrak{M}^*$) only if

$$(\exists \{d_1, \ldots, d_l\} \subseteq \{a_1, \ldots, a_n\}) [|\{a_1, \ldots, a_n\} \setminus \{d_1, \ldots, d_l\}| < k \wedge G''(\{d_1, \ldots, d_l\}) \subseteq G''(\{c_1, \ldots, c_m\})].$$

$\mathfrak{M}^*_{\bar{c}}$ becomes an $L(Q,Q^2)$-structure by interpreting Q and Q^2 as in Definition 3.

We need the following analogue of Lemma 2 in our new setting.

Lemma 6 *For any quantifier-free type p over $\omega \cup \{\bar{c}\}$ realized in $\mathfrak{M}^*_{\bar{c}}$, there is a finite $p_0 \subset p$ so that $\mathfrak{M}^*_{\bar{c}} \models p_0 \to p$.*

Proof: Just modify the proof of Lemma 2.

Lemma 7 *$\mathfrak{M}^*_{\bar{c}}$ is homogeneous for quantifier-free types over ω and $\mathrm{Th}_{L(Q,Q^2)}(\langle \mathfrak{M}^*_{\bar{c}}, \bar{c} \rangle)$ admits elimination of quantifiers.*

Proof: We first prove homogeneity. Suppose, then, that $\bar{a}_1$ and $\bar{a}_2$ are sequences of elements from $\mathfrak{M}^*_{\bar{c}}$ that satisfy the same quantifier-free type over ω. Given $b_1 \in \mathfrak{M}^*_{\bar{c}}$, we must find a $b_2 \in \mathfrak{M}^*_{\bar{c}}$ so that $\bar{a}_1 {}^\frown b_1$ and $\bar{a}_1 {}^\frown b_2$ satisfy the same quantifier-free type over ω. Applying Lemma 6, it is easy to see that there is a finite subtype of the quantifier-free type of b_1 over $\omega \cup \{\bar{a}_1\}$ that implies the whole type. Thus, as $\mathfrak{M}^*$ is existentially closed it suffices to show that we can adjoin an element b_2 to $\mathfrak{M}^*$ in such a way that the resulting structure is k-degenerate, and b_2 satisfies the appropriate finite subtype over $\{\bar{a}_2\} \cup \omega$.

To carry out the easy amalgamation, we first let b_2 be a new element and stipulate that it has the same type over $\omega \cup \{\bar{a}_2\}$ in the sense of $\mathfrak{M}^*_{\bar{c}}$ as does b_1 over $\omega \cup \{\bar{a}_1\}$. Then, we complete the ordering of b_2 with respect to the rest of $\mathfrak{M}^*$ in any consistent way, and set $\neg R(b_2, d)$, for $d \in \mathfrak{M}^*$, where possible. Since only finitely many F and G values have been prescribed to tuples containing b_2 already, we can complete the assignment of F and G values to such tuples by demanding that distinct values different from those previously assigned be given to the remaining tuples. Bearing in mind that the type of $\bar{a}_2 {}^\frown b_2$ over ω to which we are initially committed may not describe the complete type of $\bar{a}_2 {}^\frown b_2$ over ω in the full degenerate structure with universe $\mathfrak{M}^* \cup \{b_2\}$, it is a simple matter to verify that the structure just constructed is k-degenerate.

To prove quantifier elimination, as before, we first eliminate first-order quantifiers and then show how Q and Q^2 can be expressed in first-order logic.

Thus, let $\exists x\, \psi(x,y_1,\ldots,y_l,\bar{c})$ be given, where $\psi(x,y_1,\ldots,y_l,\bar{c})$ is quantifier-free. The argument here is very similar to that given in Lemma 4 and so we will only highlight it, leaving details to the reader. We assume that $\langle \mathfrak{M}^*_{\bar{c}},\bar{c}\rangle \models \exists\bar{y}\exists x\, \psi(x,\bar{y},\bar{c})$, and let $\mathcal{P} = \{p(y_1,\ldots,y_l) : p$ is a quantifier-free type over $\omega \cup \{\bar{c}\} \wedge (\exists a_1,\ldots,a_l \in \mathfrak{M}^*_{\bar{c}})\langle \mathfrak{M}^*_{\bar{c}},\bar{c}\rangle \models p(a_1,\ldots,a_l) \wedge \exists x\, \psi(x,a_1,\ldots,a_l,\bar{c})\}$. Just as in the proof of Lemma 4, we define a finite set $\mathcal{P}_1$ of finite types p_1 over $\bar{c}$ that we obtain from $\mathcal{P}$. Again we assert that

$$\langle \mathfrak{M}^*_{\bar{c}},\bar{c}\rangle \models \exists x\, \psi(x,y_1,\ldots,y_l,\bar{c}) \leftrightarrow \bigvee_{p_1\in\mathcal{P}_1} (\wedge p_1).$$

Only the only direction from right to left requires proof, so let $a_1,\ldots,a_l \in \mathfrak{M}^*_{\bar{c}}$ be such that for some $p_1 \in \mathcal{P}_1$, $\langle \mathfrak{M}^*_{\bar{c}},\bar{c}\rangle \models \wedge p_1(a_1,\ldots,a_l)$. With $d \notin \mathfrak{M}^*_{\bar{c}}$, just as before we can build a structure N^* with universe $\{d,a_1,\ldots,a_l,c_1,\ldots,c_m\} \cup \omega$ so that for any quantifier-free formula $\theta(y_1,\ldots,y_l,z_1,\ldots,z_m)$,

$$N^* \models \theta(a_1,\ldots,a_l,c_1,\ldots,c_m) \text{ iff } \mathfrak{M}^*_{\bar{c}} \models \theta(a_1,\ldots,a_l,c_1,\ldots,c_m),$$

and

$$N^* \models \psi(d,a_1,\ldots,a_l,c_1,\ldots,c_m).$$

By defining F for n-tuples from N^* where it is not already defined, we can make N^* into a k-degenerate structure N so that $N \restriction \{a_1,\ldots,a_l,\bar{c}\} \cup \omega = \mathfrak{M}^* \restriction \{a_1,\ldots,a_l,\bar{c}\} \cup \omega$. Then, after amalgamating N and $\mathfrak{M}^*$ over $\{a_1,\ldots,a_l,\bar{c}\} \cup \omega$ into a k-degenerate structure, $\mathfrak{M}^*$ being existentially closed implies that $\mathfrak{M}^* \models \exists x\, \psi(x,a_1,\ldots,a_l,\bar{c})$. This completes the argument.

Now we must show how to eliminate Q and Q^2. (Here the role of G in Definition 4 comes into play.) First we deal with Q. Let $Qx\, \psi(x,y_1,\ldots,y_l,c_1,\ldots,c_m)$ be given, where ψ is quantifier free. Let $\psi'(x,y_1,\ldots,y_l,c_1,\ldots,c_m)$ be obtained from ψ by replacing each of $F(u_1,\ldots,u_n) = v$, where $u_1,\ldots,u_n$, $v \in \{x,y_1,\ldots,y_l,c_1,\ldots,c_m\}$ by the formula

$$\begin{gathered}F(u_1,\ldots,u_n) = v \wedge \text{“}\exists\{w_1,\ldots,w_p\} \subseteq \{u_1,\ldots,u_n\}[|\{u_1,\ldots,u_n\}\setminus\{w_1,\ldots,w_p\}| \\ < k \wedge G''(\{w_1,\ldots,w_p\}) \subseteq G''(\{c_1,\ldots,c_m\})]\text{”}.\end{gathered}$$

Note that the expression in quotations is a formula as the "quantification" is really a finite disjunction. Now, following part (a) of Lemma 5, we assert that

$$\begin{gathered}\mathfrak{M}^*_{\bar{c}} \models Qx\, \psi(x,y_1,\ldots,y_l,c_1,\ldots,c_m) \leftrightarrow \exists x\Bigg[\psi'(x,y_1,\ldots,y_l,c_1,\ldots,c_m) \\ \wedge \neg P(x) \wedge \bigwedge_{1\le i\le l} (\neg P(y_i) \to x > y_i) \wedge \bigwedge_{1\le i\le m} (\neg P(c_i) \to x > c_i)\Bigg].\end{gathered}$$

As usual, only the direction from right to left requires proof. Suppose that $a_1,\ldots,a_l \in \mathfrak{M}^*_{\bar{c}}$ satisfies the right-hand side of the equivalence above in $\mathfrak{M}^*_{\bar{c}}$. It follows that $(a_1,\ldots,a_l)$ satisfies the same formula in $\mathfrak{M}^*$. Then by quantifier elimination in $\mathfrak{M}^*$, $\mathfrak{M}^* \models Qx\, \psi'(x,a_1,\ldots,a_l,c_1,\ldots,c_m)$. But by the way that ψ' has been defined, the elements that witness ψ' in $\mathfrak{M}^*$ will witness ψ in $\mathfrak{M}^*_{\bar{c}}$. Thus, $\mathfrak{M}^*_{\bar{c}} \models Qx\, \psi(x,a_1,\ldots,a_l,c_1,\ldots,c_m)$. The argument for eliminating

Q^2 is similar—just modify suitably the right-hand side of the equivalence in part (b) of Lemma 5, and then appeal to quantifier elimination in $\mathfrak{M}^*$.

For the proof of the main lemma, Lemma 9, we will need the following, rather technical lemma. It isolates exactly those conditions we will need to satisfy in the proof of Lemma 9 to extend a *fixed* witnessing sequence for an $L(Q^2_{\aleph_1})$-formula.

Lemma 8 *Let α be a countable ordinal and $\bar{a} \in \mathfrak{M}^*_{\bar{c}}$. Suppose that $I^{\varphi}_{\bar{a}} = \{b_i : i < \alpha\} \subset \mathfrak{M}^*_{\bar{c}}$ satisfies Definition 3(b) for the quantifier-free formula $\varphi(x,y,\bar{a},\bar{c})$ except possibly that $I^{\varphi}_{\bar{a}}$ is not unbounded in $\mathfrak{M}^*_{\bar{c}}$ and that $G''(I^{\varphi}_{\bar{a}}) \cap G''(\{\bar{a},\bar{c}\}) = \phi$. Then there exists an existentially closed end extension $\mathfrak{N}$ of $\mathfrak{M}^*_{\bar{c}}$ and $d \in \mathfrak{N} \setminus \mathfrak{M}^*_{\bar{c}}$ so that $I^{\varphi}_{\bar{a}} \cup \{d\}$ satisfies Definition 3(b) for $\varphi(x,y,\bar{a},\bar{c})$ except that $I^{\varphi}_{\bar{c}} \cup \{d\}$ is not unbounded in $\mathfrak{N}$.*

Proof: Our strategy is to construct a k-degenerate structure $\mathfrak{N}^*$ so that $\mathfrak{N} = \mathfrak{N}^*_{\bar{c}}$ is the existentially closed end extension as required in the conclusion of the lemma.

To begin, we consider the following quantifier-free diagram Δ in a language with constants for elements of $\mathfrak{M}^*_{\bar{c}}$ and a new constant d:

$$\begin{aligned}\Delta = {} & \text{q.f. diagram}(\mathfrak{M}^*_{\bar{c}}) \cup \{d > c : c \in \mathfrak{M}^*_{\bar{c}}\} \\ & \cup \{\varphi(d,b,\bar{a},\bar{c}) : b \in I^{\varphi}_{\bar{a}}\} \\ & \cup \{\theta(\bar{b},\bar{a},\bar{c},\bar{n}) \leftrightarrow \theta(\bar{b}',\bar{a},\bar{c},\bar{n}) : \bar{b},\bar{b}' \in (I^{\varphi}_{\bar{a}} \cup \{d\})^{<k} \wedge \\ & \quad \bar{n} \in \omega \wedge \theta(\bar{x},\bar{y},\bar{z},\bar{w}) \text{ q.f.}\}.\end{aligned}$$

We now show that

Claim 1 *Δ can be completed to the diagram Δ' of a k-degenerate structure with the same universe.*

Proof: To prove this, we first assert that

$$S = \{n \in \omega : (\exists \bar{b} \in I^{\varphi}_{\bar{a}} \cup \{\bar{c},\bar{a},d\} \setminus I^{\varphi}_{\bar{a}} \cup \{\bar{c},\bar{a}\}) F(\bar{b}) = n \in \Delta\}$$

is finite. To see this, we first observe that the only way for F to be defined by Δ for an n-tuple containing d is by one of the last two sets of sentences in Δ. However, it is obvious that each of these sets of sentences in Δ can contribute only finitely many values to S.

To show that Δ can be completed to the diagram of a k-degenerate structure with the same universe, we begin by letting $\neg R(a,d)$ hold for all $a \in \mathfrak{M}^*_{\bar{c}}$ such that this formula is not in Δ. We next assign distinct values from $\omega \setminus S$ to $F(\bar{b})$ for each n-tuple $\bar{t} \in \mathfrak{M}^*_{\bar{c}} \cup \{d\}$ for which Δ does not determine $F(\bar{t})$. Lastly, values of G can be assigned arbitrarily. We now check that the resulting quantifier-free diagram is the diagram of a k-degenerate structure. So we suppose that

$$t_1 < \ldots < t_n < t_{n+1} < \ldots < t_{2n}$$

are constants from $\mathfrak{M}^*_{\bar{c}} \cup \{d\}$, and prove that they cannot violate k-degeneracy. The argument divides into two cases.

Case I: For all $j = 1,\ldots,2n$, $t_j \in \mathfrak{M}^*_{\bar{c}}$. By the way that F has been defined for n-tuples for which it is not defined by Δ, we see that at least one of

$F(t_1,\ldots,t_n)$, $F(t_{n+1},\ldots,t_{2n})$ must be defined by Δ, say $F(t_1,\ldots,t_n)$. Then $F(t_{n+1},\ldots,t_{2n})$ cannot be determined by Δ, since $\mathfrak{M}_{\bar{c}}^*$ obviously can be completed to the diagram of a k-degenerate structure. Hence, by the condition given in Definition 4 under which F is defined, we see that $(t_1,\ldots,t_n,t_{n+1},\ldots,t_{2n})$ could not satisfy the conditions on G necessary to violate k-degeneracy.

Case II: (Necessarily) $t_{2n} = d$. By the way that R has been defined, we see that $t_j \in I_{\bar{a}}^{\varphi} \cup \{\bar{a},\bar{c}\}$ for $j = 1,\ldots,2n-1$. Hence, since all elements of $I_{\bar{a}}^{\varphi}$ are greater (with respect to $<$) than each element of $\{\bar{a},\bar{c}\}$, we observe that if $t_j \in I_{\bar{a}}^{\varphi}$ for some $j = 1,\ldots,n$, then $t_j \in I_{\bar{a}}^{\varphi}$ for all $j = n+1,\ldots,2n-1$.

Subcase a: $(t_1,\ldots,t_n)$ contains no element of $I_{\bar{a}}^{\varphi}$. By the way that we have defined F, either $F(t_1,\ldots,t_n)$ or $F(t_{n+1},\ldots,t_{2n})$ must be determined by Δ. Suppose first that $F(t_1,\ldots,t_n)$ is defined by Δ. Since $G''(I_{\bar{a}}^{\varphi}) \cap G''(\{\bar{a},\bar{c}\}) = \varnothing$, it follows that $|\{t_{n+1},\ldots,t_{2n}\} \setminus \{\bar{a},\bar{c}\}| < k$. But then by indiscernibility, we could find $t'_{n+1} < \ldots < t'_{2n}$ all in $I_{\bar{a}}^{\varphi} \cup \{\bar{a},\bar{c}\}$ so that $(t_1,\ldots,t_n,t'_{n+1},\ldots,t'_{2n})$ violates k-degeneracy, which is impossible (N.B. indiscernibility is applied only to $t_{2n-l+1},\ldots,t_{2n}$, where $l < k$). Now assume that $F(t_1,\ldots,t_n)$ is not defined by Δ, but that $F(t_{n+1},\ldots,t_{2n})$ is. Because $t_{2n} = d$, we must have that $F(t_{n+1},\ldots,t_{2n}) \in S$, and so $F(t_1,\ldots,t_n) \neq F(t_{n+1},\ldots,t_{2n})$ by construction.

Subcase b: $(t_1,\ldots,t_n)$ contains an element of $I_{\bar{a}}^{\varphi}$. In this case, we see that $F(t_1,\ldots,t_n)$ must be defined by Δ. But since $t_{n+1},\ldots,t_{2n} \in I_{\bar{a}}^{\varphi} \cup \{d\}$ and $G''(I_{\bar{a}}^{\varphi} \cup \{d\}) \cap G''(\{\bar{c}\}) = \varnothing$, once again it is not possible for $(t_1,\ldots,t_n,$ $t_{n+1},\ldots,t_{2n})$ to violate k-degeneracy. This completes the proof of Claim 1.

Next we prove

Claim 2 *Let* L$'$ *be the result of augmenting* L *by constants for each* $a \in \mathfrak{M}_{\bar{c}}^*$ *and by new constants* $\bar{d} = (d,d_1,\ldots,d_m)$. *Suppose that* $\Delta' = \Delta \cup \{\theta(\bar{d},\bar{a}')\}$ *can be completed to the quantifier-free diagram of a k-degenerate structure, where for each* $i = 1,\ldots,m$, $\theta(\bar{d},\bar{a}') \vDash d_i > d$, $\{\bar{c},\bar{a}\} \subset \{\bar{a}'\} \subset \mathfrak{M}_{\bar{c}}^*$, *and* $\theta(\bar{y},\bar{z})$ *is a quantifier-free formula with parameters from* ω *implying a complete type in* $\mathfrak{D}$. *Let* $\psi(x_1,\ldots,x_i,\ldots,x_n,\bar{y},\bar{z})$ *be a quantifier-free formula with parameters from* ω *that implies a complete type in* $\mathfrak{D}$ *such that*

$$x_1,\ldots,x_i > \textit{least element of } \bar{y}$$

and

$$x_{i+1},\ldots,x_n < \textit{least element of } \bar{y}.$$

If $\Delta' \cup \{\exists\bar{x}\,\psi(\bar{x},\bar{d},\bar{a}')\}$ *can be realized in a k-degenerate structure, then there are* $b_{i+1},\ldots,b_n \in \mathfrak{M}_{\bar{c}}^*$ *and new constants* $e_1,\ldots,e_i$ *so that*

$$\Delta' \cup \{\psi(\bar{e},\bar{b},\bar{d},\bar{a}')\}$$

can be completed to the diagram of a k-degenerate structure.

Claims 1 and 2 suffice to prove the lemma, since the desired $\mathfrak{N}^*$ can be constructed recursively as the union of a chain of length ω using Claim 1 as the base stage, and Claim 2 for the inductive stages of the construction.

Proof: To prove Claim 2, we can let $\Delta' = \Delta$, since $\theta(\bar{d}, \bar{a})$ can be absorbed into $\exists \bar{x}\, \psi(\bar{x}, \bar{d}, \bar{a}')$. Now, let

$$\mathfrak{M}' \models \Delta \cup \{\exists \bar{x}\, \psi(\bar{x}, \bar{d}, \bar{a}')\}$$

be an existentially closed k-degenerate structure. Take $\bar{b}_0, \bar{e}_0 \in \mathfrak{M}'$ so that

$$\mathfrak{M}' \models \psi(\bar{e}_0, \bar{b}_0, \bar{d}, \bar{a}')$$

where we identify elements of $\mathfrak{M}'$ with the interpretations of $\bar{d}$ and $\bar{a}'$.

Now $(\mathfrak{M}')^*_{\bar{c}} \cong \mathfrak{M}^*_{\bar{c}}$ over ω, so we can choose $\bar{b} \in \mathfrak{M}^*_{\bar{c}}$ such that the quantifier-free type of $\bar{b} \frown \bar{a}'$ over ω in $\mathfrak{M}^*_{\bar{c}}$ is the same as that of $\bar{b}_0 \frown \bar{a}'$ over ω in $(\mathfrak{M}')^*_{\bar{c}}$. Moreover, it is an easy exercise in amalgamation to see that we can choose $\bar{b}$ so that for all $j = i+1, \ldots, n$, and all $t \in I^{\varphi}_{\bar{a}} \cup \{\bar{a}'\}$,

$$\mathfrak{M}^*_{\bar{c}} \models \neg R(b_j, t)$$

unless ψ specifies otherwise.

Let $\psi'(x_{i+1}, \ldots, x_n, \bar{y}, \bar{z})$ be that part of ψ dealing only with the variables $x_{i+1}, \ldots, x_n$, $\bar{y}$, and $\bar{z}$. Our construction so far makes it clear that we can extend Δ consistently to $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$.

We next complete $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$ to the quantifier-free diagram of a k-degenerate structure $\mathfrak{N}_0$ with the same universe. To do this, we first stipulate that $\neg R(t_1, t_2)$ hold for any t_1, t_2 for which it is not required to hold by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$. Observe that

$$\{x : (\exists y \in \bar{d}) R(x, y)\} \subset I^{\varphi}_{\bar{a}} \cup \{\bar{a}', \bar{d}\}.$$

Next, let

$$S' = S \cup \{n \in \omega : \psi'(\bar{b}, \bar{d}, \bar{a}') \models F(\bar{u}) = n \text{ for some } \bar{u}\}$$

where S is as in the proof of Claim 1. It is clear that S' is finite. To n-tuples whose value of F is not determined, we assign distinct values from $\omega \setminus S'$. Lastly, we assign values of G arbitrarily to k-tuples for which it is not determined. We must show that this construction yields the quantifier-free diagram of a k-degenerate structure.

So we suppose that

$$t_1 < \ldots < t_n < t_{n+1} < \ldots < t_{2n}$$

are from N_0, and prove that they cannot violate k-degeneracy. The argument divides once again into two main cases.

Case I: For all $j = 1, \ldots, 2n$, $t_j \in \mathfrak{M}^*_{\bar{c}}$. By the way that F has been defined for n-tuples for which it is not defined by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$, we see that at least one of $F(t_1, \ldots, t_n)$, $F(t_{n+1}, \ldots, t_{2n})$ must be defined by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$, say $F(t_1, \ldots, t_n)$. Suppose first that $F(t_{n+1}, \ldots, t_{2n})$ is not determined by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$. Then, since $F(t_{n+1}, \ldots, t_{2n}) \neq s$ for any $s \in S'$, it follows that $F(t_1, \ldots, t_n)$ cannot be determined by $\psi'(\bar{b}, \bar{d}, \bar{a}')$ and so must be determined by Δ. But if $F(t_{n+1}, \ldots, t_{2n})$ is not determined by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$ and $F(t_1, \ldots, t_n)$ is determined by Δ, we see from Definition 4 that $(t_1, \ldots, t_{2n})$ could not violate k-degeneracy. Hence, $F(t_{n+1}, \ldots, t_{2n})$ must be determined by $\Delta \cup \{\psi'(\bar{b}, \bar{d}, \bar{a}')\}$. But it is impossible for both $F(t_1, \ldots, t_n)$ and $F(t_{n+1}, \ldots, t_{2n})$

to be determined by $\Delta \cup \{\psi'(\bar{b},\bar{d},\bar{a}')\}$, since this quantifier-free diagram can be realized in a k-degenerate structure. This completes the argument in Case I.

Case II: Some element of $\bar{t}$ is in $\bar{d}$. By the way we have extended the definition of R, we see that $\bar{t} \subset I^{\varphi}_{\bar{a}} \cup \{\bar{a}',\bar{b},\bar{d}\}$.

Subcase a: $\bar{t} \subset I^{\varphi}_{\bar{a}} \cup \{\bar{a}',\bar{d}\}$. Again, we see that at least one of $F(t_1,\dots,t_n)$, $F(t_{n+1},\dots,t_{2n})$ must be defined by $\Delta \cup \{\psi'(\bar{b},\bar{d},\bar{a}')\}$, say $F(t_1,\dots,t_n)$. We then observe that $F(t_1,\dots,t_n) \in S'$. Hence, by the way that we have defined F, it must be true that $F(t_{n+1},\dots,t_{2n})$ also is determined by $\Delta \cup \{\psi'(\bar{b},\bar{d},\bar{a}')\}$. But then, since $\Delta \cup \{\psi'(\bar{b},\bar{d},\bar{a}')\}$ can be realized in a k-degenerate structure, it is not possible that $\bar{t}$ violates k-degeneracy.

Subcase b: $\bar{t}$ contains an element of $\bar{d}$ and an element of $\bar{b}$. By the way that we have chosen $\bar{b}$, we must have $\bar{t} \subset \{\bar{a}',\bar{b},\bar{d}\}$. But then $F(t_1,\dots,t_n)$ and $F(t_{n+1},\dots,t_{2n})$ must be determined by $\psi'(\bar{b},\bar{d},\bar{a}')$, and then it certainly is impossible for $\bar{t}$ to violate k-degeneracy.

We now embed $\mathfrak{N}_0$ into $\mathfrak{M}'$ via a mapping i so that the restriction of the embedding is the identity on $\omega \cup \{\bar{a}',\bar{d}\}$ and $i(\bar{b}) = \bar{b}_0$. The conclusion of Claim 2 follows by taking the substructure of $\mathfrak{M}'$ whose universe is $i(M^*_{\bar{c}} \cup \{\bar{d}\}) \cup \{\bar{e}\}$. With the claim proved, the proof of Lemma 8 is complete.

Lemma 9 *For any $\bar{c} \in (\mathfrak{M}^*)^m$, there exists a continuous increasing chain of* L-*structures $\langle \mathfrak{M}_\alpha : \alpha < \omega_1 \rangle$ such that*

(i) $M_0 = \mathfrak{M}^*_{\bar{c}}$ *and* $P(\mathfrak{M}_\alpha) = P(\mathfrak{M}_0) = \omega$ *for each* $\alpha < \omega_1$

(ii) *for* $\alpha,\beta < \omega_1$, $\mathfrak{M}_\alpha$ *and* $\mathfrak{M}_\beta$ *are isomorphic over* ω

(iii) *for* $\alpha < \beta < \omega_1$, $\mathfrak{M}_\alpha \prec_{L(Q,Q^2)} \mathfrak{M}_\beta$ *and* $\mathfrak{M}_\alpha \subset_{\text{end}} \mathfrak{M}_\beta$

(iv) $\mathfrak{M}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{M}_\alpha \succ_{L(Q,Q^2)} \mathfrak{M}_\alpha$, *for all* $\alpha < \omega_1$, *where* Q *and* Q^2 *have the standard* $\aleph_1$-*interpretation in* $\mathfrak{M}_{\omega_1}$, *and the nonstandard interpretation in* $\mathfrak{M}_\alpha$.

Proof: We first construct the chain of models satisfying (i), (ii), and (iii), and lastly verify that (iv) holds. At a limit ordinal α, we shall take $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$, so we are left with the construction at successor ordinals. The main task is to make sure that witnesses are added, ensuring that Q^2 has the standard interpretation in $\mathfrak{M}_{\omega_1}$. Let $\alpha = \beta + 1$ be given, and assume we have built $\mathfrak{M}_\beta$. Furthermore, for each formula $Q^2xy\ \varphi(x,y,z_1,\dots,z_l)$ (where $\varphi(x,y,z_1,\dots,z_l)$ is quantifier-free), and $a_1,\dots,a_l \in \mathfrak{M}_\beta$ such that $\mathfrak{M}_\beta \models Q^2xy\ \varphi(x,y,a_1,\dots,a_l)$, suppose that we have fixed a sequence $I^{\varphi}_{\bar{a}} \models \langle b_\xi : \xi < \gamma \rangle$ that satisfies the conditions of Definition 3(b) and two further stipulations. The first is that if δ is the least ordinal with $a_1,\dots,a_l \in \mathfrak{M}_\delta$, $\{b_\xi : \xi < \omega\} \subseteq \mathfrak{M}_\delta$ and at each successor ordinal $\delta + (\nu + 1) < \beta$, we have that $b_\nu \in \mathfrak{M}_{\delta+(\nu+1)}$. In other words, for each formula $Q^2xu\ \varphi(x,y,a_1,\dots,a_l)$ that is to hold in $\mathfrak{M}_{\omega_1}$, a fixed witness sequence has been extended at each successor stage in the construction. The second condition to be imposed on $I^{\varphi}_{\bar{a}}$ is that $G''(I^{\varphi}_{\bar{a}}) \cap G''(\{c_1,\dots,c_m\}) = \emptyset$. That such a sequence $\langle b_\xi : \xi < \omega \rangle$ can be found, which we then extend, follows from part (b) of Lemma 5.

We will build $\mathfrak{M}_{\beta+1}$ as the union of an ω-chain of structures $\mathfrak{N}_i, i < \omega$, each isomorphic over $\omega \cup \{c_1,\ldots,c_m\}$ *to* $\mathfrak{M}^*_{\bar{c}}$, with $\mathfrak{N}_0 = \mathfrak{M}_\beta$ and $\mathfrak{N}_{i+1}$ a proper end extension of $\mathfrak{N}_i$ for all $i < \omega$. It follows immediately that $\mathfrak{M}_{\beta+1} = \bigcup_{i<\omega} \mathfrak{N}_i$ satisfies (i), (ii), and the second clause in (iii). Moreover, the first assertion in (iii) follows from (ii) and the quantifier elimination given in Lemma 7. The main task in the construction at a stage $i > 0$ is to add a new witness to the *fixed* witnessing sequence $I^\varphi_{\bar{a}}$ for the ith formula $Q^2xy\ \varphi(x,y,\bar{a},\bar{c})$ that holds in $\mathfrak{M}_\beta$ according to some fixed listing, where $\varphi(x,y,\bar{z},\bar{w})$ is quantifier-free. However, Lemma 8 guarantees that precisely this can be done.

To complete the proof of Lemma 9, we now must verify (iv). We obviously have $\mathfrak{M}_{\omega_1} >_L \mathfrak{M}_\alpha$ for each $\alpha < \omega_1$. Next, assume that we are given a formula $Qx\ \varphi(x,\bar{a},\bar{c})$, where $\varphi(x,\bar{a},\bar{c})$ is quantifier-free and $\bar{a} \in (\mathfrak{M}_\alpha)^m$. We must show that

$$\mathfrak{M}_{\omega_1} \models Qx\ \varphi(x,\bar{a},\bar{c}) \text{ iff } \mathfrak{M}_\alpha \models Qx\ \varphi(x,\bar{a},\bar{c}),$$

where Q receives the standard $\aleph_1$-interpretation on the left and the nonstandard interpretation on the right. This follows by the quantifier elimination for the nonstandard interpretation, and because $\mathfrak{M}_{\omega_1}$ is the union of a chain of countable end extensions. Finally, given a formula $Q^2xy\ \varphi(x,\bar{a},\bar{c})$ with φ quantifier-free and $\bar{a} \in (\mathfrak{M}_\alpha)^m$, we must verify that

$$\mathfrak{M}_{\omega_1} \models Q^2xy\ \varphi(x,y,\bar{a},\bar{c}) \text{ iff } \mathfrak{M}_\alpha \models Q^2xy\ \varphi(x,y,\bar{a},\bar{c}),$$

where Q^2 receives the standard $\aleph_1$-interpretation on the left, and the nonstandard one on the right. Here, the direction from left to right again follows from the quantifier elimination for Q^2 and because $\mathfrak{M}_{\omega_1}$ is the union of a chain of countable end extensions. The reverse implication holds because for some α a *fixed* witnessing sequence for $\varphi(x,y,\bar{a},\bar{c})$ has been extended at each stage $\beta > \alpha$ in the construction of $\langle \mathfrak{M}_\beta : \beta < \omega_1 \rangle$.

With the difficult preparatory labors completed, we now are able to prove Theorem 1.

Proof of Theorem 1: Let $\mathfrak{M}^*$ be the countable existentially closed k-degenerate structure. Clearly, k-degeneracy is expressible in first-order logic and

$$\mathfrak{M}^* \models Q^2xyR(x,y) \wedge \neg Qx\ P(x),$$

where Q and Q^2 receive the nonstandard interpretation. Then, $\mathrm{Th}_{L(Q,Q^2)}(\mathfrak{M}^*)$ does not have a model in which Q and Q^2 receive the standard $\aleph_1$-interpretation. For suppose not, and let $\mathfrak{N}$ be such a model of power $\aleph_1$ and let $I \subseteq \mathfrak{N}$ be homogeneous for $R(x,y)$ of power $\aleph_1$. Since $\mathfrak{M}^* \models (\forall y) \neg (Qx)x < y$, it follows that $<$ is ω_1-like on $\mathfrak{N}$, and we may assume that $I = \{b_\xi : \xi < \omega_1\}$ and whenever $\eta < \nu$ that $b_\eta < b_\nu$. Next, let I_1 be a set of pairwise disjoint increasing n-tuples whose components are from I. Since $\mathfrak{N} \models \neg(Qx)P(x)$, there is some $n_0 \in P(\mathfrak{N})$ and $I_2 \subseteq I_1$ of power $\aleph_1$ such that if $\bar{b} \in I_2$, $\mathfrak{N} \models F(\bar{b}) = n_0$. Finally, again because $P(\mathfrak{N})$ is countable, there is an $I_3 \subseteq I_2$ of power $\aleph_1$ so that for any $\bar{b}_1, \bar{b}_2 \in I_3$, $\mathfrak{N} \models \bar{b}_1 \equiv \bar{b}_2(G)$. But this is impossible, as $\mathfrak{N}$ was to be k-degenerate.

We complete the proof by establishing that any true $L(Q_{\aleph_1},Q^2_{\aleph_1})$-axiom schema of quantifier depth less than k holds in $\mathfrak{M}^*$ (with the nonstandard interpretations, of course). So let $\forall\bar{x}\ \Phi(\varphi_1,\ldots,\varphi_p)$ be an instance of such a true schema, where $\varphi_i \equiv \varphi_i(\bar{v}_i,\bar{x})$, and suppose for some $\bar{c} \in \mathfrak{M}^*$ that $\mathfrak{M}^* \models \neg\Phi(\varphi_1,\ldots,\varphi_p)$. We may also assume that φ_i is quantifier-free. Since for each $i = 1,\ldots,p$, $lh(\bar{v}_i) < k$, it then follows that for each $i = 1,\ldots,p$, and $\bar{a} \in (\mathfrak{M}^*)^{lh(\bar{v}_i)}$,

$$\mathfrak{M}^*_{\bar{c}} \models \varphi_i(\bar{a},\bar{c}) \text{ iff } \mathfrak{M}^* \models \varphi_i(\bar{a},\bar{c}).$$

Therefore $\mathfrak{M}^*_{\bar{c}} \models \neg\Phi(\varphi_1,\ldots,\varphi_p)$, and hence by Lemma 9, $\mathfrak{M}_{\omega_1} \models \neg\Phi(\varphi_1,\ldots,\varphi_p)$. But this last assertion is impossible, since $\mathfrak{M}_{\omega_1}$ is a standard $L(Q_{\aleph_1},Q^2_{\aleph_1})$-model. So the theorem is proved.

REFERENCES

[1] Keisler, H. J., "Logic with the quantifier "there exists uncountably many"," *Annals of Mathematical Logic*, vol. 1 (1970), pp. 1-93.

[2] Magidor, M. and J. Malitz, "Compact extensions of L(Q)," *Annals of Mathematical Logic*, vol. 11 (1977), pp. 217-261.

[3] Shelah, S. and C. Steinhorn, "On the non-axiomatizability of some logics by finitely many schemas," *Notre Dame Journal of Formal Logic*, vol. 27 (1986), pp. 1-11.

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