

Logical Constants as Punctuation Marks

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Abstract This paper presents a proof-theoretical approach to the question “What is a logical constant?” This approach starts with the assumption that logic is the science of formal deductions, and that basic formal deductions are structural deductions, i.e. deductions independent of any constant of the language to which the premises and conclusions belong. Logical constants, on which the remaining formal deductions are dependent, may be said to serve as “punctuation marks” for some structural features of deductions; this punctuation function, exhibited in equivalences which amount to analyses of logical constants, is taken as a criterion for being a logical constant. The paper presents an account of philosophical analysis which covers the proposed analyses of logical constants. Some related assumptions concerning logic are also considered. In particular, since a logical system is completely determined by its structural deductions, alternative logical systems arise by changing structural deductions while having constants with the same punctuation function. Some other approaches to the question “What is a logical constant?”, grammatical, model-theoretical, and proof-theoretical, are briefly considered.

1 Introduction It is clear that an answer to the question “What is a logical constant?” would provide us with the means to answer the question “Where are the limits of logic?” Since the latter question is obviously very close to the ques-

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tion “What is the subject matter of logic?” one could legitimately assume that our question “What is a logical constant?” is among the central questions of the philosophy of logic.

Apart from its intrinsic philosophical interest, the problem of the demarcation of logic is obviously of crucial importance to the logicist program in the foundations of mathematics. However, no definite criterion for this demarcation seems to have come out of the work of the logicists, with which modern logic started. On the other hand, one of the main reasons why logicism was abandoned was that at some point it was felt that the limits of logic must have been transgressed in the logicist reconstruction of mathematics. The problem of the demarcation of logic is also in the background of the discussion concerning the status of second-order logic, which started with Quine’s attack upon second-order logic (see [6]).

In spite of all that, it doesn’t seem that logicians, even those who are philosophically inclined, are trying very hard to answer the question “What is a logical constant?” A much more characteristic attitude in modern logic is that of a certain skepticism as to whether the distinction between logical and nonlogical expressions can be clearly drawn. Most logicians, like so many followers of Protagoras, are content with just listing what they take as logical constants. A clear exponent of this skepticism, and probably one of those who made it the accepted position, is Tarski, in his famous paper on the notion of logical consequence [40]. It is interesting that Bolzano anticipated Tarski not only in his definition of the semantical notion of logical consequence, but also in the belief that it is doubtful that a criterion can be found for drawing the distinction between logical and nonlogical expressions (see [5], Section 148, and [23], p. 366).

Occasionally, however, attempts do arise to find this criterion. Without trying to ascertain their merits, let us just mention some of these attempts.

In [36] Quine proposed to distinguish logical constants from other expressions (his terms are respectively “particles” and “lexicon”) by saying that the grammatical categories of the latter are “infinite and indefinite” (see pp. 28–30, 59). In [43] Wang examined some proposals, including a grammatical one, linked with Quine’s, but in general he was in a rather skeptical mood (see pp. 143–165). It is interesting that in a lecture from 1966 [42] Tarski seems to have abandoned his skepticism up to a point, and to have found a criterion for the demarcation of logic by elaborating ideas suggested by Klein’s *Erlange Programm*, and some model-theoretical results of Lindenbaum and Tarski [25]. Chief among these results is that “. . . every relation between objects (individuals, classes, relations, etc.) which can be expressed by purely logical means is invariant with respect to every one-one mapping of the *world* (i.e., the class of all individuals) onto itself . . .” ([41], p. 385; a reference to Tarski’s lecture of 1966, and a critique of his views, can be found in [28]). The proposal for the demarcation of logical constants found in [14] (pp. 21–22), which is in principle grammatical (essentially, it takes as logical constants functors from sentences and predicates to sentences and singular terms), also has a footnote dealing with identity, in which a view very similar to Tarski’s conception of 1966 is propounded. Finally, Lindström in [26] and [27] suggested a rather technical model-theoretical criterion for the demarcation of logic, by showing that the Löwenheim-Skolem Theorem together with the Compactness Theorem for de-

numerable sets of sentences cannot be extended to first-order logics with generalized quantifiers which properly extend ordinary first-order logic.

The purpose of this paper is to give a summary of the philosophical part of [9], which represents yet another attempt to answer the question “What is a logical constant?” This attempt is neither grammatical nor model-theoretical, but proof-theoretical. In principle, it is not impossible that such a proof-theoretical approach be at least extensionally equivalent with another approach, grammatical or model-theoretical, in the sense that the same constants will be selected as logical in both approaches. However, we shall not investigate here the possible connections between these various approaches. After presenting our approach in the next four sections of this paper, we shall briefly consider in the last section some similar proof-theoretical programs for the demarcation of logic.

2 Thesis [I] Our proof-theoretical attempt to answer the question “What is a logical constant?” starts with the following assumption about logic:

[A] Logic is the science of formal deductions.

This conception of logic, despite its aura of antiquity, is neither the only possible nor the dominant conception of logic. There is a strong tradition in modern logic, which starts with Frege, and pervades not only model theory, but also Hilbert-style proof theory, which assumes that logic is the science of a certain kind of truths, rather than deductions. A clear assessment of the role that this conception, and the alternative conception expressed by [A], have played in modern logic can be found in [14] (pp. 432–435; cf. also [21]).

It is also rather unclear how the logic mentioned in [A] could cover the vast number of mathematical subjects studied in model theory, recursion theory, set theory, or category theory, which all go today under the label of logic. So, “logic” in [A] should probably be taken as referring only to a certain traditional core of logic. In modern logic, the conception of [A] is clearly present only in Gentzen-style proof theory: the comparatively narrow, but doubtless important, tradition which starts with Gentzen’s seminal thesis in [18].

Sometimes [A] is expressed by saying that logic is the science of *valid* formal deductions. This use of “valid” is slightly misleading, since separating valid formal deductions from invalid ones obviously involves considering both.

Assumption [A] does not leave any room for an informal logic. If someone insists (often without much reason) upon using the word “logic” for the description of the function of all sorts of words (in principle, philosophically interesting), then we must emphasize that the “logic” of [A] should be understood as “formal logic” (a subject which does not have many things in common with, for example, “the logic of color words”).

However, this leaves us with the task of specifying what the *formal deductions* we appeal to in [A] are. To accomplish this task, we first assume the following:

[B] Basic formal deductions are structural deductions.

The term “structural” in [B] should be understood in the sense this word has in Gentzen’s sequent-systems in [18]. Structural deductions are deductions which

can be described independently of the constants of the *object language*, i.e., the language from which our premises and conclusions come. In other words, in describing structural deductions, everything in the object language is schematic.

With the apparatus of Gentzen's sequent-systems a description of structural deductions is obtained as follows. Let A, B, C, \dots be schemata for formulas of an object language, and let Γ, Δ, \dots be schemata for finite collections of formulas of this language: these collections can be either sets, multisets (i.e., collections with possibly more than one occurrence of each element, in which order is irrelevant), or sequences. (The most general approach is to consider Γ as a term made up of formulas of the object language with the help of a binary comma, which need not even be associative; cf. [3] and [12].) Then we can interpret a *single-conclusion* sequent $\Gamma \vdash A$ as saying that there is a deduction from the premises in Γ to the conclusion A . If Γ is empty, $\Gamma \vdash A$ amounts to the assertion that A is a theorem. A *multiple-conclusion* sequent $\Gamma \vdash \Delta$, where Δ has more than one member, can be understood as referring to a deduction where in each line we have a finite collection of formulas, rather than a single formula. In classical logic, these collections can be sets, in which formulas are tied by an implicit disjunction (cf. [7]). Another possibility is to understand multiple-conclusion deductions in the style of Shoesmith and Smiley [39]. Multiple-conclusion sequents can also be related to a natural generalization of Tarski's semantical notion of consequence (cf. [38], pp. 413–418).

That the appropriate form of deductions in classical logic is given by multiple-conclusion deductions does not seem to be generally acknowledged. Though, on the technical side, these deductions have some clear advantages, as was demonstrated by Gentzen in [18], intuitively they may well look like an invention of logicians, not to be found in ordinary usage. But perhaps we shall find them in ordinary usage too, if we assume that there they occur in an *enthymematic* form. Usually, a deduction is enthymematic if some true premises are omitted; but we could just as well say that a deduction is enthymematic if some false alternative conclusions are omitted. For example, the single-conclusion deduction of A from $\neg\neg A$ may be matched by the enthymematic multiple-conclusion deduction obtained from:

$$\frac{\top}{\begin{array}{ccc} A & \neg A & \neg\neg A \\ \hline & & \perp \end{array}}$$

by omitting the true premise \top and the false alternative conclusion \perp ; the formulas A and $\neg A$ are derived as alternative conclusions from \top .

Structural deductions can now be described by restricting the sequent-language to *structural sequents*, i.e. schematic sequents in which no constant of the object language occurs. Valid structural deductions will be codified by retaining in a sequent-system only axiom-schemata like $A \vdash A$, and structural rules like permutation, contraction, thinning, and cut. Another structural rule (not mentioned by Gentzen) which may be added to these is the rule of *substitution for variables*:

$$\frac{\Gamma \vdash \Delta}{S_a^x \Gamma \vdash \Delta}$$

where the lower formula stands for the sequent obtained from $\Gamma \vdash \Delta$ by substituting uniformly a for the free occurrences of x ; the schematic letter a stands for an expression of the same grammatical category as x , which, as usual, contains neither a free variable which will become bound after the substitution, nor a variable-binding operator which will bind a free variable of $\Gamma \vdash \Delta$.

It seems clear that deductions which are described independently of any constant of the object language may legitimately be called *formal*. These are the *basic* formal deductions, by which all other formal deductions are determined according to the following assumption:

[C] Any constant of the object language on whose presence the description of a nonstructural formal deduction depends can be ultimately analyzed in structural terms.

In the next section we shall explain in more detail what we mean by “ultimately analyzed in structural terms”. For the time being, we shall explain [C] only by reference to a metaphor.

Assumption [C] enables us to get a uniform picture of logical form. Logical form is primarily exhibited by structural deductions, and when logical constants are introduced they serve, so to speak, as *punctuation marks* of the object language, for some structural features of deductions. Which structural punctuation function pertains to the customary logical constants of first-order logic (implication \rightarrow , conjunction \wedge , disjunction \vee , the constant true proposition \top , the constant absurd proposition \perp , the universal quantifiers $\forall x$, the existential quantifiers $\exists x$, and identity $=$), and to the necessity operator of the modal systems S4 and S5, was mentioned in [10]. Here we shall consider as our main examples the punctuation function of implication and of the universal and existential quantifiers; the rest will be mentioned only briefly.

The connective of implication in $A \rightarrow B$ says that A and B are connected in this formula of the object language like a premise and a conclusion in a deduction. This means that for implication we have assumed a *deduction theorem* and *modus ponens*. In a sequent framework, this corresponds to the assumption of the following *double-line rule*:

$$(\rightarrow) \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

where the double line means that we have two rules, one going downward and the other upward, and Γ, A stands for the collection obtained by taking the union of Γ and $\{A\}$, or by concatenating A to Γ , or something analogous. The downward rule clearly corresponds to the deduction theorem, whereas the upward rule is equivalent to $A \rightarrow B, A \vdash B$, in the presence of the axiom-schema $C \vdash C$ and cut.

The double-line rule (\rightarrow) can serve to characterize various sorts of implications: classical, intuitionistic, and relevant. What in these characterizations distinguishes various implications is not (\rightarrow) , which is always the same, but assumptions concerning structural deductions. For example, intuitionistic implication can be obtained from classical implication by abolishing *thinning on the right*:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A},$$

which has the same effect as permitting only single-conclusion sequents (this is analogous to the way Gentzen obtains his intuitionistic sequent-system from his classical sequent-system in [18]). Relevant implication, i.e. the implication of the relevant system R of Anderson and Belnap, is obtained by further abolishing *thinning on the left*:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}.$$

The situation is analogous for the other logical constants of first-order logic, and for the necessity operators, mentioned above: they can all be characterized in alternative logical systems by fixing double-line rules for them, and changing only structural assumptions. (Various families of alternative logical systems, obtained in this way, are investigated in [10], [11], and [12].)

What we want to stress here is that in the upper sequent of (\rightarrow) everything from the object language is purely schematic, so that this sequent is structural. According to (\rightarrow) , implication is a kind of substitute in the object language for the turnstile \vdash , i.e. for the deducibility relation. Implication can reduce a sequent like $A \vdash B$, which says that B can be deduced from A , to a theorem of the object language $A \rightarrow B$: with (\rightarrow) , we have that $A \vdash B$ and $\vdash A \rightarrow B$ are interdeducible. The logical form of $A \rightarrow B$ mirrors a structural feature of deductions, viz. the relation between a premise A and a conclusion B , independently of any constants which A and B may have.

The universal and existential quantifiers can be characterized by the following double-line rules:

$$(\forall) \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \forall x A}$$

$$(\exists) \frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}$$

with the following proviso: *the variable x does not occur free in either Γ or Δ* . As with implication, by varying structural rules the same double-line rules (\forall) and (\exists) can serve for various alternative logical systems, provided we have assumed the structural rule of substitution for variables (see [10]). And as with (\rightarrow) , the upper sequents of (\forall) and (\exists) are structural. Since the rule of substitution for variables permits us to read the x which may occur free in A as “any”, according to (\forall) and (\exists) , the two quantifiers express something about the place of “any” in deductions. If “any” is in a conclusion, and nowhere else, it becomes “every”, and if it is in a premise, and nowhere else, it becomes “some”. So the logical form of $\forall x A$ and $\exists x A$ mirrors a structural feature of deductions, viz. the presence of a variable in a conclusion A , or in a premise A , a variable which doesn’t occur free anywhere else in the deduction.

For the remaining constants of first-order logic, mentioned above, we have the following double-line rules:

$$\begin{array}{l}
(\wedge) \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \\
(\vee) \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \\
(\top) \frac{\vdash \Delta}{\top \vdash \Delta}, \quad (\perp) \frac{\Gamma \vdash}{\Gamma \vdash \perp} \\
(=) \frac{S_a^x \Gamma \vdash \Delta |}{\Gamma, x = a \vdash \Delta}.
\end{array}$$

The double-line rules (\wedge) and (\vee) should be interpreted as if the pairs of upper sequents were tied by a conjunction; so, for example, the upward direction of (\wedge) gives two rules, one with the conclusion $\Gamma \vdash \Delta, A$ and the other with the conclusion $\Gamma \vdash \Delta, B$. According to (\wedge) and (\vee) conjunction and disjunction serve to economize: they reduce to one deduction two deductions which differ only at one place in the conclusions or in the premises. The constant \top is a substitute for the empty collection of premises ($\vdash \Delta$ can be understood as referring to a demonstration of one of the conclusions in Δ), and \perp is a substitute for the empty collection of conclusions ($\Gamma \vdash$ can be understood as referring to a refutation of one of the premises in Γ). Identity serves to indicate substitution possibilities in a deduction: what holds for a also holds for whatever is assumed to be identical with a .

The double-line rule for the necessity operator \Box of S4 and S5 is based on sequents of higher levels, i.e. sequents which have collections of sequents on the left and right of the turnstile (see [10]; cf. [11]). Without entering into details, let us only mention that $\Box A$ can be understood as indicating that A is assumed as a theorem.

The connective of equivalence \leftrightarrow can either be explicitly defined by $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$, or we can give the following double-line rule for it:

$$(\leftrightarrow) \frac{\Gamma, A \vdash \Delta, B \qquad \Gamma, B \vdash \Delta, A}{\Gamma \vdash \Delta, A \leftrightarrow B}.$$

The assumptions [A], [B], and [C] naturally yield the following thesis concerning constants of our object language:

[I] A constant is logical if, and only if, it can be ultimately analyzed in structural terms.

In the next section we shall try to explain what we mean by the expression “ultimately analyzed in structural terms,” which occurs in assumption [C] and in thesis [I].

3 Analysis Let us first settle what “ultimately” means in “ultimately analyzed”. An expression is *ultimately* analyzed if and only if it is either analyzed or it can be explicitly defined in terms of analyzed expressions. Next, we must

say what we mean by “analyzed”. This is a more difficult matter. It presupposes a general account of philosophical analysis, which we proceed to sketch.

In the philosophical tradition which continues to be called “analytical”, but which has gone through many significant changes in this century, it seems that the most precise account of analysis which can be found, and to which many accepted opinions can still be traced, is in the writings of G.E. Moore. On the other hand, this account, which ultimately treats analysis as a kind of explicit definition, cannot be supported by many examples of significant philosophical analyses which conform to its standards. Our account of analysis will differ from Moore’s by drawing a distinction between analysis and explicit definition. (In that, and in some other respects, it follows [24], which contains a detailed critique of Moore’s account of analysis and of those accepted opinions that can be traced to his account.)

Let us suppose that we have an expression α of a language L and that we want to analyze α . Then we must specify a language M , to which α does not belong, in which we shall formulate the analytic equivalent of α , i.e. the analysans. Below we shall consider more closely the relation between L and M ; for the time being it is enough to say that M , as a vehicle of our analysis, must be a language which we understand. Our first condition concerning analysis specifies the *form* an analysis should take; it says that

- (1) An analysis consists in establishing that a sentence **A** in M plus α , in which α occurs only once, is equivalent to a sentence **B** in M .

The second condition concerning analysis specifies that an analysis must be *sound and complete*: it says that

- (2) From the equivalence of (1), and from the understanding of M and L minus α , we can infer every sentence of L which is analytically true in L and sentence of L which is not analytically true in L .

Here “analytically true in L ” means “true in virtue of the meaning of the expressions of L ”.

Our third condition concerning analysis says that an analysis must characterize *uniquely* the expression analyzed, in the following sense

- (3) The expressions α_1 and α_2 can receive the same analysis if, and only if, α_1 and α_2 have the same meaning.

An equivalence can satisfy our three conditions for analysis without amounting to an explicit definition. There are at least two additional conditions which an explicit definition is normally assumed to satisfy, and which need not be satisfied by an analysis.

The first of these additional conditions, which we may call *Pascal’s condition*, says that:

The definitional equivalence should enable us to find for every sentence of M plus α a sentence of M with the same meaning.

If M is equal to L minus α , this amounts to the requirement that in every sentence of L we must be able to eliminate a defined expression by its definiens, without changing the meaning. (Such a requirement is implied by Pascal in [29],

pp. 244, 279–282; cf. [4], pp. 504–505. A similar implication might be found in Aristotle’s *Topica*, Z.4, 142 b.) Pascal’s condition implies that for **A** in (1) there will be a **B** with the same meaning.

The second condition for explicit definitions we shall call the *conservativeness condition*; it says that

Every sentence of L minus α which is analytically true in L is analytically true in L minus α .

This implies that from the definitional equivalence, and from the understanding of M and L minus α , we should not be able to infer a sentence of L minus α , analytically true in L , but not analytically true in L minus α .

Strengthening our weaker notion of analysis to explicit definability of some sort, which should presumably satisfy Pascal’s condition and conservativeness, threatens to exclude most philosophically interesting analyses, and may reduce us to the recording of more or less lexicographical facts, like the fact that “ a is a brother” is equivalent to “ a is a male sibling”.

To illustrate this, consider first Ramsey’s analysis of the predicate “is true”, given by the equivalence:

“ A ” is true if, and only if, A .

Here, L is a fragment of English, α is “is true”, and M is our fragment of English without α , extended by the schema A for sentences of L (our fragment of English presumably contains quotation marks, or a similar device, transforming expressions of the grammatical category of sentences into expressions of the grammatical category of nouns; A is of the grammatical category of sentences). It is plausible to hold that Ramsey’s analysis satisfies (1)–(3), and even conservativeness (provided we have restricted our fragment of English so that Liar-type paradoxes cannot be derived). However, it will not satisfy Pascal’s condition, since we are unable to eliminate “is true” from a sentence like “Socrates said something true” (at least not without introducing in M further logical paraphernalia, such as propositional variables and propositional quantifiers binding these variables). It can plausibly be argued that the point of having “is true” in ordinary English is to enable us to say things which cannot be said without this predicate (cf. [19]).

Consider now Russell’s analysis of the definite description “the king of France”, given by the equivalence:

The king of France is P if, and only if, there is a single individual which is a king of France and that individual is P .

Here, L is again a fragment of English, α is “the king of France”, and M is our fragment of English without α (but with the predicate “is a king of France”), extended by the predicate schema P . Granted that Russell’s analysis satisfies (1)–(3), and even Pascal’s condition, it is clear that it does not satisfy conservativeness, since on Russell’s analysis the analytical truth “The king of France is equal to the king of France” yields “There is a king of France”, which cannot be analytically true in L minus α .

Let us call an expression α *inconsistent*, relative to a language K without α , if in K not every sentence is analytically true, but in K plus α every sentence

is analytically true. Some expressions, like the predicate “is a round square”, which it would be natural to call inconsistent, are not inconsistent in our sense. An example of an inconsistent expression in our sense is Peano’s μ (see [30] and [2]), the analysis of which is given by the equivalence:

$$\frac{a}{b} \mu \frac{c}{d} = e \text{ if and only if } \frac{a+c}{b+d} = e,$$

where a, b, c, d , and e are schematic letters for rational numbers. The operation μ is inconsistent, relative to the language K of the arithmetic of rationals: introducing μ into this arithmetic would give rise to inconsistencies, and since an inconsistency implies everything, every sentence will be made analytically true in K plus μ .

Can an inconsistent expression be analyzed? As we have just seen above with μ , the answer is yes, provided we have assumed we have such an expression in L . No doubt, inconsistent expressions are to be avoided, and languages containing them should not be constructed for actual use. But constructing a language is something we do before trying to analyze the expressions of this language, and if we have been so unreasonable, or unfortunate, as to construct a language with inconsistent expressions, the fact that these too can be analyzed is not a defect in analysis. There is no reason to require that the conditions for analysis should single out inconsistent expressions as unanalyzable, so that analysis should be impracticable with unreasonably constructed languages. On the contrary, analysis can sometimes help us in investigating an unreasonably constructed language, and in locating the source of the trouble.

Since we can analyze inconsistent expressions, *a fortiori* we can analyze an expression that does not satisfy the conservativeness condition. A language L with such an expression α , even if it does not give rise to inconsistencies, is also unreasonable in some sense. Such is, indeed, our L which contains the expression “the king of France”. The possibility of ascertaining in L supposedly analytical truths like “There is a king of France” (obtained from “The king of France is a king of France”) makes L unreasonable. Russell’s analysis can help us in locating the source of the trouble.

The situation is different with definitions. In this case there will be no pre-existing language L in which α has a meaning, but the definitional equivalence will add α to L minus α , and give meaning to α . And if we want L to be reasonable our definition must define only α , and not also something in L minus α . Hence, the conservativeness condition should be satisfied, even if Pascal’s condition is not. If the definition is to be counted as explicit, it should also satisfy Pascal’s condition. Of course, analysis will satisfy conservativeness whenever we deal with a “reasonably constructed” language L . (For a related discussion of conservativeness in the theory of meaning see [14], pp. 453–455, 396–397.)

Some other remarks should be made concerning condition (1). In it we require that α occurs in \mathbf{A} *only once*. We think that this requirement is justified, since otherwise it could not be said that we are analyzing α , rather than a series of occurrences of α . However, a more relaxed view of analysis may perhaps be envisaged, in which this restriction is lifted. In another respect, condition (1) could be strengthened, viz. we could require that the same schematic letters

should occur in **A** and **B**, or something more elaborate of this kind. (There is no absolute necessity that schematic letters occur in **A** and **B**, but in general they will, as our examples show too.) This requirement would presumably make the analytic equivalence resemble a kind of relevant mutual implication, since “variable sharing” conditions are typical for relevant implication.

Though an analysis doesn’t *give* the meaning of an expression, as an explicit definition would, it follows from conditions (2) and (3) that an analysis is very closely tied to the meaning of the expression analyzed, and that the analytic equivalence should presumably be a kind of relevant equivalence. But this presumed relevant equivalence need not yield an equivalence as strong as identity of meaning, or propositional identity: otherwise we would be in danger of excluding practically all philosophically interesting analyses. However, it is unclear how the relevance involved can be explained formally, and simple sharing of schematic letters need not capture it. Consider, for example, Church’s Thesis:

An arithmetical function f is computable if, and only if, f is Turing computable.

It may well be held that this equivalence gives an analysis of the predicate “is computable”, L being mathematical English and M the language of the theory of Turing machines. Some relevance condition is presumably satisfied by this equivalence, but the relevance involved is not exhausted by the sharing of the schematic letter f .

In general, when the relevant analytic equivalence does not amount to propositional identity, this has to do with a difference in grammatical form between **A** and **B**. Analogously, we have that A and $A \wedge A$ are, no doubt, relevantly equivalent, but they do not stand for the same proposition. In our examples of analyses, save in Church’s Thesis, there is a difference in grammatical form between **A** and **B**. (It seems plausible to say that Church’s Thesis satisfies not only (1)–(3) but also Pascal’s condition and conservativeness; hence, this thesis might be taken not only as an analysis but as an explicit definition as well. And, indeed, this is how it is sometimes understood.)

It seems that (1)–(3) cannot represent all the necessary conditions for a philosophically significant notion of analysis. In particular, they say nothing about the *clarificatory value* of an analysis. It is not enough merely to assume that the language M is a language we understand. It is usually, and rightly, further assumed that M is of a different order than L , in that it is more *basic*, in the sense that it makes fewer assumptions, and that the understanding of L is somehow dependent on a previous understanding of M , and not the other way round. If we are allowed to use the term “basic” without further explanation, a fourth condition for analysis can be stated as follows:

(4) The language M should be more basic than the language L .

Do the examples of analysis we have mentioned up until now satisfy this condition? In these examples, M differed from L in lacking a certain problematic expression we wanted to analyze, but, on the other hand, we supposed that M contained schematic letters, and it may seem that the presence of such technical devices as schematic letters makes this language less basic. However, in our understanding of “basic”, M should be taken as more basic, because schematic

letters serve only to make explicit certain grammatical regularities which are very elementary, and are indeed presupposed by the understanding of ordinary English. The possibility of generating an unlimited number of sentences like “The king of France is bald”, “The king of France is brave”, etc., as soon as we have figured out that “the king of France” functions as something in the grammatical category of nouns, amounts to an implicit understanding of the schematic letter P for predicates. The fact that in M these implicit features are made explicit does not mean that M is less basic. In the same sense, we could say that a book of English grammar is more basic than a novel in English. It may seem less basic to a native speaker of English, but not so to a foreigner: the book of grammar makes fewer assumptions, and the understanding of the novel is in a certain sense dependent on a previous understanding of the grammar, and not the other way around.

So we suppose that condition (4) is satisfied in all of our examples of analysis, including Church’s Thesis, where M , the language of the theory of Turing machines, is more basic in a mathematical sense.

Finally, note that nothing in what we have said excludes the possibility that two literally different equivalences both represent an analysis of the same expression α .

Even if we cannot pretend that (1)–(4) are *all* the necessary conditions that a certain notion of analysis should satisfy, it can still be argued that they are a plausible approximation, and that they could be strengthened to yield a sufficient condition by developing something which they already contain implicitly.

What we have tried to do here is to give necessary conditions only for a certain notion of analysis, and not for all possible notions. Many philosophically interesting analyses are not given exactly in the form of a single equivalence. Indeed, they need not be given in the form of equivalences at all, but, presumably, analysis in the form of equivalence has a certain primacy among all possible notions of analysis.

Let us see now, in the light of this account of analysis, how we can analyze logical constants in structural terms. We shall concentrate only on the analysis of implication given by the double-line rule (\rightarrow). Here implication is directly analyzed, whereas in [10] other constants are only *ultimately* analyzed, i.e. explicitly defined in terms of analyzed constants: for example, negation \neg is defined in terms of implication \rightarrow and the absurd \perp by $\neg A =_{df} A \rightarrow \perp$, and \rightarrow and \perp are directly analyzed.

In the analysis of implication given by (\rightarrow) the language L is the language of propositional, or first-order, logic, and M is the deductive metalanguage in which we speak about structural deductions; more precisely, M is the language of structural sequents. The sentence \mathbf{A} is the lower sequent of (\rightarrow), and \mathbf{B} is the upper sequent of (\rightarrow). Since the double line stands for an equivalence our analysis has the form prescribed in condition (1). In [10] it was shown that (\rightarrow) serves to characterize implication, soundly, completely, and uniquely, in classical, intuitionistic, and relevant logical systems; i.e. conditions (2) and (3) are satisfied. Finally, we suppose that our structural analysis of implication satisfies condition (4), since we suppose that the language M of structural sequents is more basic than the language L of propositional, or first-order, logic. This last supposition can be justified by referring to our assumptions [A] and [B]. If formal

deductions are the subject matter of logic, and basic formal deductions are structural deductions, the language M is indeed more basic than a language L in which formal deductions are not explicit anymore, and where we get only non-structural truths based on logical constants. (Some further remarks on M and L will be made in the next section.)

On the other hand, the structural analysis of implication given by (\rightarrow) need not satisfy Pascal's condition, since in a single-conclusion sequent-system we may be unable to eliminate implication from the sequent $A \rightarrow B \vdash C$. Conservativeness can also fail in some cases: for example, if we add (\rightarrow) to a sequent-system with thinning on the right only, we can derive thinning on the left, and this may make L nonconservative with respect to L minus \rightarrow . However, in the analysis of classical, intuitionistic, and relevant implication, conservativeness will be satisfied.

4 Assumptions [D] and [E] We shall now consider two further assumptions concerning logic, relevant to the discussion of Section 2. The first assumption is the following:

[D] Logic is independent of subject matter.

Though this is an assumption which is found quite often in the philosophy of logic, it is far from being clear. A possible paraphrase of at least part of the import of [D] is that only logical constants are essential for a logical system — everything else can be schematic. In this sense, [D] seems uncontroversial, almost a truism. But sometimes [D] is meant to express more: it is meant to imply that, independently of what we are dealing with, we should rely on the same logical laws. It is very questionable to what extent this stronger version of [D] can be sustained.

Of course, [D] cannot mean that logic is independent of the subject matter of logic, and if we are right in assuming by [A] that formal deductions are the subject matter of logic, we can claim that the same logical laws are always in force only if a single type of formal deduction is recognized as valid. Since by [C] all formal deductions are determined by structural deductions, the question arises whether only one type of structural deduction is valid. Without trying to answer this question with precision, we surmise that the type of valid structural deductions may depend on the object language, and on the subject matter of that language.

For example, consider an object language where sentences can be asserted if and only if they are *provable*. Presumably, the corresponding valid structural deductions should be codified by a single-conclusion sequent-system. And if sentences of our object language can be asserted if and only if they are *true*, the corresponding valid structural deductions should be codified by a multiple-conclusion sequent-system. Whereas the connection between *proof* and single-conclusion sequents seems to be rather clear, the connection between *truth* and multiple-conclusion sequents involves an element of discovery (a discovery which should be ascribed to Gentzen). In some other cases, for example, the language of quantum phenomena, where a sentence can be asserted if and only if it is *verifiable with existing instruments*, it is not clear that any corresponding form of valid structural deductions can, or indeed should, be discovered.

Sometimes [D] is invoked to justify the introduction of quantificational logical systems *free* of existential assumptions, i.e. whose theorems are valid in the empty domain too. Now we have seen in Section 2 that first-order quantificational logic can be understood as the logic of “any”, or, semantically, as the logic of arbitrary individuals. The assumption that there are some individuals can then be understood as proceeding directly from the subject matter of this logic. Presumably, quantificational logic cannot be independent of its own subject matter. The producing of a free logic, though it need not involve an actual contradiction, may well seem unreasonable: on the one hand, we want a logic whose subject matter consists of arbitrary individuals, but on the other hand, we want this logic to be applicable even when its subject matter is missing. (To make an analogy, it is like giving library rules that are meant to be applied even in libraries without books: the question whether an empty domain is a domain is like the question whether a library without books is a library.) Since logic cannot be independent of the subject matter of logic, [D] understood in such a way that it justifies the introduction of free logics does not seem warranted.

The second additional assumption concerning logic which we shall consider is the following:

[E] The level of discourse of logic is higher than the level of discourse in which we treat of a particular subject matter relying on logical principles.

Part of what [E] can mean is contained in the uncontroversial part of [D], which says that in a logical system everything except logical constants is schematic, i.e. made of meta-variables. But another ingredient of [E] may be that a logical system is formulated properly in a *higher* language, a deductive metalanguage, like the language of sequents, whereas we treat of a particular subject matter in a *lower* language, an object language from which we draw our premises and conclusions.

Understood in this latter sense, assumption [E] probably expresses the lesson taught by Achilles and the Tortoise in [8]. The Tortoise (its head low on the ground) is unable to see anything above the lower language. Disregarding [E] is the misleading aspect of the picture of language in Quine’s “Two Dogmas” [35], and of the general conception that logic is the science of logical truths of the object language (cf. [14], p. 596, and [15], p. 353). (However, [E] seems to be assumed by Quine in his critique of conventionalism in [34].)

Assumption [E] is in perfect harmony with [A], [B], [C], and thesis [I]. Logical constants are expressions of the lower language which have their *raison d’être* in some features of the higher language. When we treat of a particular subject matter, some structural features of the higher language are only implicit in the activity of making deductions. Logical constants serve to make explicit these features in the lower language: they help us to reduce structural truths of the higher language to truths of the lower language. In other words, they reduce truths of the language *M* of structural deductions to truths of the language *L* of propositional, or first-order, logic. This is why we can say metaphorically that logical constants are punctuation marks for some structural features of deductions. This metaphor underlies thesis [I].

The analogy between logical constants and punctuation marks should be

taken with a certain reservation. Ordinary punctuation marks usually directly exhibit certain features of the activity of speech in all contexts in which they may occur, whereas logical constants directly exhibit certain structural features of the activity of deducing only when they occur in a particular way in a deduction, viz. as the main constant of a conclusion or of a premise. Some logical constants are analyzed by double-line rules in which they occur on the right-hand side of the turnstile, and others by double-line rules in which they occur on the left-hand side of the turnstile; for some other constants, like \forall , \exists , \top , \perp , and $=$, there may be additional conditions concerning the respective double-line rules. When a constant occurs in the lower language, or in a context of the higher language, where it cannot be eliminated by using the respective double-line rule, we may still say that it mirrors the corresponding punctuation function, but only in a derived sense. Generalizing condition (2) for analysis, we may assume that whatever holds for this constant, in any context of the lower or higher language, should be derivable from the corresponding double-line rule. The fact that logical constants cannot always be eliminated, i.e. the fact that they need not satisfy Pascal's condition, indicates that there might be a real gain in introducing them not only in the lower language but also in the higher language: they may enable us to say things which cannot be said without them.

To conclude our discussion of thesis [I], let us examine how effective the criterion which it provides is. It can be inferred from [10] and [11] that the customary logical constants of first-order logic with identity, and the modal constants of classical and intuitionistic S4 and S5, are all logical. (It is also not difficult to give an analysis in structural terms of some other constants, like the Hilbert ϵ -operator.) In general, thesis [I] can effectively be applied to show that a constant *is* logical: if it is ultimately analyzed in structural terms we can claim that it is logical. On the other hand, this thesis does not seem to be effective in showing that a constant is *not* logical, for it is not clear on what grounds we could claim that an ultimate analysis in structural terms is impossible. So the effective use of thesis [I] in settling disputed cases may be limited, if we don't state more precisely what form a structural analysis can possibly take. But, even in this imperfect form, thesis [I] can serve to show what is common to all the constants which are usually accepted as logical without dispute, and which are logical according to this thesis too. Moreover, together with the underlying assumptions about logic which we have discussed it shows that this common characteristic of logical constants proceeds from a certain conception of logic.

5 Thesis [II] We stated in Section 2 that the double-line rule (\rightarrow) can serve to characterize implication in various logical systems: classical, intuitionistic, and relevant. In these characterizations (\rightarrow) is always the same: only assumptions concerning structural deductions are changed. We also mentioned that the situation is analogous for other logical constants we have considered. This situation naturally leads us to postulate the following thesis:

[II] Two logical systems are alternative if, and only if, they differ only in their assumptions on structural deductions.

This thesis can also be viewed as a consequence of assumptions [A], [B], and [C]. According to these assumptions, different systems of formal deductions can arise because they have either

- (i) different structural deductions and logical constants ultimately analyzed in the same way, or
- (ii) the same structural deductions and logical constants ultimately analyzed in different ways, or
- (iii) both different structural deductions and logical constants ultimately analyzed in different ways.

We suppose that the situation is best described by saying that in case (i) we are confronted with *alternative* logical systems, in case (ii) with logical systems that are *supplements*, and in case (iii) with logical systems that are both alternatives and supplements. For example, classical first-order logic and intuitionistic first-order logic are alternative logical systems, whereas the first-order modal logics S4 and S5 are supplements of classical first-order logic.

If the language M used for giving an analysis of α can be understood in two different ways (as happens when we assume different structural rules to be valid in M), so that in fact we have two languages, M_1 and M_2 , then the analysis based on M_1 cannot be the same as the analysis based on M_2 . However, it is clear that these two different analyses will have a common core. The expression α_1 with the analysis based on M_1 will not have the same meaning as the expression α_2 with the analysis based on M_2 (according to condition (3) for analysis, in Section 3), but we can say that α_1 and α_2 are *analytically identical*. This analytic identity is a kind of identity of function; in the case of logical constants this is identity of structural deductive function. This identity may be taken as the “common denominator” of alternative logical systems. For example, we are justified in saying that both classical and intuitionistic implication are implications, because for both of them we assume the same structural deductive function, i.e. we assume a *deduction theorem* and *modus ponens*. Usually, this “common denominator” of alternative logical systems is not clearly characterized, and a more or less vague resemblance between the alternatives is considered sufficient.

Two logical systems can “differ only in their assumptions on structural deductions” not only when these assumptions are explicit, as in sequent-systems, but also when these assumptions are only implicit, as in Hilbert-style axiomatizations.

We shall summarize our discussion of thesis [II] by considering how it proceeds from thesis [I]. The import of thesis [I] is that a logical system is completely determined by its assumptions on structural deductions. Hence, to change logic, and not merely to supplement it by new “punctuation marks”, we must change structural deductions. An alternative logical system is obtained when the same “punctuation marks” work in a different structural context.

6 Similar programs Our attempt to analyze logical constants and thesis [I] should not be confused with the program of *defining* logical constants by syntactical means. First, the main goal of this program is to show that the *mean-*

ing of logical constants can be given syntactically, whereas our analyses are neutral with respect to this claim, and are equally compatible with the view that the meaning of logical constants is to be given in a more conventional semantical framework. Second, the search for a criterion for being a logical constant does not always have a very important place in this program. If the problem of finding this criterion is considered at all, it is usually taken that logical constants will be just those expressions whose meaning can be given by syntactical means, which makes the search for a criterion dependent upon the main goal of the program. Thesis [I] is an attempt to formulate such a criterion without tying it to a thesis on the meaning of logical constants. On the other hand, it is clear that our analyses of logical constants and thesis [I] are congenial to this program. Presumably, the “deductive structural function”, which we think is demonstrated by our analyses, must be closely tied with the meaning of logical constants, even if we need not assume that it coincides exactly with this meaning.

The origins of this program can be traced to Gentzen, who made the following remark concerning the introduction and elimination rules of his natural deduction systems: “Die Einführungen stellen sozusagen die „Definitionen“ der betreffenden Zeichen dar, und die Beseitigungen sind letzten Endes nur Konsequenzen hiervon, . . .” ([18], Section 5.13).

In a series of papers (which he later called “bad and ill fated”) Popper attempted to use a certain sort of sequent-system to define logical constants, and to find a criterion for the demarcation of logic. (For references see [37], which reconstructs in great detail Popper’s theory by divorcing it from the attempt to define logical constants syntactically, and tries only to derive from Popper’s work a syntactical criterion for the demarcation of logic; it also contains an extensive bibliography on matters related to the program we are considering.) A program similar to Popper’s was formulated by Kneale in [22] (see also [23]), who tried to formulate for that purpose a natural deduction multiple-conclusion system. The program has been more recently pursued by Hacking in [20]. Hacking is concerned in principle with standard Gentzen-style sequent-systems with rules for introducing constants on the left and on the right of the turnstile, but he also tries to introduce a certain notion pertaining to the parametric parts of these rules which apparently makes it impossible for him to deal with modal constants. One of Hacking’s aims is to show that quantum logic is logic.

That part of the proof-theoretical program of Prawitz (see [31] and [32]) which is relevant to our discussion is not so much concerned with the demarcation of logic, as with pursuing the idea expressed in Gentzen’s remark on natural deduction introduction rules. This also holds for the relevant views of Dummett (see [16] and [17], pp. 389–403).

Prior criticized this general program by giving in [33] natural deduction rules for what we would call “an inconsistent constant” (see Section 3). In the reply by Belnap [2], the requirements of conservativeness and uniqueness are stated for rules which pretend to define a constant (cf. [13]).

Our replacement of the term “definition” by the term “analysis” in the above program is not merely a verbal move. One substantial difference is that the requirement of conservativeness ceases to play for us the role it has to play in the former program. We don’t want our double-line rules to give the meaning of the logical constants, but only a philosophically significant analysis. It

is possible that something not very far from our notion of analysis was on Gentzen's mind when he made his brief remark on the *sozusagen Definitionen*.

The metaphor we have used to describe the function of logical constants is reminiscent of the following remarks of Wittgenstein from [44]: "Die logischen Operationszeichen sind Interpunktionen" (5.4611). ". . . in der Logik ist jeder Satz die Form eines Beweises" (6.1264). However, one would probably have to take these remarks out of their immediate context in order to connect them with the views we have expressed here.

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