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Cut-Free Systems for Three-Valued Modal Logics

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Abstract Cut-free formal systems for some of the three-valued modal propositional logics are given. This refines Morikawa's work.

Introduction In [1] Morikawa introduced (the three-valued version of) the sequent calculi for the three-valued modal propositional logics **3-K**$_3$, **3-K**$_2$, **3-M**$_3$, **3-M**$_2$, **3-S4**$_3$, **3-S4**$_2$, **3-S5**$_3$, and **3-S5**$_2$. But, as he showed in [1] (Theorem 1), none of them enjoys the cut-elimination property. So we will formulate, in this paper, another sequent calculi (for those logics) which admit elimination of cuts. In the first section, we introduce among others the notion of validity in those logics; to make this article self-contained, we will repeat here many of the definitions in [1]. In the next section, our formal systems are presented in the style of (the three-valued version of) the sequent calculi. The facts that provability implies validity and that the latter implies cut-free provability will be shown in Sections 3 and 4, respectively. In view of the equivalence between validity and cut-free provability of our systems, decision procedures for those logics are easily obtained.

We let G be a variable varying through the logics **3-K**$_3$, **3-K**$_2$, **3-M**$_3$, **3-M**$_2$, **3-S4**$_3$, **3-S4**$_2$, **3-S5**$_3$, and **3-S5**$_2$. Then, put $i(G) = 3$ or $i(G) = 2$, according as $G \in \{\mathbf{3\text{-}K_3}, \mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}, \mathbf{3\text{-}S5_3}\}$ or $G \in \{\mathbf{3\text{-}K_2}, \mathbf{3\text{-}M_2}, \mathbf{3\text{-}S4_2}, \mathbf{3\text{-}S5_2}\}$.

1 Preliminaries We put $T = \{1,2,3\}$, and will use T as the set of *truth values*. Intuitively, the truth values 1, 2, and 3 stand for 'true', 'undefined', and 'false', respectively. We let $\lambda, \mu, \nu, \ldots$ denote truth values. We mean by $\mu\hat{}$ the set $T - \{\mu\}$.

Formulas are constructed from propositional variables by means of propositional connectives and the necessity operator $\Box$; we assume that for each propositional connective F, the arity $\alpha(F) \geq 0$ and the truth function $f_F: T^{\alpha(F)} \to T$ are predetermined. *Subformulas* of a formula are defined as usual. We use $A, B, C, \ldots$ as syntactical variables which vary through formulas.

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Definition A *matrix* is a finite set consisting of valued formulas, where a *valued formula* is any pair of a formula and a truth value.

A matrix represents a generalized disjunction of constituent formulas. In [1], it has been defined to be a triplet of finite sets of formulas, as the three-valued version of Gentzen's sequent. But for convenience we modify the definition as above. Thus a matrix is a set of the form:

$$\{(A_1,1),\ldots,(A_l,1),(B_1,2),\ldots,(B_m,2),(C_1,3),\ldots,(C_n,3)\},$$

and this corresponds to the matrix in the sense of [1]:

$$\{A_1,\ldots,A_l\}_1 \cup \{B_1,\ldots,B_m\}_2 \cup \{C_1,\ldots,C_n\}_3,$$

where $l, m, n \geq 0$. When S is a subset of T, the matrix $\{(A,\mu) \mid \mu \in S\}$ is abbreviated as (A,S); the matrix $L \cup \{(A,\mu)\}$ is written simply as $L \cup (A,\mu)$. We use $K, L, \ldots$ as syntactical variables which vary through matrices.

Definition A *(three-valued) Kripke frame* is the triplet (W, R, φ), where W is a nonempty set, R a binary relation on W, and φ a mapping which assigns a truth value to each pair of a propositional variable and an element of W.

Definition Suppose that (W, R, φ) is a Kripke frame and $i \in \{3,2\}$. We call the triplet (W, R, φ^i) a *(three-valued) Kripke structure of type i (generated from* (W, R, φ)*)*, if φ^i is the mapping which assigns a truth value to each pair of a formula and an element of W, and is defined by recursion as follows:

$\varphi^i(p,s) = \varphi(p,s)$, where p is a propositional variable;

$\varphi^i(F(A_1,\ldots,A_{\alpha(F)}),s) = f_F(\varphi^i(A_1,s),\ldots,\varphi^i(A_{\alpha(F)},s))$;

$$\varphi^3(\Box A,s) = \begin{cases} 1, \text{ if } sRt \text{ implies } \varphi^3(A,t) = 1 \text{ for every } t \in W; \\ 2, \text{ if } sRt \text{ and } \varphi^3(A,t) = 2 \text{ for some } t \in W; \\ 3, \text{ otherwise;} \end{cases}$$

$$\varphi^2(\Box A,s) = \begin{cases} 1, \text{ if } sRt \text{ implies } \varphi^2(A,t) = 1 \text{ for every } t \in W; \\ 3, \text{ otherwise.} \end{cases}$$

Definition *Models* of G are defined as follows. Models of $\mathbf{K}_i$ are nothing but the Kripke structures of type i; whereas a model of $\mathbf{M}_i$, $\mathbf{S4}_i$, or $\mathbf{S5}_i$ is a Kripke structure (W, R, φ^i) of type i such that R is reflexive, reflexive and transitive, or forms an equivalence relation, respectively.

Definition A Kripke structure (W, R, φ^i) of type i *certifies* (*rejects*) a matrix L at $s \in W$, when $(B, \varphi^i(B,s)) \in L$ for some formula B (for no formula B, resp.).

Definition A matrix is *valid* in G if it is certified by every model $(W, R, \varphi^{i(G)})$ of G at every element of W.

2 Formal systems for three-valued modal logics

In this section, we introduce our formal systems for the logics **3-K**$_3$, **3-K**$_2$, **3-M**$_3$, **3-M**$_2$, **3-S4**$_3$, **3-S4**$_2$, **3-S5**$_3$, and **3-S5**$_2$.

Beginning matrices and structural rules Each of our systems has (A, T) as the *beginning matrix* (namely, the 0-premise rule), where A is any formula, and the following rules as the *structural rules*:

$$\text{Weakening:} \quad \frac{L}{K}, \text{ where } L \subseteq K.$$

$$\text{Cut:} \quad \frac{L \cup (A, \mu) \qquad K \cup (A, \nu)}{L \cup K}, \text{ where } \mu \neq \nu.$$

Propositional rules The following rule $(F; \mu)$ is used in each of our systems as the *propositional rule*, for every propositional connective F and every $\mu \in T$.

$$(F; \mu): \quad \frac{\begin{array}{c} L \cup (A_1, \nu_1\hat{\ }) \cup \ldots \cup (A_{\alpha(F)}, \nu_{\alpha(F)}\hat{\ }) \\ \text{for every } (\nu_1, \ldots, \nu_{\alpha(F)}) \in f_F^{-1}(\mu\hat{\ }) \end{array}}{L \cup (F(A_1, \ldots, A_{\alpha(F)}), \mu)}.$$

$\Box$-operation and modal rules To formulate the modal rules of our systems, we need the $\Box$-operation on matrices. The definition of the $\Box$-*operation* is taken from the following list depending on logics; the choice for each logic is described later.

(D1) $L^{\Box} = \{(B,2) \mid (\Box B,2) \in L\} \cup \{(B,3) \mid (\Box B,1\hat{\ }) \subseteq L\}$.
(D2) $L^{\Box} = \{(B,\nu) \mid \nu \in 1\hat{\ }, (\Box B,3) \in L\}$.
(D3) $L^{\Box} = \{(\Box B,2) \mid (\Box B,2) \in L\} \cup \{(\Box B,3) \mid (\Box B,1\hat{\ }) \subseteq L\}$.
(D4) $L^{\Box} = \{(\Box B,3) \mid (\Box B,3) \in L\}$.
(D5) $L^{\Box} = \{(\Box B,\nu) \mid \nu \in T, (\Box B,\nu) \in L\}$.

Some of the following rules are selected as the *modal rules* of our systems; the list of the rules for each logic is specified later as well.

$$(\Box;1): \quad \frac{L^{\Box} \cup (A,1)}{L \cup (\Box A,1)}.$$

$$(\Box;2): \quad \frac{L \cup (A,2)}{L \cup (\Box A,2)}.$$

$$(\Box;3)_1: \quad \frac{L^{\Box} \cup (A,2\hat{\ }) \qquad L \cup (\Box A,2)}{L \cup (\Box A,3)}.$$

$$(\Box;3)_2: \quad \frac{L \cup (A,1\hat{\ })}{L \cup (\Box A,3)}.$$

$$(\Box;3)_3: \quad \frac{L^{\Box} \cup (A,2\hat{\ }) \qquad L \cup (A,3)}{L \cup (\Box A,3)}.$$

$$(\Box;1\hat{\ }): \quad \frac{L \cup (A,3)}{L \cup (\Box A,1\hat{\ })}.$$

$$(\Box;\mu)^*: \quad \frac{L \cup (A,\nu\hat{\ }) \text{ for every } \nu \in T}{L \cup (\Box A,\mu)}.$$

Now the choice of the definition of the $\Box$-operation and the modal rules for each logic is made as follows:

3-K$_3$: (D1); $(\Box;1)$, $(\Box;3)_1$.
3-K$_2$: (D2); $(\Box;1)$.
3-M$_3$: (D1); $(\Box;1)$, $(\Box;2)$, $(\Box;3)_1$, $(\Box;\hat{1})$.
3-M$_2$: (D2); $(\Box;1)$, $(\Box;3)_2$.
3-S4$_3$: (D3); $(\Box;1)$, $(\Box;2)$, $(\Box;3)_1$, $(\Box;\hat{1})$.
3-S4$_2$: (D4); $(\Box;1)$, $(\Box;3)_2$.
3-S5$_3$: (D5); $(\Box;1)$, $(\Box;2)$, $(\Box;3)_3$, $(\Box;\mu)^*$ for every $\mu \in T$.
3-S5$_2$: (D5); $(\Box;1)$, $(\Box;3)_2$, $(\Box;\mu)^*$ for every $\mu \in T$.

Definition A matrix is *provable* (*strictly provable*) in G, when it is obtained from beginning matrices by a finite number of applications of rules (rules except cut, resp.) of our system for G.

Note that the modal rule $(\Box;\mu)^*$ is derivable by means of cut and weakening. But it is included in our systems for **3-S5$_3$** and **3-S5$_2$** to eliminate cut.

Hence the main theorem of this article, which claims the deductive completeness and the cut-elimination property of our systems, is formulated as below. It will be routine, by this theorem, to decide whether a given matrix is valid in G or not.

Theorem *Let G be* **3-K$_3$**, **3-K$_2$**, **3-M$_3$**, **3-M$_2$**, **3-S4$_3$**, **3-S4$_2$**, **3-S5$_3$**, *or* **3-S5$_2$**. *The following properties on any matrix L are mutually equivalent*:
(a) *L is valid in G.*
(b) *L is provable in G.*
(c) *L is strictly provable in G.*

Clearly, (c) implies (b). We will show in Section 3 that (a) is a consequence of (b), and in Section 4 the fact that (c) follows from (a) will be proved via a (generalized) canonical model construction.

3 Soundness of our systems In this section, we will show the soundness property of our systems, that is, the fact that every provable matrix is valid. We will first prove the following lemma.

Lemma 1 *Let $\mathcal{M} = (W, R, \varphi^{i(G)})$ be a model of G. If sRt, and $\mathcal{M}$ certifies $L^{\Box}$ at t, then it certifies L at s; in other words, if sRt, and $\mathcal{M}$ rejects L at s, then it rejects $L^{\Box}$ at t.*

Proof: We suppose that sRt, and $\mathcal{M}$ certifies $L^{\Box}$ at t.

Case 1: $G \in \{$**3-K$_3$**, **3-M$_3$**$\}$, so that $i(G) = 3$ and the $\Box$-operation is defined by (D1). By the assumption, either $\varphi^3(B,t) = 2$ and $(\Box B, 2) \in L$, or $\varphi^3(B,t) = 3$ and $(\Box B, \hat{1}) \subseteq L$, for some B. Since $\varphi^3(\Box B, s) = 2$ in the former case, whereas $\varphi^3(\Box B, s) \neq 1$ in the latter, it follows $(\Box B, \varphi^3(\Box B, s)) \in L$ in either case; so $\mathcal{M}$ certifies L at s.

Case 2: $G \in \{$**3-K$_2$**, **3-M$_2$**$\}$, so that $i(G) = 2$ and the $\Box$-operation is defined by (D2). By the assumption, $\varphi^2(B,t) \neq 1$ and $(\Box B, 3) \in L$ for some B. Since $\varphi^2(\Box B, s) = 3$, it follows $(\Box B, \varphi^2(\Box B, s)) \in L$; so $\mathcal{M}$ certifies L at s.

Case 3: $G \in \{\mathbf{3\text{-}S4_3}\}$, so that $i(G) = 3$, the $\Box$-operation is defined by (D3), and R is transitive. Since $\varphi^3(\Box B, t) = 2$ ($\varphi^3(\Box B, t) = 3$) implies $\varphi^3(\Box B, s) = 2$($\varphi^3(\Box B, s) \neq 1$, resp.), one can manage this case similarly to Case 1.

Case 4: $G \in \{\mathbf{3\text{-}S4_2}\}$, so that $i(G) = 2$, the $\Box$-operation is defined by (D4), and R is transitive. Since $\varphi^2(\Box B, t) = 3$ implies $\varphi^2(\Box B, s) = 3$, one can manage this case similarly to Case 2.

Case 5: $G \in \{\mathbf{3\text{-}S5_3}, \mathbf{3\text{-}S5_2}\}$, so that the $\Box$-operation is defined by (D5), and R is symmetric and transitive. By the assumption, $\varphi^{i(G)}(\Box B, t) = \nu$ and $(\Box B, \nu) \in L$ for some B and some $\nu \in T$. Then by a simple calculation, it follows that $\varphi^{i(G)}(\Box B, s) = \nu$. Hence $(\Box B, \varphi^{i(G)}(\Box B, s)) \in L$; so $\mathcal{M}$ certifies L at s.

Now we will demonstrate the following lemma, which states that the property (b) formulated in the Theorem implies the property (a).

Lemma 2 *If a matrix is provable in G, it is valid in G.*

Proof: The proof is by induction on the length of the proof of the matrix. It is clear that any beginning matrix having the form (A, T) for some A is valid, and that the structural rules (namely, weakening and cut) preserve the validity of matrices. So we are left to mention the propositional and the modal rules in the following.

It suffices to show that, if a model $(W, R, \varphi^{i(G)})$ of G rejects the lower matrix of a propositional or a modal rule at some element of W, then it also rejects some upper matrix of the rule at some element of W. Hence, we let $\mathcal{M} = (W, R, \varphi^{i(G)})$ be a model of G, and assume that $\mathcal{M}$ rejects at $s \in W$ the lower matrix of a rule.

Case 1: Propositional rule $(F; \mu)$ with the upper matrices $L \cup (A_1, \nu_1\hat{\ }) \cup \ldots \cup (A_{\alpha(F)}, \nu_{\alpha(F)}\hat{\ })$ for each $(\nu_1, \ldots, \nu_{\alpha(F)}) \in f_F^{-1}(\mu\hat{\ })$ and the lower matrix $L \cup (F(A_1, \ldots, A_{\alpha(F)}), \mu)$. By the assumption, $\varphi^{i(G)}(F(A_1, \ldots, A_{\alpha(F)}), s) \neq \mu$. Put $\nu_k = \varphi^{i(G)}(A_k, s)$ for $k = 1, \ldots, \alpha(F)$. Then it follows $f_F(\nu_1, \ldots, \nu_{\alpha(F)}) \neq \mu$. Hence $L \cup (A_1, \nu_1\hat{\ }) \cup \ldots \cup (A_{\alpha(F)}, \nu_{\alpha(F)}\hat{\ })$ constitutes an upper matrix and is rejected by $\mathcal{M}$ at s.

Case 2: Modal rule $(\Box; 1)$ with the upper matrix $L^{\Box} \cup (A, 1)$ and the lower matrix $L \cup (\Box A, 1)$. By the assumption, $\varphi^{i(G)}(\Box A, s) \neq 1$, so sRt and $\varphi^{i(G)}(A, t) \neq 1$ for some $t \in W$. Hence $\mathcal{M}$ rejects the upper matrix at t by Lemma 1.

Case 3: Modal rule $(\Box; 2)$ with the upper matrix $L \cup (A, 2)$ and the lower matrix $L \cup (\Box A, 2)$, and moreover $G \in \{\mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}, \mathbf{3\text{-}S5_3}\}$ so that $i(G) = 3$ and R is reflexive. Since $\varphi^3(\Box A, s) \neq 2$ by the assumption, $\varphi^3(A, s) \neq 2$. Hence $\mathcal{M}$ rejects the upper matrix at s.

Case 4: Modal rule $(\Box; 3)_1$ with the upper matrices $L^{\Box} \cup (A, 2\hat{\ })$ and $L \cup (\Box A, 2)$ and the lower matrix $L \cup (\Box A, 3)$, and moreover $G \in \{\mathbf{3\text{-}K_3}, \mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}\}$ so that $i(G) = 3$. By the assumption, $\varphi^3(\Box A, s) \neq 3$. If $\varphi^3(\Box A, s) = 1$, then $\mathcal{M}$ rejects the upper matrix $L \cup (\Box A, 2)$ at s; whereas if $\varphi^3(\Box A, s) = 2$, then sRt and $\varphi^3(A, t) = 2$ for some $t \in W$, hence $\mathcal{M}$ rejects the upper matrix $L^{\Box} \cup (A, 2\hat{\ })$ at t by Lemma 1.

Case 5: Modal rule $(\Box;3)_2$ with the upper matrix $L \cup (A,1^{\hat{}})$ and the lower matrix $L \cup (\Box A,3)$, and moreover $G \in \{\mathbf{3\text{-}M_2}, \mathbf{3\text{-}S4_2}, \mathbf{3\text{-}S5_2}\}$ so that $i(G) = 2$ and R is reflexive. Note that $\varphi^2(\Box A, s) \neq 3$ implies $\varphi^2(A,s) = 1$. Hence, $\mathcal{M}$ rejects the upper matrix at s.

Case 6: Modal rule $(\Box;3)_3$ with the upper matrices $L^{\Box} \cup (A,2^{\hat{}})$ and $L \cup (A,3)$ and the lower matrix $L \cup (\Box A,3)$, and moreover $G \in \{\mathbf{3\text{-}S5_3}\}$ so that $i(G) = 3$ and R is reflexive. Note that $\varphi^3(\Box A,s) = 1$ implies $\varphi^3(A,s) = 1$. Then one can manage this case similarly to Case 4.

Case 7: Modal rule $(\Box;1^{\hat{}})$ with the upper matrix $L \cup (A,3)$ and the lower matrix $L \cup (\Box A,1^{\hat{}})$, and moreover $G \in \{\mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}\}$ so that $i(G) = 3$ and R is reflexive. Similar to Case 3.

Case 8: Modal rule $(\Box;\mu)^*$ with the upper matrices $L \cup (A,\nu^{\hat{}})$ for each $\nu \in T$ and the lower matrix $L \cup (\Box A,\mu)$, and moreover $G \in \{\mathbf{3\text{-}S5_3}, \mathbf{3\text{-}S5_2}\}$. Put $\nu = \varphi^{i(G)}(A,s)$. Then $\mathcal{M}$ rejects the upper matrix $L \cup (A,\nu^{\hat{}})$ at s.

Since models of G certainly exist, it is not the case that the empty matrix (namely, the empty set as a matrix) is valid in G. Hence the following corollary holds.

Corollary *The empty matrix is unprovable in G.*

4 *Strict completeness of our systems*

In this final section, we will prove following Schütte [2] the strict completeness property of our systems, that is, the fact that every valid matrix is strictly provable.

Definition A matrix K is *G-subsidiary* to a matrix L, when for every valued formula (A,μ) in K, the matrix L contains a valued formula (B,ν) such that A is a subformula or a proper subformula of B according as $G \in \{\mathbf{3\text{-}K_3}, \mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}\}$ or not (that is, $(\Box;3)_1$ is a rule of G or not).

Definition A matrix L forms a *partial valuation* of G, when (i) L is not strictly provable in G, and (ii) if K is G-subsidiary to L, but the matrix $L \cup K$ is not strictly provable in G, then $K \subseteq L$.

Lemma 3 *Suppose that s forms a partial valuation of G.*

(1) *If $(F(A_1,\ldots,A_{\alpha(F)}),\mu) \in s$, then $(A_1,\nu_1^{\hat{}}) \cup \ldots \cup (A_{\alpha(F)},\nu_{\alpha(F)}^{\hat{}}) \subseteq s$ for some $(\nu_1,\ldots,\nu_{\alpha(F)}) \in f_F^{-1}(\mu^{\hat{}})$.*
(2) *Let $G \in \{\mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}, \mathbf{3\text{-}S5_3}\}$. If $(\Box A,2) \in s$, then $(A,2) \in s$.*
(3) *Let $G \in \{\mathbf{3\text{-}M_2}, \mathbf{3\text{-}S4_2}, \mathbf{3\text{-}S5_2}\}$. If $(\Box A,3) \in s$, then $(A,1^{\hat{}}) \subseteq s$.*
(4) *Let $G \in \{\mathbf{3\text{-}M_3}, \mathbf{3\text{-}S4_3}\}$. If $(\Box A,1^{\hat{}}) \subseteq s$, then $(A,3) \in s$.*
(5) *Let $G \in \{\mathbf{3\text{-}S5_3}, \mathbf{3\text{-}S5_2}\}$. If $(\Box A,\mu) \in s$, then $(A,\nu^{\hat{}}) \subseteq s$ for some $\nu \in T$.*

Proof: (1): Note that $(F;\mu)$ forms a rule of each G. Suppose $(F(A_1,\ldots, A_{\alpha(F)}),\mu) \in s$. It follows that $s \cup (A_1,\nu_1^{\hat{}}) \cup \ldots \cup (A_{\alpha(F)},\nu_{\alpha(F)}^{\hat{}})$ is not strictly provable in G for some $(\nu_1,\ldots,\nu_{\alpha(F)}) \in f_F^{-1}(\mu^{\hat{}})$, since otherwise s becomes strictly provable in G by means of the rule $(F;\mu)$. Hence $(A_1,\nu_1^{\hat{}}) \cup \ldots \cup (A_{\alpha(F)},\nu_{\alpha(F)}^{\hat{}})$ is included in s, for it is G-subsidiary to s. (2)–(5): Note that $(\Box;2)$, $(\Box;3)_2$, $(\Box;1^{\hat{}})$, or $(\Box;\mu)^*$ forms a rule of G, respectively. Then one can manage these similarly to (1).

Lemma 4 *Suppose that $G \in \{$**3-S5**$_3$, **3-S5**$_2\}$, and s forms a partial valuation of G. If $(B,\mu) \in s$, and C is a proper subformula of B, then $(C, \nu^{\hat{}}) \subseteq s$ for some $\nu \in T$.*

Proof: We prove this by induction on the construction of B.

Case 1: B is a propositional variable. This is impossible, since B must have a proper subformula C.

Case 2: B has the form $F(A_1, \ldots, A_{\alpha(F)})$. Since C is a proper subformula of B, it is a subformula of A_k for some $k = 1, \ldots, \alpha(F)$. From $(B,\mu) \in s$, it follows by Lemma 3(1) that, $(A_1, \nu_1^{\hat{}}) \cup \ldots \cup (A_{\alpha(F)}, \nu_{\alpha(F)}^{\hat{}}) \subseteq s$ for some $(\nu_1, \ldots, \nu_{\alpha(F)}) \in f_F^{-1}(\mu^{\hat{}})$, especially $(A_k, \nu_k^{\hat{}}) \subseteq s$. Hence if C is A_k itself, the conclusion of the lemma has been obtained; whereas if C is a proper subformula of A_k, the conclusion follows from the hypothesis of induction.

Case 3: B has the form $\Box A$. Similar to Case 2 by Lemma 3(5).

Lemma 5 *Suppose that L is not strictly provable in G. Then, L can be extended to a partial valuation t of G. Moreover, if $G \in \{$**3-S5**$_3$, **3-S5**$_2\}$, t maintains the following additional property: if s forms a partial valuation of G, $s^\Box \subseteq L$, and the matrix $L - s^\Box$ is G-subsidiary to $s^\Box$, then $s^\Box = t^\Box$.*

Proof: Let $K_1, \ldots, K_n$ be the matrices which are G-subsidiary to L. We define $L_1, \ldots, L_n, L_{n+1}$ by recursion as follows. Put $L_1 = L$. If $1 \le k \le n$ and L_k has been defined, we let L_{k+1} be the matrix L_k or $L_k \cup K_k$ according as $L_k \cup K_k$ is strictly provable in G or not.

Now we shall show that L_{n+1} constitutes the desired matrix. Clearly, $L = L_1 \subseteq L_2 \subseteq \ldots \subseteq L_{n+1}$ and none of $L_1, L_2, \ldots,$ or L_{n+1} is strictly provable in G. Suppose that K is G-subsidiary to L_{n+1}, but $L_{n+1} \cup K$ is not strictly provable in G. It is easy to see that K is also G-subsidiary to L, so $K = K_k$ for some $k = 1, \ldots, n$. Then $L_k \cup K$ is not strictly provable in G, since $L_k \subseteq L_{n+1}$. Hence $L_{k+1} = L_k \cup K$, so $K \subseteq L_{n+1}$. Thus L_{n+1} constitutes an extension of L and a partial valuation of G.

Now we suppose $G \in \{$**3-S5**$_3$, **3-S5**$_2\}$ and will show the additional property of L_{n+1}. Assume that s forms a partial valuation of G, $s^\Box \subseteq L$, and $L - s^\Box$ is G-subsidiary to $s^\Box$. Recall that the $\Box$-operation is defined by (D5). Clearly, $s^\Box \subseteq L^\Box \subseteq L_{n+1}^\Box$. Hence to derive $s^\Box = L_{n+1}^\Box$, it suffices to deduce a contradiction from the assumption $(C,\lambda) \in L_{n+1}^\Box - s^\Box$. Since $L_{n+1} - L$ is G-subsidiary to L by the construction of L_{n+1}, it follows that there is a valued formula $(B,\mu) \in s^\Box \subseteq s$ such that C is a proper subformula of B. Then by Lemma 4, $(C, \nu^{\hat{}}) \subseteq s$ for some $\nu \in T$. If $\lambda = \nu$, then from $(C,\nu) = (C,\lambda) \in L_{n+1}$ and $(C, \nu^{\hat{}}) \subseteq s^\Box \subseteq L_{n+1}$, it follows $(C, T) \subseteq L_{n+1}$ and so L_{n+1} becomes strictly provable in G, which is a contradiction. On the other hand, if $\lambda \ne \nu$, then $(C,\lambda) \in (C, \nu^{\hat{}}) \subseteq s$, which contradicts the assumption. Hence $s^\Box = L_{n+1}^\Box$.

Lemma 6 *Suppose that s forms a partial valuation of G.*

(1) *If $(\Box A, 1) \in s$, then $s^\Box \subseteq t$ and $(A, 1) \in t$ for some partial valuation t of G. Moreover, if $G \in \{$**3-S5**$_3$, **3-S5**$_2\}$, then t can be taken so that $s^\Box = t^\Box$.*

(2) *Suppose $G \in \{$**3-K**$_3$, **3-M**$_3$, **3-S4**$_3\}$. If $(\Box A, 3) \in s$, then either $s^\Box \subseteq t$ and $(A, 2^{\hat{}}) \subseteq t$ for some partial valuation t of G, or $(\Box A, 2) \in s$.*

(3) *Suppose* $G \in \{\textbf{3-S5}_3\}$. *If* $(\Box A,3) \in s$, *then either* $s^\Box = t^\Box$ *and* $(A,2\hat{\ }) \subseteq t$ *for some partial valuation* t *of* G, *or* $s^\Box = t^\Box$ *implies* $(A,3) \in t$ *for every partial valuation* t *of* G.

Proof: (1): Note that $(\Box;1)$ forms a rule of each G. Suppose $(\Box A,1) \in s$. It follows that $s^\Box \cup (A,1)$ is not strictly provable in G, since otherwise s becomes strictly provable in G by means of the rule $(\Box;1)$. Hence $s^\Box \subseteq t$ (and $s^\Box = t^\Box$ when $G \in \{\textbf{3-S5}_3, \textbf{3-S5}_2\}$) and $(A,1) \in t$ for some partial valuation t of G by Lemma 5.

(2): Note that $(\Box;3)_1$ forms a rule of G. Suppose $(\Box A,3) \in s$. It follows in view of the rule $(\Box;3)_1$ that, at least one of $s^\Box \cup (A,2\hat{\ })$ and $s \cup (\Box A,2)$ is not strictly provable in G. In the former case, $s^\Box \subseteq t$ and $(A,2\hat{\ }) \subseteq t$ for some partial valuation t of G by Lemma 5; whereas in the latter, $(\Box A,2) \in s$ since the matrix $\{(\Box A,2)\}$ is G-subsidiary to s.

(3): Note that $(\Box;3)_3$ forms a rule of G and the $\Box$-operation is defined by (D5). Suppose $(\Box A,3) \in s$. If $s^\Box \cup (A,2\hat{\ })$ is not strictly provable in G, then $s^\Box = t^\Box$ and $(A,2\hat{\ }) \subseteq t$ for some partial valuation t of G by Lemma 5. So we assume that $s^\Box \cup (A,2\hat{\ })$ is strictly provable in G. We claim that $s^\Box = t^\Box$ implies $(A,3) \in t$ for every partial valuation t of G. To show this, let t be a partial valuation of G such that $s^\Box = t^\Box$. Then $(\Box A,3) \in t$, and $t^\Box \cup (A,2\hat{\ })$ is strictly provable in G. It follows in view of the rule $(\Box;3)_3$ that $t \cup (A,3)$ is not strictly provable in G. Hence $(A,3) \in t$, since the matrix $\{(A,3)\}$ is G-subsidiary to t.

Definition The Kripke frame (W_G, R_G, φ_G) is characterized as follows:

(i) W_G is the set of partial valuations of G. By Lemma 5 and the corollary of Lemma 2, W_G is nonempty.

(ii) For any $s,t \in W_G$: sR_Gt iff $s^\Box \subseteq t$, when $G \in \{\textbf{3-K}_3, \textbf{3-K}_2, \textbf{3-M}_3, \textbf{3-M}_2, \textbf{3-S4}_3, \textbf{3-S4}_2\}$; whereas sR_Gt iff $s^\Box = t^\Box$, when $G \in \{\textbf{3-S5}_3, \textbf{3-S5}_2\}$.

(iii) Let $s \in W_G$ and p be any propositional variable. Since s is a matrix that is not strictly provable, it is not the case that $(p,T) \subseteq s$. So, we let $\varphi_G(p,s)$ be one of the truth value μ such that $(p,\mu) \notin s$.

Lemma 7 *The Kripke structure* $(W_G, R_G, \varphi_G^{i(G)})$ *of type* $i(G)$ *generated from* (W_G, R_G, φ_G) *forms a model of* G.

Proof: We will prove this by cases.

Case 1: $G \in \{\textbf{3-K}_3, \textbf{3-K}_2\}$. Nothing is left to be proved.

Case 2: $G \in \{\textbf{3-M}_3\}$. Since the $\Box$-operation is defined by (D1), it follows that R_G is reflexive by Lemma 3(2) and (4).

Case 3: $G \in \{\textbf{3-M}_2\}$. Similar to Case 2 by Lemma 3(3).

Case 4: $G \in \{\textbf{3-S4}_3, \textbf{3-S4}_2\}$. Since the $\Box$-operation is defined by (D3) or (D4) respectively, R_G is clearly reflexive and transitive.

Case 5: $G \in \{\textbf{3-S5}_3, \textbf{3-S5}_2\}$. The binary relation R_G clearly forms an equivalence relation.

Lemma 8 *In the model* $(W_G, R_G, \varphi_G^{i(G)})$ *of* G, $(B, \varphi_G^{i(G)}(B,s)) \notin s$ *for every formula* B *and every* $s \in W_G$.

Proof: We prove this by induction on the construction of B.

Case 1: B is a propositional variable. Clear by the choice of φ_G.

Case 2: B has the form $F(A_1,\ldots,A_{\alpha(F)})$. Suppose $(B,\mu)\in s$, where $\mu=\varphi_G^{i(G)}(B,s)$. It follows by Lemma 3(1) that $(A_1,\nu_1\hat{\ })\cup\ldots\cup(A_{\alpha(F)},\nu_{\alpha(F)}\hat{\ })\subseteq s$ for some $(\nu_1,\ldots,\nu_{\alpha(F)})\in f_F^{-1}(\mu\hat{\ })$. Then $\nu_k\neq\varphi_G^{i(G)}(A_k,s)$ for some $k=1,\ldots,\alpha(F)$, since $\mu=f_F(\varphi_G^{i(G)}(A_1,s),\ldots,\varphi_G^{i(G)}(A_{\alpha(F)},s))$. So $(A_k,\varphi_G^{i(G)}(A_k,s))\in(A_k,\nu_k\hat{\ })\subseteq s$, which contradicts the hypothesis of induction. Hence $(B,\mu)\notin s$.

Case 3: B has the form $\Box A$. It suffices to derive a contradiction from the assumption $(B,\mu)\in s$, where $\mu=\varphi_G^{i(G)}(B,s)$.

Subcase 3.1: $\mu=1$. From the assumption $(\Box A,1)\in s$, it follows by Lemma 6(1) that sR_Gt and $(A,1)\in t$ for some $t\in W_G$. Since $\mu=1$ and sR_Gt, it follows that $\varphi_G^{i(G)}(A,t)=1$. So $(A,\varphi_G^{i(G)}(A,t))\in s$, which contradicts the hypothesis of induction.

Subcase 3.2: $\mu=2$ and $G\in\{\textbf{3-K}_3,\textbf{3-M}_3,\textbf{3-S4}_3,\textbf{3-S5}_3\}$, so that $i(G)=3$. From $\mu=2$, it follows that sR_Gt and $\varphi_G^3(A,t)=2$ for some $t\in W_G$. Then since $(\Box A,2)\in s$ and $s^\Box\subseteq t$ (and by Lemma 3(2) when $G\in\{\textbf{3-S4}_3,\textbf{3-S5}_3\}$), it follows that $(A,2)\in t$ and so $(A,\varphi_G^3(A,t))\in t$, which contradicts the hypothesis of induction.

Subcase 3.3: $\mu=2$ and $G\in\{\textbf{3-K}_2,\textbf{3-M}_2,\textbf{3-S4}_2,\textbf{3-S5}_2\}$, so that $i(G)=2$. This is impossible, since $\varphi_G^2(\Box A,s)$ does not take the value 2.

Subcase 3.4: $\mu=3$ and $G\in\{\textbf{3-K}_3,\textbf{3-M}_3,\textbf{3-S4}_3\}$, so that $i(G)=3$. Since $\mu=3$, sR_Gt implies $\varphi_G^3(A,t)\neq 2$ for every $t\in W_G$, whereas sR_Gt_0 and $\varphi_G^3(A,t_0)=3$ for some $t_0\in W_G$. From $(\Box A,3)\in s$, it follows by Lemma 6 (2) that, either sR_Gt and $(A,2\hat{\ })\subseteq t$ for some $t\in W_G$, or $(\Box A,2)\in s$. In the first case, it follows that $(A,\varphi_G^3(A,t))\in(A,2\hat{\ })\subseteq t$, which contradicts the hypothesis of induction. So in the meantime, we suppose $(\Box A,2)\in s$. Then since $(\Box A,1\hat{\ })\subseteq s$ and $s^\Box\subseteq t_0$ (and by Lemma 3(4) when $G\in\{\textbf{3-S4}_3\}$), it follows that $(A,3)\in t_0$ and so $(A,\varphi_G^3(A,t_0))\in t_0$, which is a contradiction, too.

Subcase 3.5: $\mu=3$ and $G\in\{\textbf{3-K}_2,\textbf{3-M}_2,\textbf{3-S4}_2,\textbf{3-S5}_2\}$, so that $i(G)=2$. From $\mu=3$, it follows that sR_Gt and $\varphi_G^2(A,t)\neq 1$ for some $t\in W_G$. Then since $(\Box A,3)\in s$ and $s^\Box\subseteq t$ (and by Lemma 3(3) when $G\in\{\textbf{3-S4}_2,\textbf{3-S5}_2\}$), it follows that $(A,1\hat{\ })\subseteq t$ and so $(A,\varphi_G^2(A,t))\in t$, which contradicts the hypothesis of induction.

Subcase 3.6: $\mu=3$ and $G\in\{\textbf{3-S5}_3\}$, so that $i(G)=3$. From $(\Box A,3)\in s$, it follows by Lemma 6(3) that, either sR_Gt and $(A,2\hat{\ })\subseteq t$ for some $t\in W_G$, or sR_Gt implies $(A,3)\in t$ for every $t\in W_G$. In the former case, from $\mu=3$ and sR_Gt, it follows that $(A,\varphi_G^3(A,t))\in(A,2\hat{\ })\subseteq t$, which contradicts the hypothesis of induction. So in the meantime, we consider the latter case. From $\mu=3$, it follows that sR_Gt and $\varphi_G^3(A,t)=3$ for some $t\in W_G$. Then $(A,3)\in t$ and so $(A,\varphi_G^3(A,t))\in t$, which is a contradiction, too.

At last, we are prepared to prove the following lemma, which states that the property (c) formulated in the Theorem is implied by the property (a).

Lemma 9 *If a matrix is valid in G, it is strictly provable in G.*

Proof: Assume that a matrix L is not strictly provable in G. Then $L \subseteq s$ for some $s \in W_G$ by Lemma 5. Hence in the model $(W_G, R_G, \varphi_G^{i(G)})$ of G, $(B, \varphi_G^{i(G)}(B, s)) \in L$ for no formula B by Lemma 8. So L is not valid in G.

REFERENCES

[1] Morikawa, O., "Some modal logics based on a three-valued logic," *Notre Dame Journal of Formal Logic*, vol. 30 (1989), pp. 130–137.

[2] Schütte, K., *Proof Theory*, Springer-Verlag, Berlin, 1977.

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