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Connection Structures: Grzegorczyk's and Whitehead's Definitions of Point

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Abstract Whitehead, in his famous book *Process and Reality*, proposed a definition of point assuming the concepts of "region" and "connection relation" as primitive. Several years after and independently Grzegorczyk, in a brief but very interesting paper, proposed another definition of point in a system in which the inclusion relation and the relation of being separated were assumed as primitive. In this paper we compare their definitions and we show that, under rather natural assumptions, they coincide.

1 Introduction When facing the literature on the axiomatic foundation of geometry, we notice surprisingly that, although the primitive relations and the axioms may vary, there is a primitive term that remains in all cases the same: the point. On the other hand, since it is evident that nature does not provide objects without dimensions (a property that geometry ascribes to points), it should be of some interest conceiving axiomatic systems in which the concept of point is defined from primitive terms more easily interpretable in nature (information about the attempts in this direction can be found in Gerla [6]).

Now, an interesting possibility is to consider as primitive the regions, the inclusion between regions, and the "connection relation," that is, the relation between two regions that overlap or have at least a common boundary point. Structures of such a type, which we call *connection structures*, were first examined by Laguna [4] in 1922 . Successively, in 1929 Whitehead [8] put the connection relation on the basis of a very extensive analysis of the abstraction process leading to the concepts of point, line and surface. Whitehead listed a very large sequence of properties which a connection relation has to verify—In Chapter 2 Whitehead exposed 31 assumptions!—but no attempt was made to frame his analysis into a mathematical theory. In particular, no attempt was made to reduce his system of assumptions and definitions to a logical minimum. A first step in this direction was made in Gerla and Tortora [5]; a

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rather different version of Whitehead's system was proposed by Clarke [2] and [3] and criticized by Biacino and Gerla [1].

Independently from Whitehead's work, Grzegorczyk [7] assumed as primitive the inclusion relation and the relation of being separated. Like Whitehead, the purpose of Grzegorczyk is "to make more precise the well-known conviction that geometry may be built without speaking about points." Indeed, by a very simple system of axioms he was able to obtain a representation theorem relating his "topology without points" with the classical topology theory. Now, as a matter of fact, if we substitute the relation of being separated by its negation, it becomes clear that Grzegorczyk's work is very close to Whitehead's ideas and therefore that it furnishes a very powerful mathematical treatment of connection structures.

In this note we confine ourselves to rewrite in a more manageable manner the system of axioms of Grzegorczyk and to compare the definitions of point given by Whitehead and Grzegorczyk respectively.

2 Preliminaries We begin by considering Grzegorczyk's axiom system for the geometry without points. Grzegorczyk [7] assumed as primitive a set $\mathcal{R}$ whose elements are called *spatial bodies*, the *inclusion relation* and the relation of *being separated*. However, in order to emphasize the similarity with the analysis proposed by Whitehead [8], we prefer to assume the negation of the relation of being separated, which we call *connection relation*. We will rewrite Grzegorczyk's axioms in accordance with such a choice. So, we consider structures $(\mathcal{R}, \leq, C)$, which we call *connection structures*, such that $\leq$ and C are binary relations in $\mathcal{R}$ satisfying the axioms

G_0 $(\mathcal{R}, \leq)$ is a mereological field;

G_1 xCx for every $x \in \mathcal{R}$;

G_2 $xCy \Rightarrow yCx$ for every $x, y \in \mathcal{R}$;

G_3 $x \leq y \Rightarrow C(x) \subseteq C(y)$;

where

- a *mereological field* is the structure obtained from a complete Boolean algebra $\mathbf{B}$ by deleting the zero-element, i.e., $\mathcal{R} = \mathbf{B} - \{0\}$;
- $C(z) = \{x \in \mathcal{R} \mid zCx\}$ for every $z \in \mathcal{R}$.

We call *regions* the elements of $\mathcal{R}$, *inclusion relation* the relation $\leq$, and *connection relation* the relation C. As an immediate consequence of G_1 and G_3, we have that every region is connected with the unity 1 of $\mathcal{R}$. In the following we say that a region x *overlaps* a region y and write xOy if a region z exists such that $z \leq x$ and $z \leq y$ (i.e., $x \wedge y \neq 0$). Also, we set $O(z) = \{x \in \mathcal{R} \mid zOx\}$.

Proposition 2.1 *For every pair of regions x, y*

1. $xCy, y \leq z \Rightarrow xCz$;
2. $x \leq y \Rightarrow xCy$;
3. $xOy \Rightarrow xCy$.

Proof: (1) is a consequence of G_3, and (2) follows from G_1 and (1). To prove (3) assume that xOy and therefore that $z \in \mathcal{R}$ exists such that $z \leq x$ and $z \leq y$. As a consequence, since $C(z) \subseteq C(x)$, we have zCx and, by G_2, xCz. Since $C(z) \subseteq C(y)$, this implies xCy. $\square$

Proposition 2.1.3 says that C contains the relation O. Let $\mathbf{B}$ be any complete Boolean algebra and set $\mathcal{R} = \{x \in \mathbf{B} \mid x \neq 0\}$ then a "minimal" connection structure in $\mathcal{R}$ is obtained by setting C equal to the overlapping relation O. A "maximal" connection structure is obtained by setting C equal to the relation $\mathcal{R} \times \mathcal{R}$, i.e., the relation satisfied by any pair (x, y) of regions.

Definition 2.2 We say that x is nontangentially enclosed in y and we write $x \ll y$ if $C(x) \subseteq O(y)$.

The following proposition gives a simple characterization of the relation $\ll$.

Proposition 2.3 *For every pair of regions x and y, with $y \neq 1$,*

$$x \ll y \iff x \not{C} - y.$$

Proof: Let $x \ll y$, then $C(x) \subseteq O(y)$ and therefore, since $-y$ does not overlap y, $-y$ does not belong to $C(x)$, that is $x \not{C} - y$. Conversely, assume that $x \not{C} - y$ then $x \not{C} z$ for every $z \leq -y$. So, if z is connected with x, we have $z \neq -y$ and therefore $z \wedge y \neq 0$. Thus $C(x) \subseteq O(y)$ that is, $x \ll y$. $\square$

Proposition 2.4 *The following statements hold for every $x, y, z, v \in \mathcal{R}$,*

1. $x \ll y \iff -y \ll -x$ *(provided that $x \neq 1$ and $y \neq 1$),*
2. $x \ll y \Rightarrow x \leq y$,
3. $x \ll y$ *and* $y \leq z \Rightarrow x \ll z$,
4. $x \leq y$ *and* $y \ll z \Rightarrow x \ll z$,
5. $x \ll u$ *and* $u \ll y \Rightarrow x \ll y$,
6. $x \ll 1$.

Proof: To prove (1) notice that by Proposition 2.3,

$$x \ll y \iff x \not{C} - y \iff -y \not{C} x \iff -y \ll -x.$$

Implication (2) is obvious in the case $y = 1$, otherwise, from $-y \not{C} x$, we have that $-y$ does not overlap x and therefore $x \leq y$. (3) follows from Proposition 2.1.1. To prove (4) assume that $x \leq y$ and $y \ll z$, then $-x \geq -y$ and $-y \gg -z$. Consequently, by (3), $-x \gg -z$ and therefore $x \ll z$. Finally, (5) is a consequence of (3) and (2), and (6) follows from the equality $O(1) = \mathcal{R}$.

The converse of (2) is false, in general. As an example, if $C = \mathcal{R} \times \mathcal{R}$ then since, $C(x) = \mathcal{R}$, for every $x \in \mathcal{R}$, we have that $x \ll y$ only if $y = 1$. As a matter of fact, as Proposition 2.3 shows, only if C coincides with the overlapping relation then $\ll$ is equal to $\leq$. From (6) it follows that $1 \ll 1$ and therefore that $\ll$ is not a "strict" order. $\square$

Remark 2.5 Perhaps it is worth noting that we can assume as primitive the nontangential inclusion instead of the connection relation. Indeed, assume that $\mathcal{R}$ is a mereological field, then Proposition 2.4 says that the nontangential inclusion satisfies the properties

A_1 $x \ll y \iff -y \ll -x$ (provided that $x \neq 1$ and $y \neq 1$)

A_2 $x \ll y \Rightarrow x \leq y$

A_3 $x \ll y$ and $y \leq z \Rightarrow x \ll z$.

Conversely, let $\ll$ be a binary relation in a mereological field $\mathcal{R}$ satisfying A_1, A_2, and A_3. Then we can define a relation C by setting xCy provided that $y = 1$ or $x \not\ll -y$. It is easily proven that $(\mathcal{R}, \leq, C)$ is a connection structure. Indeed, by definition, we have that $1C1$ while, in the case $x \neq 1$, since by A_2 we have that $x \not\ll -x$, it is xCx. To prove G_2, assume that xCy, then if $x = 1$ it is immediate that yCx. If $x \neq 1$ and $y = 1$ then $y \not\ll -x$ since otherwise by A_2 we have $-x = 1$ and therefore $x = 0$. So, we can conclude that yCx. If $x \neq 1$ and $y \neq 1$ then by A_1,

$$xCy \Rightarrow x \not\ll -y \Rightarrow y \not\ll -x \Rightarrow yCx.$$

Finally to prove G_3, assume that $x \leq y$ and that zCx. We have to prove that zCy. Now, in the case $x = 1$, since it is also $y = 1$, we have that zCy. In the case $x \neq 1$ and $y = 1$ it is immediate that zCy. In the case $x \neq 1$ and $y \neq 1$ we have that $z \not\ll -x$ and therefore since $-y \leq -x$ by A_3 it is also $z \not\ll -y$ and therefore zCy.

Now, denote by $\ll'$ the relation of nontangential inclusion associated with the structure defined above. Then if $y \neq 1$ we have

$$x \ll' y \iff x \not{C} - y \iff x \ll y.$$

In the case $y = 1$, we have that $x \ll' 1$ but it is possible that $x \ll 1$ does not hold. As an example, let $\ll$ be the empty relation, i.e., no pair of regions x and y satisfies $x \ll y$. Then A_1, A_2 and A_3 are trivially satisfied but, while $x \ll' 1$ we have that $x \not\ll 1$. (Notice that the associated relation C defines the maximal model in $\mathcal{R}$.) So, if we want $\ll'$ to coincide with $\ll$ we have to add the axiom

A_4 $x \ll 1$.

3 ***The definition of point in Grzegorczyk*** By following Grzegorczyk, we say that a set p of regions is *a representative of a point* if:

A. p is without minimum and totally ordered with respect to $\ll$;

B. given two regions u and v, uOx and vOx for every $x \in p$ implies uCv.

A *point* is a filter P generated by a representative of a point p, i.e., $P = \{y \in \mathcal{R} | y \geq x$ for a suitable $x \in p\}$. A point P *belongs* (is *adherent*) to a region r provided that r is an element of P (r overlaps with all the elements of P). If p represents the point P and $z \in p$, then the "cut" $\{x \in p | x \leq z\}$ represents P too. Consequently, if P belongs to the region r we may represent P by a chain of regions all contained in r. We denote by $\mathcal{P}$ the set of points and by $P(x)$ the set of points belonging to the region x.

Proposition 3.1 *Let z, z' be regions, $z \neq 1$ and $z' \neq 1$, and P a point. Then the following statements hold.*

1. *$P \notin z \iff P$ is adherent to $-z$;*
2. *P adherent to z and $z \ll z' \Rightarrow P \in z'$.*

Proof: (1) $P \notin z \iff \forall x \in P (x \not\leq z) \iff \forall x \in P (x \wedge -z \neq 0) \iff P$ is adherent to $-z$. (2) Suppose P adherent to z, $z \ll z'$ and $P \notin z'$. Then by (1) P is adherent to $-z'$ so, by (B), we have $-z'Cz$. This is an absurdity since $z \ll z'$ is equivalent to $-z' \not\mathrel{C} z$. □

The following two axioms concern the existence of points:

- G_4 every region has a point;
- G_5 $xCy \Rightarrow$ a point P exists such that P is adherent to x and y.

Observe that (B) entails that if there is a point adherent to both regions x and y, then x is connected with y. Axiom G_5 claims that the converse implication holds, too. Also, notice that, since a representative of a point is an infinite class of regions, the existence of a point entails that $\mathcal{R}$ is infinite. Consequently, each minimal connection structure in a finite mereological field is a model of G_0–G_3 in which neither G_4 nor G_5 is satisfied. This shows that these axioms are independent from G_0–G_3.

Remark 3.2 Axioms G_4 and G_5 enable us to prove that, as a matter of fact, the implication in G_3 is an equivalence. Indeed, suppose $C(x) \subseteq C(y)$ but $x \not\leq y$. Then $x \wedge -y \neq 0$ and, by Axiom G_4, $\exists P \in x \wedge -y$. So $\exists r \in P$ such that $r \ll x \wedge -y \leq -y$ and, by Proposition 2.3, $r \not\mathrel{C} y$. On the other hand $r \in C(x)$ and, since $C(x) \subseteq C(y)$, rCy, an absurdity. This means that, by following Whitehead, it should be possible to assume as primitive only the connection relation and to define the order relation by setting $x \leq y$ provided that $C(x) \subseteq C(y)$.

Grzegorczyk proves two basic theorems. Although these theorems are not used in this paper, we will enunciate them for their intrinsic interest. In fact, they state that the pointless theory of the connection structures is, in a sense, equivalent to the point-based theory of topological spaces. Recall that a subset x of a topological space is called (open) *regular* provided that $x = \overset{\circ}{\bar{x}}$. The first theorem shows how to obtain a connection structure by starting from a topological space.

Theorem 3.3 *Let $\mathcal{T}$ be a Hausdorff topology on the set S, R the class of the nonempty regular elements of $\mathcal{T}$ and put, for every $x, y \in R$, xCy if $\bar{x} \cap \bar{y} \neq \varnothing$. Then $(R, \subseteq, C)$ is a connection structure in which $x \ll y$ means $\bar{x} \subseteq y$. Moreover, if every point is the intersection of a strictly decreasing (with respect to $\ll$) family of open sets, then $(R, \subseteq, C)$ satisfies G_4–G_5, too.*

In any connection structure we have that $1 \ll 1$. Now, in the structure $(R, \subseteq, C)$ defined above, a region $x \neq 1$ exists such that $x \ll x$ if and only if the topology $\mathcal{T}$ is not connected. Indeed, recall that $\mathcal{T}$ is not connected if and only if a coplen set x (a set that is both open and closed) exists different from $\varnothing$ and S. Moreover, it is immediate that every coplen set x is a regular set such that $\bar{x} \subseteq x$ and that every regular set x such that $\bar{x} \subseteq x$ is a coplen. Notice also that if P is an element of S then by hypothesis a representative of a point p exists such that P coincides with the intersection $\cap p$. In other words, every element of S is associated with a point in the sense of Grzegorczyk. The converse implication is not true, in general, since if $\mathcal{T}$ is not compact and p is a representative of point, then $\cap p$ can be empty or not.

The second theorem shows that every connection structure can be obtained by starting from a suitable topological space.

Theorem 3.4 *Assume that* $(R, \leq, C)$ *is a connection structure satisfying* G_4–G_5 *and let* $\mathcal{T}$ *be the topology on* $\mathcal{P}$ *generated by* $\{P(x) \mid x \in R\}$, *then*

1. $\{P(x)|x \in R\}$ *is the class of the nonempty regular elements of* $\mathcal{T}$;
2. $x \leq y \iff P(x) \subseteq P(y)$; $x \ll y \iff \overline{P(x)} \subseteq P(y)$;
3. $xCy \iff \overline{P(x)} \cap \overline{P(y)} \neq \emptyset$; *P is adherent to x* $\iff P \in \overline{P(x)}$.

4 Some consequences of Grzegorczyk's system In the following proposition we give some immediate consequences of axioms G_4 and G_5 that will be useful in the sequel.

Proposition 4.1 *The following statements hold.*

1. $xC(z \vee z') \iff xCz$ *or* xCz'.
2. $x \ll z, y \ll z' \Rightarrow x \wedge y \ll z \wedge z', x \vee y \ll z \vee z'$.

Proof: At first we will prove that

$$z \ll z' \Rightarrow \forall x(x \in C(z) \Rightarrow x \wedge z' \in C(z)).$$

Indeed, let $x \in C(z)$, then by G_5 there exists a point P adherent to z and x. So for every region r in P, rOz and rOx. Since P is adherent to z and $z \ll z'$, from Proposition 3.1.2 it follows that $P \in z'$. Then we may represent P by a chain p of regions contained in z'. Since $r \wedge x \neq 0$ for every $r \in p$, we have also $r \wedge x \wedge z' \neq 0$ for every $r \in p$ and therefore P is adherent to $z \wedge z'$. Since P is adherent to z also, by (B) we have that $x \wedge z' \in C(z)$.

(1) Assume that $xC(z \vee z')$ and that $x \not{C} z$, $x \not{C} z'$. Then $x \ll -z$ and $x \ll -z'$, that is $C(x) \subseteq O(-z)$ and $C(x) \subseteq O(-z')$. Let $y \in C(x)$, then by the above proven implication we have that $y \wedge -z \in C(x)$. Since $C(x) \subseteq O(-z')$ it is $y \wedge -z \wedge -z' \neq 0$. Thus, for every $y \in C(x)$, we have that $y \wedge -(z \vee z') \neq 0$, that is $yO-(z \vee z')$. This means that $x \ll -(z \vee z')$ and so $x \not{C}(z \vee z')$ despite the hypothesis.

(2) Let $x \ll z$ and $y \ll z'$, then, by Proposition 2.3, $x \not{C} - z$ and $y \not{C} - z'$ and so $x \wedge y \not{C} - z$ and $x \wedge y \not{C} - z'$. By (1) $x \wedge y \not{C} - z \vee -z'$ that is $x \wedge y \not{C} - (z \wedge z')$ and this means that $x \wedge y \ll z \wedge z'$. In a similar way one proves that $x \vee y \ll z \vee z'$. $\square$

The following proposition shows an interesting property of the regions that was emphasized by Whitehead. In particular, from this property it follows that no region is an atom, i.e., that the Boolean algebra under consideration is not atomic.

Proposition 4.2 *Every region contains two subregions that are not connected.*

Proof: Let r be a region, then by G_4 a point P exists belonging to r and if p represents P then $x \in p$ exists such that $x \ll r$. Since p is without minimum there is $x' \in p$ such that $x' \ll x$, and $x' \neq x$. So $x - x'$ is a region. By G_4 a point P' exists belonging to $x - x'$ and, if p' represents P', then $y \in p'$ exists such that $y \ll x - x' = x \wedge (-x')$. Therefore $y \not{C} - (x - x')$ that is $y \not{C}(-x \vee x')$. Thus, $y \not{C} x'$ so y and x' are two subregions of r that are not connected. $\square$

5 The definition of point in Whitehead Whitehead [8] defined a connection structure as a pair $(\mathcal{R}, C)$ where $\mathcal{R}$ is a set whose elements are called *regions* and C is a binary relation in $\mathcal{R}$, the *connection relation.* The basic subject of Whitehead's book is the abstractive process enabling us to define points, lines and areas by starting from the primitive concepts of region and connection relation. The properties assigned to the connection structure look different from the ones listed in Grzegorczyk's system. For example, the inclusion relation is not primitive, but is defined by the equivalence

$$x \leq y \iff C(x) \subseteq C(y).$$

Also, the class of regions does not constitute a mereological field. Indeed, while the idea of region in Grzegorczyk is related to the whole class of the regular open subsets, Whitehead seems to confine his attention only to the connected subsets. On the other hand, the class of connected subsets is not closed with respect to the unions. However, since the differences are not substantial (see for example the remark in Section 3), we will refer to Grzegorczyk's system of axioms, just comparing, in such a frame, the definitions of point proposed by Whitehead and Grzegorczyk.

Whitehead defined an *abstractive set* as a class α of regions such that

(j) α is totally ordered with respect to $\ll$;
(jj) there is no region included in every element of α.

An abstractive set α *covers* an abstractive set β, in brief $\alpha \succeq \beta$, if for every $x \in \alpha$ there exists $y \in \beta$ such that $x \geq y$. The covering relation is a preorder and therefore it defines an equivalence relation $\equiv$ in the following way.

$$\alpha \equiv \beta \iff \alpha \succeq \beta \text{ and } \beta \succeq \alpha.$$

Given an abstractive process α, we denote by $[\alpha]$ the related complete class of equivalence. Whitehead calls such a class a *geometrical element*. The covering relation induces an order relation on the set of geometrical elements. A *point* is a geometrical element minimal with respect to such a relation. We call a *W-representative* of point every abstractive set α such that $[\alpha]$ is a point. Then a W-representative of point is a class α of regions such that (j), (jj) are satisfied and the following holds

(jjj) $\alpha' \preceq \alpha \Rightarrow \alpha' \equiv \alpha$ for every abstractive set α'.

The term *G-representative* will be used to denote a representative of point as defined by Grzegorczyk. We will compare the two concepts above and study under what hypotheses they coincide. To this purpose, we associate every abstractive set α with the filter $F_\alpha = \{x \in \mathcal{R} \mid \exists y \in \alpha : x \geq y\}$ generated by α. It is immediate that

$$\alpha \succeq \beta \iff F_\alpha \subseteq F_\beta$$

and therefore that

$$\alpha \equiv \beta \iff F_\alpha = F_\beta.$$

As a consequence the correspondence associating a geometrical element $[\alpha]$ with the filter F_α is injective and we may define the geometrical elements as the filters generated by suitable abstractive sets. In particular, we may define the points as the filters that are maximal in the class of the filters generated by the abstractive sets.

Theorem 5.1 *If p is a G-representative then p is a W-representative and therefore the points as defined by Grzegorczyk are also points in the sense of Whitehead.*

Proof: Assume that p is a G-representative; then Condition (j) is satisfied, obviously. In order to prove (jj) we notice that, if a region r exists such that $r \leq x$ for every $x \in p$ then, by G_4 and Proposition 4.2, r admits two subregions u and v that are not connected. The fact that uOx and vOx for every $x \in p$ contradicts (B).

We now prove (jjj). Let q be an abstractive set such that $q \preceq p$ that is

$$\forall z \in p \exists w \in q : z \geq w. \tag{1}$$

We will prove that $q \equiv p$, that is $\forall x \in q \exists y \in p : x \geq y$. Suppose that this is not the case, then $\exists x \in q : \forall y \in p$ we have $x \not\geq y$, that is $yO-x$. This means that p is adherent to $-x$. Now, by (jj) we have that x is not contained in every element of q, that is $x' \in q$ exists such that x is not contained in x' and, since q is totally ordered with respect to $\ll$, $x' \ll x$. Then, by Proposition 2.4.1, $-x' \gg -x$ and, by Proposition 4.1.2, $p \in -x'$. Therefore, a region $z \in p$ exists such that $z \leq -x'$. We claim that, for every $w \in q$, $z \not\geq w$, indeed otherwise from $z \geq w$ we have that $-x' \geq w$ with $x' \in q$ and $w \in q$. Now, either $x' \ll w$ or $w \ll x'$. Both these inequalities are incompatible with $-x' \geq w$. This contradicts Equation 1. □

To establish a converse of the previous proposition we have to consider the following axiom we call the *normality* axiom.

(G_6) For every x and y such that $x \ll y$, a region z exists such that $x \ll z \ll y$.

Such an axiom is satisfied by a very large class of connection structures as the following proposition shows.

Proposition 5.2 *The connection structure associated with a Hausdorff topology $\mathcal{T}$ satisfies G_6 if $\mathcal{T}$ is normal. In particular the connection structure associated with a Euclidean space satisfies G_6.*

Proof: Recall that a Hausdorff space $\mathcal{T}$ is normal if whenever we consider two disjoint closed subsets C_1 and C_2 an open set A exists such that $A \supseteq C_1$ and $\bar{A}$ is disjoint from C_2. Let X and Y be regular sets such that $\bar{X} \subseteq Y$; then $\bar{X}$ is disjoint from $(S-Y)$ and therefore an open set A exists such that $A \supseteq \bar{X}$ and $\bar{A} \cap (S-Y) = \varnothing$, that is $\bar{A} \subseteq Y$. Set $Z = \overset{\circ}{\bar{A}}$, then Z is a regular set containing A and therefore $\bar{X}$, i.e., $Z \gg X$. Also, since $Z \subseteq \bar{A}$, we have that $\bar{Z} \subseteq \bar{A}$ and therefore, since $\bar{A} \subseteq Y$, $\bar{Z} \subseteq Y$, i.e.,$Z \ll Y$. □

Theorem 5.3 *If* (R,C) *satisfies G_6 then if a sequence is a W-representative then it is a G-representative, too.*

Proof: Let $p = (p_i)_{i \in N}$ be a W-representative. To prove that p is a G-representative it is enough to prove that, given $u, v \in R$

$$uOp_i \text{ and } vOp_i \text{ for every } i \in N \Rightarrow uCv.$$

Assume, by absurdity, that u and v exist such that uOp_i and vOp_i for every $i \in N$ but $u \not\!C v$. Then $u \ll -v$ and by G_6 a sequence $(u_i)_{i \in N}$ exists, decreasing with respect to $\ll$ such that $u_1 = -v$ and $\wedge\{u_i | i \in N\} \geq u$. Observe that for every $i, j \in N$, $u_i \wedge p_i \neq 0$ and $u_i \wedge p_i \not\geq p_j$, since $p_j O v$ and $u_i \wedge p_i \leq -v$. Now, by Proposition 4.1.2, $(u_i \wedge p_i)_{i \in N}$ is a sequence decreasing with respect to $\ll$ and, since $u_i \wedge p_i \leq p_i$, no region exists contained in all the regions $u_i \wedge p_i$. Thus the sequence p is not minimal and this contradicts the hypothesis. $\square$

Although Theorem 5.1 holds for every G-representative, the just proved theorem holds for W-representatives that are expressible by sequences. We do not know if this result holds in any case. However, since our task is to give a pointless foundation of Euclidean geometry rather than a pointless foundation of the topological spaces, we are not too much interested in this question. Indeed, it is possible to define directly a representative of a point as a suitable sequence (rather than a class) of regions, in accordance with the fact that our intuition of the abstraction activity leading to the concept of point is a step-by-step process. In this way, a new theory is obtained since the meaning of axioms G_4 and G_5 is modified. Nevertheless, it is immediate that the connection structure associated with the Euclidean space (and with every topological space satisfying the first axiom of enumerability) satisfies this theory and therefore it furnishes a good basis for a pointless foundation of the Euclidean geometry. In such a theory, Theorems 5.1 and 5.3 show that Grzegorczyk's and Whitehead's definitions of point coincide.

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