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Reverse Mathematics and Fully Ordered Groups

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Abstract We study theorems of ordered groups from the perspective of reverse mathematics. We show that RCA_0 suffices to prove Hölder's Theorem and give equivalences of both WKL_0 (the orderability of torsion free nilpotent groups and direct products, the classical semigroup conditions for orderability) and ACA_0 (the existence of induced partial orders in quotient groups, the existence of the center, and the existence of the strong divisible closure).

1 Introduction The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of ordinary mathematics. In this case, we consider theorems about ordered groups. Whereas this section gives some background material on reverse mathematics, it is not intended as an introduction to the subject. The reader who is unfamiliar with this area is referred to Simpson [15] or Friedman, Simpson, and Smith [4] for more details. This article is, however, self-contained with respect to the material on ordered groups.

We will be concerned with three subsystems of second-order arithmetic: RCA_0, WKL_0, and ACA_0. RCA_0 contains the ordered semiring axioms for the natural numbers plus Δ^0_1 comprehension, Σ^0_1 formula induction and the set induction axiom

$$\forall X\left(\left(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)\right) \rightarrow \forall n(n \in X)\right).$$

The Δ^0_1 comprehension scheme consists of all axioms of the form

$$\forall n\left(\varphi(n) \longleftrightarrow \psi(n)\right) \rightarrow \exists X\,\forall n\left(n \in X \longleftrightarrow \varphi(n)\right)$$

where φ is a Σ^0_1 formula, ψ is a Π^0_1 formula, and X does not occur freely in either φ or ψ. The Σ^0_1 formula induction scheme contains the following axiom for each Σ^0_1 formula φ,

$$\left(\varphi(0) \wedge \forall n\left(\varphi(n) \rightarrow \varphi(n+1)\right)\right) \rightarrow \forall n\left(\varphi(n)\right).$$

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We will use $\mathbb{N}$ to denote the set defined by the formula $x = x$. Notice that in the comprehension scheme φ may contain free set variables other than X as parameters.

The computable sets form the minimum ω-model of RCA_0 and any ω-model of RCA_0 is closed under Turing reducibility. RCA_0 is strong enough to prove the existence of a set of unique codes for the finite sequences of elements from any set X. We use Fin_X to denote this set of codes. Also, we use $\langle a, b\rangle$, or more generally $\langle x_0, \dots, x_n\rangle$, to denote pairs, or longer sequences, of elements of $\mathbb{N}$. For any sequences σ and τ, we denote the length of σ by $\mathrm{lh}(\sigma)$, the k^{th} element of σ by $\sigma(k)$, and the concatenation of σ and τ by $\sigma * \tau$. The empty sequence is denoted by $\langle\rangle$ and has length 0. The i^{th} column of X is denoted X_i and consists of all n such that $\langle n, i\rangle \in X$.

Definition 1.1 (RCA_0) A *binary branching tree* is a set $T \subseteq \mathrm{Fin}_{\{0,1\}}$ which is closed under initial segments. A *path* through T is a function $f : \mathbb{N} \to \{0, 1\}$ such that for all n, $f[n] = \langle f(0), \dots, f(n-1)\rangle \in T$.

Lemma 1.2 (Weak König's Lemma) *Every infinite binary branching tree has a path.*

WKL_0 consists of RCA_0 plus Weak König's Lemma and ACA_0 consists of RCA_0 plus arithmetic comprehension. Any ω-model of ACA_0 is closed under the Turing jump and the arithmetic sets form the minimum ω-model of ACA_0. The ω-models of WKL_0 are exactly the Scott sets and by the Low Basis Theorem each must contain a set of low Turing degree (on the Low Basis Theorem, see Jockusch and Soare [8]).

We use RCA_0 as our base system, which means that if $RCA_0 \vdash T$, we will not look for a proof of T in a weaker subsystem. However, if we find a proof of T in ACA_0 or WKL_0 and not in RCA_0, then we will try to show that $RCA_0 + T$ suffices to prove the extra axioms in ACA_0 or WKL_0. When proving such a reversal, the following theorems are extremely useful (for proofs, see [15]).

Theorem 1.3 (RCA_0) *The following are equivalent:*

1. WKL_0
2. *For every pair of functions f, g such that for all m, n, $f(n) \neq g(m)$, there exists a set X such that for all m, $f(m) \in X$ and $g(m) \notin X$.*

Theorem 1.4 (RCA_0) *The following are equivalent:*

1. ACA_0
2. *The range of every $1-1$ function exists.*

Given the characterizations of the ω-models of RCA_0, WKL_0, and ACA_0 in terms of Turing degrees, it is not surprising that equivalences in reverse mathematics have immediate consequences in computable mathematics. Any theorem provable in RCA_0 is effectively true, whereas the effective version of any theorem equivalent to WKL_0 or ACA_0 is not true. Results in computable mathematics are stated as corollaries throughout this article.

In Section 2, we present the basic definitions for partially and fully ordered groups. The main result is that RCA_0 suffices to prove the existence of the induced order on the quotient of a fully ordered group by a convex normal subgroup, but ACA_0 is required for the induced order on the quotient of a partially ordered group.

Sections 3 and 4 deal with group conditions that imply full orderability. Downey and Kurtz [3] were the first to explore the computational content of the classical theorem stating that every torsion free abelian group is fully orderable. They constructed a computable torsion free abelian group with no computable full order. Hatzikiriakou and Simpson [7] went on to show that this theorem is equivalent to WKL_0. In Section 3, we show that WKL_0 is in fact equivalent to the theorem that every torsion free nilpotent group is fully orderable. In Section 4, we consider direct products of fully ordered groups. RCA_0 suffices to prove that any finite direct product of fully orderable groups is fully orderable, but because of uniformity issues, WKL_0 is required for countable products.

As a side issue from the work on nilpotent groups, we examine the center of a group in Section 5. Not surprisingly, the existence of the center is equivalent to ACA_0. As a corollary, we show that the center of a computable nilpotent group can be as complicated as $0'$, even if the length of the lower central series is three and the group is computably fully orderable. This result illustrates the computation difference between finitely and infinitely generated nilpotent groups (see Baumslag et al. [1]).

In addition to studying group conditions, algebraists have looked at semigroup conditions that imply orderability. We consider three of these conditions in Section 6 and prove that each is equivalent to WKL_0.

Hölder's Theorem states that every Archimedean fully ordered group is order isomorphic to a subgroup of the additive group of the real numbers under the standard order. In Section 7 we show that Hölder's Theorem is provable in RCA_0 and hence is effectively true.

Finally, we turn to the divisible closure of an abelian group. There are three interesting questions to ask about divisible closures in reverse mathematics: which axioms are required to prove that they exist, which are required to prove that they are unique, and which are required to prove that the original group is isomorphic to a subgroup of the divisible closure. In the context of ordered groups, we can also ask if the answer to any of these question is affected by having a full order on the group. Smith [16] proved that each computable group has a computable divisible closure. Friedman, Simpson, and Smith [4] showed that RCA_0 suffices to prove the divisible closure exists and that ACA_0 is equivalent to its uniqueness. Downey and Kurtz [3] proved that each computably fully ordered computable group has a unique computably fully ordered computable divisible closure whose order extends that of the original group. In Section 8, we consider the notion of the strong divisible closure and prove that the existence of a strong divisible closure is equivalent to ACA_0, even if the group is fully ordered.

The notation for objects from computability theory will follow Soare [17]. For example, we use $\leq_T$ to denote Turing reducibility and $0'$ for the Turing jump of the empty set. The notation for ordered groups will follow Fuchs [6] and Kokirin and Kopytov [10].

2 *Ordered quotient groups*

The main result of this section is that RCA_0 suffices to prove the existence of the induced order on the quotient of a fully ordered group, but ACA_0 is required if the group is only partially ordered.

Definition 2.1 (RCA_0) A *group* is a set $G \subseteq \mathbb{N}$ together with a constant, 1_G (or sometimes 0_G), and an operation, $\cdot_G$, which obeys the usual group axioms.

Definition 2.2 (RCA_0) A *partial order* is a set X together with a binary relation $\leq_X$ which satisfies the standard axioms for a partial order.

Definition 2.3 (RCA_0) A *partially ordered (p.o.) group* is a pair $(G, \leq_G)$ where G is a group, $\leq_G$ is a partial order on the elements of G, and for any $a, b, c \in G$, if $a \leq_G b$ then $a \cdot_G c \leq_G b \cdot_G c$ and $c \cdot_G a \leq_G c \cdot_G b$. If the order is a linear order, the pair $(G, \leq_G)$ is called a *fully ordered (f.o.) group*. A group for which there exists some full order is called an *O-group*.

Except for cases when they are needed to avoid confusion, the subscripts on $\cdot_G$ and $\leq_G$ are dropped.

Example 2.4 The additive groups $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{Z}, +)$ with the standard orders are f.o. groups. Let $\mathbb{Q}^+$ and $\mathbb{R}^+$ be the strictly positive rational and real numbers. The multiplicative groups $(\mathbb{R}^+, \cdot)$ and $(\mathbb{Q}^+, \cdot)$ are f.o. groups under the standard orders.

Example 2.5 The most important example for our purposes is the free abelian group on ω generators. Let G be the free abelian group with generators a_i for $i \in \omega$. Elements of G have the form $\sum_{i \in I} r_i a_i$ where $I \subseteq \omega$ is a finite set, $r_i \in \mathbb{Z}$ and $r_i \neq 0$. To compare the element above with $\sum_{j \in J} q_j a_j$, let $K = I \cup J$. For each $k \in K$, define $r_k = 0$ if $k \in J \setminus I$ and $q_k = 0$ is $k \in I \setminus J$. Let k be the maximum element of K such that $r_k \neq q_k$. The order is given by: $\sum_{i \in I} r_i a_i \leq \sum_{j \in J} q_j a_j$ if and only if $r_k \leq q_k$. This order makes G into an f.o. group.

As expected, RCA_0 suffices to prove many basic facts about p.o. groups.

Lemma 2.6 (RCA_0) *Let $(G, \leq)$ be a p.o. group.*

1. *If $a < b$ then $ac < bc$ and $ca < cb$.*
2. *If $a < b$ then $c^{-1}ac < c^{-1}bc$.*
3. *If $a < b$ then $b^{-1} < a^{-1}$.*
4. *If $a < b$ and $c < d$ then $ac < bd$.*

Defining a partial order can sometimes be notationally complicated. It is frequently easier to specify only the elements which are greater than the identity. Such a specification uniquely determines the order.

Definition 2.7 (RCA_0) The *positive cone*, $P(G, \leq_G)$ of a p.o. group is the set of elements which are greater than or equal to the identity.

$$P(G, \leq_G) = \{g \in G \mid 1_G \leq_G g\}$$

Each element $x \in P(G, \leq_G)$ is called *positive*. Sometimes we consider the *strict positive cone* which contains only the elements strictly greater than the identity.

When the intended order $\leq_G$ is clear, $P(G)$ is used instead of $P(G, \leq_G)$. Because $P(G)$ has a Σ^0_0 definition, RCA_0 is strong enough to prove its existence. Conversely, the relationship between any two elements can be defined in RCA_0 using $P(G)$ as a parameter because $a \leq b$ if and only if $a^{-1}b \in P(G)$. Hence, RCA_0 suffices to prove that each positive cone uniquely determines an order on G. Notice that if G is a computable group, we have $\deg(P(G)) = \deg(\leq_G)$ for any partial order $\leq_G$ and its associated positive cone.

Example 2.8 The complex numbers $(\mathbb{C}, +)$ with the set of positive elements $P(G) = \{x + yi \,|\, x > 0 \vee (x = 0 \wedge y \geq 0)\}$ forms an f.o. group. The group $(\mathbb{Q}^+, \cdot)$ with the order determined by $P(G) = \mathbb{N}^+$ is a p.o. group. Unraveling the definition of the positive cone shows that if $a, b \in \mathbb{Q}^+$ then $a \leq b$ if and only if a divides b. This order is not a full order but does form a lattice.

There are classical algebraic conditions which determine if an arbitrary subset of a group is the positive cone for some full or partial order on that group.

Definition 2.9 (RCA_0) If $X \subseteq G$, then $X^{-1} = \{g^{-1} \mid g \in X\}$. X is a *full subset* of G if $X \cup X^{-1} = G$ and X is a *pure subset* of G if $X \cap X^{-1} \subseteq \{1_G\}$.

Theorem 2.10 (RCA_0) *A subset P of a group G is the positive cone of some partial order on G if and only if P is a normal pure semigroup with identity. Furthermore, P is the positive cone of a full order if and only if in addition P is full.*

Proof: The standard proof of this theorem carries through in RCA_0. For details, see [10] or [6]. □

One can state a similar result for the strict positive cone. P is the strict positive cone of a full order if and only if P is a normal semigroup, $P \cup P^{-1} = G \setminus \{1_G\}$, and $P \cap P^{-1} = \emptyset$.

In the study of ordered groups, it is natural to ask which theorems of group theory hold for ordered groups and which theorems either fail completely or require extra conditions. For example, if H is a normal subgroup of G, then G/H inherits a group structure from G. However, if G is partially ordered, then H must also be convex (defined below) for the partial order on G to induce a natural partial order on G/H. To formulate this statement in second-order arithmetic, we first need a definition for the quotient group. Unique representatives of each coset gH in G/H are chosen by picking the $\leq_{\mathbb{N}}$-least element of gH. These choices can be made in RCA_0 because $mH = nH$ if and only if $m^{-1}n \in H$.

Definition 2.11 (RCA_0) If G is a group and H is a normal subgroup of G, then the *quotient group* G/H is defined by the set

$$\{n \mid n \in G \wedge \forall m < n\,(m \notin G \vee m^{-1} \cdot n \notin H)\}$$

and the operation $a \cdot_{G/H} b = c$ if and only if $a, b, c \in G/H$ and $c^{-1} \cdot_G a \cdot_G b \in H$.

Definition 2.12 (RCA_0) A subset X of a partial order Y is *convex* if

$$\forall a, b, x \in Y \big((a, b \in X \wedge a \leq x \leq b) \rightarrow x \in X\big).$$

A subgroup H of a p.o. group G is *convex* if it is convex as a subset of G.

Definition 2.13 Let $(G, \leq)$ be a p.o. group and H a convex normal subgroup. The *induced order*, $\leq_{G/H}$, on G/H is defined by $a \leq_{G/H} b$ if and only if $\exists h \in H(a \leq_G bh)$.

A useful variant of this definition is that $P(G/H)$ is the image of $P(G)$ under the canonical map $G \to G/H$.

$$P(G/H) = \{ g \in G/H \,|\, \exists h \in H\, (gh \in P(G)) \}$$

As above, the subscript on $\leq_{G/H}$ is dropped as long as it is clear whether a and b are being compared as elements of G or G/H. When the context is not clear, we denote elements of G/H by aH and bH.

We would like to know which set existence axioms are required to form the induced order on G/H. It turns out that the answer depends on whether we have a full or partial order on G. The condition in Definition 2.13 is Σ^0_1, so Σ^0_1 comprehension certainly suffices. The following theorem shows that in the case of fully ordered groups, we can do better than the Σ^0_1 definition.

Theorem 2.14 (RCA_0) *Let $(G, \leq)$ be an f.o. group and H a convex normal subgroup. The induced order on G/H exists.*

Proof: Let $a, b \in G/H$ and $a \neq b$. Because a and b are representatives of different cosets, $ab^{-1} \notin H$.

Claim 2.15 *$\exists h \in H\, (a \leq bh)$ if and only if $a \leq b$.*

If $a \leq b$ then, because $1_G \in H$, it follows that $\exists h \in H\, (a \leq bh)$. For the other direction, suppose $\exists h \in H(a \leq bh)$ and $b < a$. Then $b < a \leq bh$ and so $1_G < b^{-1}a \leq h$. Since H is convex, $b^{-1}a \in H$ which gives a contradiction. The induced order can now be given by a Σ^0_0 condition: $aH \leq bH$ if and only if $aH = bH$ or $a < b$. □

Corollary 2.16 *If $(G, \leq_G)$ is a computably fully ordered computable group and H is a computable convex normal subgroup, then the induced order on G/H is computable.*

It is also important to know when we can combine full orders on G/H and H to form a full order on G under which H is convex and the induced orders on H and G/H are the ones with which we started. Notice that an order on H is not necessarily preserved under conjugation by arbitrary elements of G, but that any order on G must have this property. Hence a necessary condition for an order on H to extend to all of G, is that $a \leq_H b$ implies $gag^{-1} \leq_H gbg^{-1}$ for all $g \in G$. This condition is also sufficient.

Definition 2.17 (RCA_0) Let H be a normal subgroup of G and $\leq$ a full order on H. $(H, \leq)$ is *fully G-ordered* if for any $a, b \in H$ and $g \in G$, $a \leq b$ implies $gag^{-1} \leq gbg^{-1}$.

Theorem 2.18 (RCA_0) *Let $(H, \leq_H)$ be a fully G-ordered normal subgroup and $(G/H, \leq_{G/H})$ an f.o. group. G admits a full order under which the induced orders on H and G/H correspond to those given and H is convex.*

Proof: The standard proof goes through in RCA_0. The idea is that given $a, b \in G$, we define $a \leq_G b$ if and only if either $aH \leq_{G/H} bH$ or $aH = bH$ and $a^{-1}b \in P(H)$. For more details, see [10]. □

Next we show that ACA_0 is equivalent to the existence of the induced order on the quotient of a p.o. group. By Theorem 1.4, ACA_0 is equivalent to the existent of the range of an arbitrary $1-1$ function. Given such a function, the strategy is to code its range into a group in such a way that it can be recovered from the order on the quotient group. The torsion free abelian group A on generators a_i, b_i for $i \in \mathbb{N}$ is used to do the coding. The first step is to present this group formally. Because A is an abelian group, we use additive notation.

The elements of A are quadruples of finite sets (I, q, J, p) where I and J are finite subsets of $\mathbb{N}$ and p and q represent functions

$$q : I \to \mathbb{Z} \setminus \{0\} \qquad \text{and} \qquad p : J \to \mathbb{Z} \setminus \{0\}.$$

The element represented by (I, q, J, p) is denoted $\sum_{i\in I} q_i a_i + \sum_{j\in J} p_j b_j$. The elements represented by (I, q, J, p) and (I', q', J', p') are equal if and only if $I = I'$, $J = J'$, $q = q'$ and $p = p'$. The sum

$$\left(\sum_{i\in I} q_i a_i + \sum_{j\in J} p_j b_j\right) + \left(\sum_{k\in K} r_k a_k + \sum_{l\in L} s_l b_l\right)$$

is $\sum_{m\in M} t_m a_m + \sum_{n\in N} u_n b_n$ where $M = (I \cup K) \setminus \{x \in I \cap K \mid q_x + r_x = 0\}$ and

$$t_m = \begin{cases} q_m & \text{if} \quad m \in I \setminus K \\ r_m & \text{if} \quad m \in K \setminus I \\ q_m + r_m & \text{if} \quad m \in K \cap I. \end{cases}$$

N and u_n are defined similarly. The identity element, 0_A, is represented by $(\varnothing, \varnothing, \varnothing, \varnothing)$ and if g is represented by (I, q, J, p), then g^{-1} is the sum

$$\sum_{i\in I} -q_i a_i + \sum_{j\in J} -p_j b_j.$$

Theorem 2.19 (RCA_0) *The following are equivalent:*

1. ACA_0
2. *For every p.o. group $(G, \leq_G)$ and every convex normal subgroup H, the induced order $\leq_{G/H}$ on G/H exists.*

Proof:

Case 1: $(1) \Longrightarrow (2)$:

For $x, y \in G/H$, use Σ^0_1 comprehension in ACA_0 to define the relation

$$x \leq_{G/H} y \longleftrightarrow \exists h \in H\ (x \leq_G yh).$$

Case 2: $(2) \Longrightarrow (1)$:

Let $f : \mathbb{N} \to \mathbb{N}$ be a $1-1$ function. By Theorem 1.4, it suffices to show that the range of f exists. Define $P(A)$ to be the semigroup generated by the a_i's using Σ^0_0 comprehension.

$$P(A) = \left\{ \sum_{i\in I} q_i a_i + \sum_{j\in J} p_j b_j \;\middle|\; J = \varnothing \wedge \forall i \in I (q_i > 0) \right\}$$

This definition is Σ^0_0 because $\forall i \in I$ is a bounded quantifier.

Claim 2.20 *$P(A)$ is the positive cone for a partial order on A.*

It suffices to show that $P(A)$ is a pure normal semigroup with identity. By definition, $0_A \in P(A)$. $P(A)$ is normal because it is a subset of an abelian group and $P(A)$ is a semigroup since it is closed under componentwise addition. Finally, since $P^{-1}(A)$ is defined by

$$P^{-1}(A) \;=\; \left\{ \sum_{i\in I} q_i a_i \;+\; \sum_{j\in J} p_j b_j \;\middle|\; J = \varnothing \wedge \forall i \in I (q_i < 0) \right\},$$

it is clear that $P(A)$ is pure.

Let H be the subgroup generated by elements of the form $-a_n + b_m$ where $f(n) = m$. Formally, $\sum_{i\in I} q_i a_i \;+\; \sum_{j\in J} p_j b_j$ is in H if and only if either $I = J = \varnothing$ or $I \neq \varnothing$ and

$$\forall i \in I (f(i) \in J \wedge q_i = -p_{f(i)}) \;\wedge\; \forall j \in J \; \exists i \in I (f(i) = j \wedge q_i = -p_j).$$

This condition is Σ^0_0 since all the quantification is bounded. H is normal because the group is abelian.

Claim 2.21 *H is convex.*

It suffices to show that there are no nontrivial intervals in H. That is, for any $c, d \in H$, $c \leq d$ implies $c = d$. Notice that any $c, d \in H$ can be expressed as

$$c = \sum_{i\in I} -q_i a_i \;+\; \sum_{i\in I} q_i b_{f(i)} \qquad \text{and} \qquad d = \sum_{j\in J} -p_j a_j \;+\; \sum_{j\in J} p_j b_{f(j)}.$$

If $c \leq d$, then $-c + d \in P(A)$. Since $P(A)$ is generated by the a_i's, the b_i part of the sums must cancel out. Hence $\sum_{i\in I} -q_i b_{f(i)} \;+\; \sum_{j\in J} p_j b_{f(j)} \;=\; 0$. Since 0 is represented by the quadruple $(\varnothing, \varnothing, \varnothing, \varnothing)$, we have $I = J$ and $q = p$. Hence $c = d$ as required.

Now that A, $P(A)$, and H are defined, all that remains to show is how the range of f can be defined from the induced order $\leq_{A/H}$ on A/H. This definition follows from the final two claims.

Claim 2.22 *The existence of $\leq_{A/H}$ implies the existence of $P(A) + H$.*

Given $x \in A$, we need to decide if $x \in P(A) + H$. Let $n \in A/H$ be such that $n + H = x + H$. Since x and n differ by an element of H, $x \in P(A) + H$ if and only if $n \in P(A) + H$. However,

$$0_{A/H} \leq_{A/H} n \;\longleftrightarrow\; \exists h \in H \; (n + h \in P(A)) \longleftrightarrow\; n \in P(A) + H.$$

Thus, $P(A) + H$ is definable from $\leq_{A/H}$ in RCA_0.

Claim 2.23 $b_m \in P(A) + H \;\longleftrightarrow\; m \in$ *range*(f)

First assume that $b_m = p + h$ for some $p \in P(A)$ and $h \in H$. Then b_m can be written as

$$b_m \;=\; \sum_{i\in I} q_i a_i \;+\; \left(\sum_{j\in J} -p_j a_j \;+\; \sum_{j\in J} p_j b_{f(j)} \right).$$

The parts of the equation with a_i's must cancel out, leaving $I = J$. Furthermore, because only b_m appears on the left of the equation, $J = \{n\}$ where $f(n) = m$ and $p_n = 1$. Hence m is in the range of f.

For the other direction, assume that m is in the range of f. For some n, $f(n) = m$, and hence $-a_n + b_m \in H$ and $a_n \in P(A)$. Adding these equations shows that $b_m \in P(A) + H$. □

Corollary 2.24 *There is a computably partially ordered computable group* $(G, \leq_G)$ *and a computable convex normal subgroup H such that the degree of the induced order on G/H is* $0'$.

Proof: Let f be a computable $1-1$ function that enumerates $0'$. Since f is computable, the p.o. group in the proof of Theorem 2.19 is a computably partially ordered computable group. The range of f is computable from the induced order on G/H, so $0' \leq_T \deg(\leq_{G/H})$. On the other hand, $\leq_{G/H}$ has a Σ^0_1 definition, so $\deg(\leq_{G/H}) \leq_T 0'$. □

3 Group conditions for orderability Any group can be partially ordered: take the trivial partial order under which no two distinct elements are comparable. Determining when a group admits a full order is more complicated question. Being torsion free is a necessary condition, but unfortunately not a sufficient one. If G is the group presented by the letters a and b with the relation $aba^{-1} = b^{-1}$, then G is torsion free but not orderable. Indeed, if $b > 1_G$ then $aba^{-1} = b^{-1}$ forces $b^{-1} > 1_G$ and if $b < 1_G$ then $aba^{-1} = b^{-1}$ forces $b^{-1} < 1_G$.

The simplest group condition that implies full orderability is being torsion free and abelian. A proof of this fact can be found in [6] or [10].

Theorem 3.1 *Every torsion free abelian group is an O-group.*

The effective content of Theorem 3.1 was first explored in [3]. They constructed a computable group classically isomorphic to $\bigoplus_\omega \mathbb{Z}$ which has no computable full order.

Theorem 3.2 (Downey and Kurtz) *There is a computable torsion free abelian group with no computable full order.*

Hatzikiriakou and Simpson [7] used a similar proof in the context of reverse mathematics to show that Theorem 3.1 is equivalent to WKL_0. By the Low Basis Theorem, this fact implies that every computable torsion free abelian group must have a full order of low Turing degree.

Theorem 3.3 (Hatzikiriakou and Simpson) (RCA_0) *The following are equivalent:*

1. WKL_0
2. *Every torsion free abelian group is an O-group.*

Theorem 3.1 is generalized in [10] to torsion free nilpotent groups.

Theorem 3.4 *Every torsion free nilpotent group is an O-group.*

The goal of this section is to use arguments similar to those in [7], to show that Theorem 3.4 is equivalent to WKL_0. Notice that as long as RCA_0 suffices to prove that every abelian group is nilpotent, Theorem 3.3 already shows that Theorem 3.4 implies WKL_0. To state the result precisely, we need a formal definition of nilpotent groups in second-order arithmetic.

In keeping with standard mathematical notation, if H is a normal subgroup of G, we let $\pi : G \to G/H$ denote the projection function. That is, π picks out the $<_{\mathbb{N}}$-least representative of gH. Frequently, we write gH instead of $\pi(g)$.

Definition 3.5 The *center* of a group G is defined as

$$C(G) \;=\; \{\, g \in G \,|\, \forall x \in G\, (gx = xg)\,\}.$$

In general, the existence of the center is equivalent to ACA_0, as we shall see in Section 5. However if $C(G)$ is given, the next two lemmas can be proved in RCA_0.

Lemma 3.6 (RCA_0) *If $C(G)$ exists then $C(G)$ is a normal subgroup of G.*

Lemma 3.7 (RCA_0) *If H is a normal subgroup of G, $\pi : G \to G/H$ and $C(G/H)$ exists, then $K = \{g \in G \mid \pi(g) \in C(G/H)\} = \pi^{-1}(C(G/H))$ is a normal subgroup of G.*

Definition 3.8 Let G be a group. The *upper central series* of G is the series of subgroups $\zeta_0 G \leq \zeta_1 G \leq \zeta_2 G \leq \cdots$ defined by $\zeta_0 G = \langle 1_G \rangle$, $\zeta_1 G = C(G)$, and $\zeta_{i+1} G = \pi^{-1}\big(C(G/\zeta_i G)\big)$ where $\pi : G \to G/\zeta_i G$. G is *nilpotent* if $\zeta_n G = G$ for some $n \in \omega$.

Notice that $\zeta_{i+1}G/\zeta_i G \cong C(G/\zeta_i G)$. In order to use nilpotent groups in RCA_0, we need to define a code for them that explicitly gives the information contained in the upper central series.

Definition 3.9 (RCA_0) The pair $N \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ is a *code for a nilpotent group* G if the first $n+1$ columns of N satisfy

1. $N_0 = \langle 1_G \rangle$
2. $N_1 = C(G)$
3. $N_n = G$
4. For $0 \leq i < n$, if $\pi : G \to G/N_i$, then $N_{i+1} = \pi^{-1}(C(G/N_i))$.

A group G is *nilpotent* is there is such a code (N, n) for G.

Lemma 3.10 (RCA_0) *Every abelian group is nilpotent.*

Proof: If G is abelian then we can define a code for G as a nilpotent group by setting $n = 1$ and $N \subseteq \mathbb{N}$ with $N_0 = \langle 1_G \rangle$ and $N_1 = G$. □

Lemma 3.11 (RCA_0) *If (N, n) is the code for a nilpotent group G then for all $0 \leq i < n$, N_{i+1}/N_i is abelian.*

Proof: By definition, $N_{i+1} = \pi^{-1}(C(G/N_i))$ with $\pi : G \to G/N_i$. Therefore, $N_{i+1}/N_i \cong C(G/N_i)$. □

Theorem 3.12 (RCA_0) *The following are equivalent.*

1. WKL_0
2. *Every torsion free nilpotent group is an O-group.*

The idea of the proof is that a nilpotent group is formed from a finite number of abelian quotients N_{i+1}/N_i. These quotients are torsion free, so each is fully orderable by Theorem 3.3. We need to put these orders together using a finite number of applications of Theorem 2.18. Notice that if (N, n) is the code for a torsion free nilpotent group G and $n \geq 1$, then N_1 must be torsion free since it is a subgroup of G.

Definition 3.13 The *commutator* of x and y, denoted $[x, y]$, is the element $x^{-1}y^{-1}xy$.

Lemma 3.14 (RCA_0) *Let (N, n) be a code for a nilpotent group G. If $0 \leq i < n$ and $x \in N_{i+1}$, then $[x, g] \in N_i$ for all g.*

Proof: Notice that for $i = 0$, the lemma follows trivially because N_1 is the center of G. Assume $i \geq 1$. By definition, $x \in N_{i+1}$ means $xgN_i = gxN_i$ for all g. For any particular g, there is a $c \in N_i$ such that $xg = gxc$ and hence also $cg^{-1}x^{-1} = x^{-1}g^{-1}$. Let h be any element of G.

$$[x, g] \cdot h = x^{-1}g^{-1}xg \cdot h = x^{-1}g^{-1}gxch = ch$$

Since $c \in N_i$, we know that $ch = hc\tilde{c}$ for some $\tilde{c} \in N_{i-1}$. We now have:

$$ch = hc\tilde{c} = hcg^{-1}x^{-1}xg\tilde{c} = hx^{-1}g^{-1}xg\tilde{c}.$$

Thus, we have $[x, g] \cdot h = h \cdot [x, g] \cdot \tilde{c}$ for some $\tilde{c} \in N_{i-1}$ and hence

$$[x, g] \cdot hN_{i-1} = h \cdot [x, g]N_{i-1}.$$

This equality implies that $[x, g]N_{i-1}$ is in the center of G/N_{i-1} and hence that $[x, g] \in N_i$. □

Lemma 3.15 (RCA_0) *Let (N, n) be a code for a nilpotent group G. If $1 \leq i < n$ and $x \in N_{i+1}$, then for all $m > 0$, $[x, g]^m N_{i-1} = [x^m, g]N_{i-1}$.*

Proof: Because $[x, g]^m N_{i-1} = [x^m, g]N_{i-1}$ is a Σ^0_0 statement, we can prove this lemma in RCA_0 by induction on m. The case for $m = 1$ is trivial, so assume the equality holds for m and we prove it for $m + 1$. Since $[x, g]^{m+1} = [x, g]^m \cdot [x, g]$, we can apply the induction hypothesis in the form $[x, g]^m = [x^m, g] \cdot c$ for some $c \in N_{i-1}$. We now have

$$[x, g]^{m+1} = [x^m, g] \cdot c \cdot [x, g] = x^{-m}g^{-1}x^m gc \cdot [x, g].$$

By Lemma 3.14, $x \in N_{i+1}$ implies $[x, g] \in N_i$ and so $[x, g]$ commutes with elements of G modulo N_{i-1}. Therefore, for some $\tilde{c} \in N_{i-1}$ we have

$$\begin{aligned} x^{-m} \cdot g^{-1}x^m gc \cdot [x, g] &= x^{-m} \cdot [x, g] \cdot g^{-1}x^m gc\tilde{c} \\ &= x^{-m-1}g^{-1}xgg^{-1}x^m gc\tilde{c} \\ &= [x^{m+1}, g] \cdot c\tilde{c}. \end{aligned}$$

Because $c\tilde{c} \in N_{i-1}$, this calculation establishes the induction case. □

Lemma 3.16 (Mal'cev) (RCA_0) *Let* (N,n) *be a code for a torsion free nilpotent group* G*. For every* $0 \leq i < n$*,* N_{i+1}/N_i *is torsion free.*

Proof: We prove this theorem by bounded induction on i. Because $N_0 = \langle 1_G \rangle$ we have $N_1/N_0 = N_1$, which establishes the theorem for $i = 0$. Assume $i \geq 1$ and the theorem holds for $i-1$. The induction hypothesis tells us that N_i/N_{i-1} is torsion free. Let $x \in N_{i+1}$ and suppose that $x^m \in N_i$ for some $m > 0$. We need to show that $x \in N_i$. For any $g \in G$, Lemma 3.15 implies that $[x,g]^m N_{i-1} = [x^m, g]N_{i-1}$. By Lemma 3.14, $x^m \in N_i$ implies that $[x^m, g] \in N_{i-1}$. Therefore, $[x,g]^m \in N_{i-1}$. Applying Lemma 3.14 to $x \in N_{i+1}$ tells us that $[x,g] \in N_i$. Putting these facts together, we have $[x,g]N_{i-1} \in N_i/N_{i-1}$ and $[x,g]^m N_{i-1} = 1_G N_{i-1}$. Since N_i/N_{i-1} is torsion free, it must be that $[x,g] \in N_{i-1}$. However, this fact implies that $xgN_{i-1} = gxN_{i-1}$ for all g and so $x \in N_i$ as required. $\square$

Lemma 3.17 (WKL_0) *Let* (N,n) *be a code for a torsion free nilpotent group* G*. For every* $0 \leq i < n$*,* N_{i+1}/N_i *is a fully* G/N_i*-orderable group.*

Proof: We need to show that there is a full order on N_{i+1}/N_i such that for all $a, b \in N_{i+1}/N_i$ and $g \in G/N_i$, if $aN_i < bN_i$ then $gag^{-1}N_i < gbg^{-1}N_i$. By Lemmas 3.11 and 3.16, N_{i+1}/N_i is a torsion free abelian group and hence by Theorem 3.3, WKL_0 proves that it is fully orderable.

Let $\leq$ be any full order on N_{i+1}/N_i, let $a < b$ be elements of N_{i+1}/N_i and let $g \in G/N_i$. Since $N_{i+1}/N_i \cong C(G/N_i)$, we have $gag^{-1}N_i = aN_i$ and $gbg^{-1}N_i = bN_i$. Hence, $aN_i < bN_i$ implies $gag^{-1}N_i < gbg^{-1}N_i$. $\square$

We are now ready to prove Theorem 3.12

Proof:

Case 1: $(2) \Longrightarrow (1)$

Assume every torsion free nilpotent group is an O-group. By Lemma 3.10, this assumption implies that every torsion free abelian group is an O-group. From here, Theorem 3.3 implies (1).

Case 2: $(1) \Longrightarrow (2)$

For each $1 \leq i \leq n$, let $\hat{P}_i$ be the strict positive cone of a full G/N_{i-1}-order on N_i/N_{i-1}. Set $P_i = \{x \in N_i \mid xN_{i-1} \in \hat{P}_i\}$ and $P = (\cup_{i=1}^n P_i) \cup \{1_G\}$.

The following series of claims proves that P is the positive cone for a full order on G.

Claim 3.18 *P is a semigroup with identity.*

It suffices to show P is closed under multiplication. Let $x, y \in P$ with $x, y \neq 1_G$. There are i, j such that $x \in P_i$ and $y \in P_j$. If $i = j$ then $xN_{i-1}, yN_{i-1} \in \hat{P}_i$ and so $xyN_{i-1} \in \hat{P}_i$ and $xy \in P_i$. If $i \neq j$ then, without loss of generality, assume that $i < j$. Since $x \in P_i$, it follows that $x \in N_i$ and hence $x \in N_{j-1}$. But then, $xyN_{j-1} = yN_{j-1}$ and so $xy \in P_j$.

Claim 3.19 *P is normal.*

Let $x \in P$, $x \neq 1_G$ and $g \in G$. There is an i such that $x \in P_i$. Since $\hat{P}_i$ is the strict positive cone of a full G/N_{i-1}-order on N_i/N_{i-1}, we have that $xN_{i-1} \in \hat{P}_i$ implies that $gxg^{-1}N_{i-1} \in \hat{P}_i$. Hence $gxg^{-1} \in P_i$.

Claim 3.20 *P is pure.*

Let $x \in P$ and $x \neq 1_G$. We need to show that $x^{-1} \notin P$. There is an i such that $x \in P_i$. Because $\hat{P}_i$ is the strict positive cone on N_i/N_{i-1}, we know that $x \in N_i$ and $x \notin N_{i-1}$. Hence $x^{-1} \in N_i$ and $x^{-1} \notin N_{i-1}$. However, because $xN_{i-1} \in \hat{P}_i$, it follows that $x^{-1}N_{i-1} \notin \hat{P}_i$ and so $x^{-1} \notin P_i$. To show $x^{-1} \notin P_j$ for $j > i$, notice that since $x^{-1} \in N_i$, we also have $x^{-1} \in N_{j-1}$. Therefore $x^{-1}N_{j-1} = 1_G N_{j-1}$ and hence $x^{-1} \notin P_j$. Finally, assume for a contradiction that $j < i$ and $x^{-1} \in P_j$. It follows that $x^{-1} \in N_{i-1}$. However, above we showed that $x^{-1} \notin N_{i-1}$. Thus, $x^{-1} \notin P_j$ for any j.

Claim 3.21 *P is full.*

Let $x \in P$ and $x \neq 1_G$. We need to show that either $x \in P$ or $x^{-1} \in P$. There is an i such that $x \in N_i$ and $x \notin N_{i-1}$. Since $\hat{P}_i$ is a full order on N_i/N_{i-1}, either $xN_{i-1} \in \hat{P}_i$ or $x^{-1}N_{i-1} \in \hat{P}_i$. Thus, either $x \in P_i$ or $x^{-1} \in P_i$. □

4 Direct products Groups are frequently constructed by means of a direct product. These constructions preserve full orderability. A proof of the following theorem can be found in either [6] or [10].

Theorem 4.1 *Any direct product of O-groups is an O-group.*

To examine this theorem in reverse mathematics, we need to distinguish between finite and restricted countable direct products. The finite direct product $A_0 \times A_1 \times \cdots \times A_{n-1}$ consists of sequences of length n such that the i^{th} element of each sequence is in A_i. Multiplication is componentwise. The elements of the restricted direct product of A_i for $i \in \mathbb{N}$ are finite sequences σ such that for all $i < \mathrm{lh}(\sigma)$, $\sigma(i) \in A_i$. The idea is that the element represented by σ has 1_{A_j} as its j^{th} component for all $j \geq \mathrm{lh}(\sigma)$. In order to make each sequence represent a distinct element, we add the requirement that the last element in the sequence is not an identity element. The formal definitions are given below.

Definition 4.2 (RCA_0) If $n \in \mathbb{N}$ and for all $i < n$, A_i is a group, then the *finite direct product* $G = \prod_{i=0}^{n-1} A_i$ is defined by:

$$G = \{\sigma \in \mathrm{Fin}_{\mathbb{N}} \mid \mathrm{lh}(\sigma) = n \wedge \forall i < n\, (\sigma(i) \in A_i)\}$$

$$1_G = \langle 1_{A_0}, 1_{A_1}, \ldots, 1_{A_{n-1}} \rangle$$

$$\sigma \cdot_G \tau = \langle \sigma(0) \cdot_{A_0} \tau(0), \ldots, \sigma(n-1) \cdot_{A_{n-1}} \tau(n-1) \rangle.$$

Theorem 4.3 (RCA_0) *If $n \in \mathbb{N}$ and for all $i < n$, A_i is an O–group, then $G = \prod_{i=0}^{n-1} A_i$ is an O–group.*

Proof: Let $P^+(A_i)$ be the strict positive cone of a full order on A_i. Order G lexicographically:

$$P^+(G) = \{\sigma \in G \mid \exists i < n\ (\sigma(i) \in P^+(A_i) \wedge \forall j < i(\sigma(j) = 1_{A_j}))\}$$

$$P(G) = P^+(G) \cup \{ \langle 1_{A_0}, \ldots, 1_{A_{n-1}} \rangle \}.$$

From this definition, $P(G)$ is clearly full, pure, and contains the identity. It remains to check that it is a normal semigroup. Since $P(G)$ is closed under multiplication, it is a semigroup. To see that it is normal, let $\sigma \in P(G)$ have its first nonidentity element at $\sigma(i)$. If $\tau = \langle g_0, \ldots, g_{n-1} \rangle \in G$ then $\tau\sigma\tau^{-1}$ is

$$\langle g_0, \ldots, g_{n-1} \rangle \cdot_G \langle 1_{A_0}, \ldots, 1_{A_{i-1}}, a_i, \ldots, a_{n-1} \rangle \cdot_G \langle g_0^{-1}, \ldots, g_{n-1}^{-1} \rangle.$$

The first nonidentity element in this product is $g_i a_i g_i^{-1}$. Because $a_i \in P^+(A_i)$, we have $g_i a_i g_i^{-1} \in P^+(A_i)$ and hence $\tau\sigma\tau^{-1} \in P^+(G)$. $\square$

Corollary 4.4 *The direct product of a finite number of computably fully orderable computable groups is computably fully orderable.*

Definition 4.5 (RCA_0) Let A be a set such that for each i, the i^{th} column A_i is a group. The *restricted direct product* $G = \prod_{n \in \mathbb{N}} A_n$ is defined by:

$$G = \{\sigma \in \mathrm{Fin}_{\mathbb{N}} \mid \forall i < \mathrm{lh}(\sigma)\ (\sigma(i) \in A_i \wedge \sigma(\mathrm{lh}(\sigma) - 1) \neq 1_{A_{\mathrm{lh}(\sigma)-1}})\}$$

$$1_G = \langle\rangle$$

where $\langle\rangle$ is the empty sequence. Multiplication is componentwise, removing any trailing identity elements.

Theorem 4.6 (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If* $\forall i\ (A_i$ *is an O-group*$)$ *then* $G = \prod_{i \in \mathbb{N}} A_i$ *is an O-group.*

Proof:

Case 1: $(1) \Longrightarrow (2)$:

We know $\forall i\, \exists Y\, (Y$ is a positive cone on $A_i)$ and by Theorem 4.3, for each $n \in \mathbb{N}$, RCA_0 suffices to prove that there exists a positive cone on $\prod_{i=0}^{n-1} A_i$.

A uniform (strict) order on the A_i's is a set P such that its i^{th} column P_i is the (strict) positive cone of a full order on A_i. To prove that G is an O-group, it suffices to prove the existence of a uniform order on the A_i. From a uniform order, we can define the lexicographic order on G as in Theorem 4.3. To show the existence of a uniform order, we build a tree T such that any path on the tree codes such an order. T is built in stages such that at the end of stage s, all nodes of length s are defined. Each node on T keeps a guess at an approximation to a uniform strict order. Suppose σ is a node on T at level s, $s+1 = \langle e, i \rangle$, $e \neq 1_{A_i}$, and P_σ is σ's approximation. At stage $s+1$ we check if $1_{A_j} \in P_\sigma$ for any j. Since P_σ is a finite set, this can be done computably. If $1_{A_j} \in P_\sigma$, then P_σ cannot be a subset of a uniform strict order, so we terminate this branch. Otherwise, we define two extensions of P_σ: one by adding $e \in A_i$ to P_σ and the other by adding $e^{-1} \in A_i$ to P_σ. These sets are each closed under one step multiplication and conjugation by elements less than s. One extension becomes $P_{\sigma*0}$ and the other becomes $P_{\sigma*1}$. This construction is presented formally below. T_s is the set of nodes of T of length s.

Construction

Stage 0: Set $T_0 = \{\langle\rangle\}$ and $P_{\langle\rangle} = \varnothing$.

Stage $\mathbf{s+1}$: Assume $s = \langle e, i\rangle$. For each $\sigma \in T_s$ do the following:

1. Check if 1_{A_j} appears in P_σ for any j. If so, σ has no extensions on T, so move on to the next node in T_s. If not, add $\sigma * 0$ and $\sigma * 1$ to T_{s+1} and move on to 2.
2. If $e = 1_{A_i}$ or e does not represent an element of A_i, then set $P_{\sigma*0} = P_{\sigma*1} = P_\sigma$ and move on to the next node in T_s. Otherwise, move on to 3.
3. If $e \in A_i$ and $e \neq 1_{A_i}$ define

$$\tilde{P}_{\sigma*0} = P_\sigma \cup \{\langle e^{-1}, i\rangle\} \qquad \text{and} \qquad \tilde{P}_{\sigma*1} = P_\sigma \cup \{\langle e, i\rangle\}$$

   Extend these by:

$$\begin{aligned} \langle k, j\rangle \in P_{\sigma*0} \quad \longleftrightarrow \quad & \langle k, j\rangle \in \tilde{P}_{\sigma*0} \ \vee \\ & \exists \langle m, j\rangle, \langle n, j\rangle \in \tilde{P}_{\sigma*0}(m \cdot_{A_j} n = k) \ \vee \\ & \exists n \leq s\, \exists \langle m, j\rangle \in \tilde{P}_{\sigma*0}(n \in A_j \wedge n \cdot_{A_j} m \cdot_{A_j} n^{-1} = k) \end{aligned}$$

$$\begin{aligned} \langle k, j\rangle \in P_{\sigma*1} \quad \longleftrightarrow \quad & \langle k, j\rangle \in \tilde{P}_{\sigma*1} \ \vee \\ & \exists \langle m, j\rangle, \langle n, j\rangle \in \tilde{P}_{\sigma*1}(m \cdot_{A_j} n = k) \ \vee \\ & \exists n \leq s\, \exists \langle m, j\rangle \in \tilde{P}_{\sigma*1}(n \in A_j \wedge n \cdot_{A_j} m \cdot_{A_j} n^{-1} = k). \end{aligned}$$

End of Construction

Claim 4.7 *T is infinite.*

For a contradiction, suppose that T is not infinite and hence there is some level n at which T has no nodes. Because the standard coding for pairs satisfies the inequality $\langle x, y\rangle \geq y$, we know that if $\langle x, y\rangle$ occurs in the construction before stage n, then $y \leq n$. That is, at stage n, T has only considered elements from A_0 through A_n. By Theorem 4.3, RCA_0 suffices to prove that $\prod_{i=0}^n A_i$ is an O-group. Let X be the strict positive cone for a full order on this finite product and let $P^+(A_i)$ be defined by

$$x \in P^+(A_i) \ \longleftrightarrow \ \langle 1_{A_0}, \ldots, 1_{A_{i-1}}, x, 1_{A_{i+1}}, \ldots, 1_{A_n}\rangle \in X.$$

For each $k \leq n$, $k = \langle x, i\rangle$ for some $i \leq n$. Define $\sigma \in \mathrm{Fin}_{\mathbb{N}}$ with $\mathrm{lh}(\sigma) = n$ by

$$\sigma(k) = \begin{cases} 1 & \text{if } k = \langle x, i\rangle \wedge x \in P^+(A_i) \\ 0 & \text{otherwise} \end{cases}$$

From the definition it is clear that

$$\sigma(k) = 0 \longleftrightarrow x = 1_{A_i} \vee x^{-1} \in P^+(A_i) \vee x \notin A_i. \tag{1}$$

To prove the claim, it suffices to show that $\sigma \in T$. We show by induction that for all $k \leq n$, $\sigma[k] = \langle \sigma(0), \ldots, \sigma(k-1)\rangle \in T$ and $P_{\sigma[k]} \subseteq X$. Clearly, $\sigma[0] = \langle\rangle \in T$ and $P_{\sigma[0]} = \varnothing \subseteq X$. Assume that $\sigma[k] \in T$ and $P_{\sigma[k]} \subseteq X$. Because $1_{A_j} \notin P_{\sigma[k]}$ we know that $\sigma[k+1] \in T$. From the definition of σ and equation 1, it is clear that $\tilde{P}_{\sigma[k+1]} \subseteq X$.

Because $P_{\sigma[k+1]}$ is obtained by multiplying and conjugating elements of $\tilde{P}_{\sigma[k+1]}$, it follows that $P_{\sigma[k+1]} \subseteq X$. Thus, $\sigma[n] = \sigma \in T$.

Since T is infinite, WKL_0 provides a path f through T. Let $f[n]$ denote the sequence $\langle f(0), \dots, f(n-1) \rangle$ and define

$$\tilde{Z} = \bigcup_{n \in \mathbb{N}} P_{f[n]}$$

$$Z = \tilde{Z} \cup \{\langle 1_{A_i}, i \rangle \mid i \in \mathbb{N}\}.$$

$\tilde{Z}$ has a Σ^0_1 definition, but for $x \neq 1_{A_i}$, we have $\langle x, i \rangle \in \tilde{Z} \longleftrightarrow \langle x^{-1}, i \rangle \notin \tilde{Z}$. Thus, $\tilde{Z}$ has a Δ^0_1 definition and so both $\tilde{Z}$ and Z exist. It remains to show that Z_i is the positive cone for a full order on A_i.

To show Z_i is full, consider any $x \in A_i$, $x \neq 1_{A_i}$. Let $\sigma = f[n]$ with $\mathrm{lh}(\sigma) = \langle x, i \rangle$. Since f is a path, either $\sigma * 0 = f[n+1]$ or $\sigma * 1 = f[n+1]$.

$$\sigma * 0 = f[n+1] \Longrightarrow \langle x, i \rangle \in P_{\sigma * 0} \Longrightarrow x \in Z_i$$

$$\sigma * 1 = f[n+1] \Longrightarrow \langle x^{-1}, i \rangle \in P_{\sigma * 1} \Longrightarrow x^{-1} \in Z_i$$

To show Z_i is pure, suppose $x \neq 1_{A_i}$ and $x, x^{-1} \in Z_i$. It follows that for some n, both $\langle x, i \rangle$ and $\langle x^{-1}, i \rangle$ are in $P_{f[n]}$. From the construction, 1_{A_i} appears in both $P_{f[n]*0}$ and $P_{f[n]*1}$ so neither $f[n]*0$ nor $f[n]*1$ has an extension. This contradicts the fact that f is a path.

Z_i is a semigroup since if $x, y \in Z_i$ then there is an n such that $\langle x, i \rangle, \langle y, i \rangle \in P_{f[n]}$. By the one step multiplicative closure, $\langle x \cdot_{A_i} y, i \rangle \in P_{f[n+1]}$ and hence $x \cdot_{A_i} y \in Z_i$. Showing Z_i is normal is similar but uses the one step closure under conjugates. Thus Z_i is a full order on A_i and we have constructed the desired uniform order.

Case 2: (2) $\Longrightarrow$ (1):

Assume the restricted countable direct product of O-groups is an O-group. Let f, g be functions such that for all n, m, $f(n) \neq g(m)$. By Theorem 1.3 it suffices to prove the existence of a set S such that

$$\mathrm{range}(f) \subseteq S \ \wedge \ \mathrm{range}(g) \subseteq \mathbb{N} \setminus S.$$

Recall from the first half of this proof that an order on the direct product is equivalent over RCA_0 to a uniform order on the components A_i. The idea of this proof is to give abelian groups A_n each of which has two generators, a_n and b_n. If n is in the range of f, we force a_n and b_n to have the same sign in any order on A_n. That is, either both are positive or both are negative. If n is in the range of g, we force a_n and b_n to have different signs in any order. If neither of these holds, then we let A_n be a free abelian group on two generators. Since the groups are abelian, we use additive notation. The groups look like:

$$A_{f(n)} = \langle a_{f(n)}, b_{f(n)} \mid a_{f(n)} = p_n b_{f(n)} \rangle$$

$$A_{g(n)} = \langle a_{g(n)}, b_{g(n)} \mid a_{g(n)} = -p_n b_{g(n)} \rangle$$

where p_n is the n^{th} odd prime. If n is not in the range of f or g then

$$A_n = \langle a_n, b_n \mid - \rangle.$$

Formally, the elements of A_n are formal combinations $ca_n + db_n$ where $c, d \in \mathbb{Z}$ and

$$\neg\exists i(p_i < 2|d| \wedge f(i) = n) \ \wedge \ \neg\exists i(p_i < 2|d| \wedge g(i) = n).$$

To add $ca_n + db_n$ and $c'a_n + d'b_n$ we check whether $(c + c')\, a_n + (d + d')\, b_n$ violates either of these conditions. If there is an i such that $p_i < 2|d + d'|$ and $f(i) = n$, then we use the relation $a_n = p_i b_n$ to rewrite $(d + d')\, b_n$ as $c''a_n + d''b_n$ where $|d''| < p_i/2$. If the second condition is violated, we do the same thing except we use the relation $a_n = -p_i b_n$.

Because the definition of A_n is uniform in n, the sequence A_n exists. It remains to show that each A_n is orderable and that the separating set is definable from a uniform order of the A_n.

Claim 4.8 *Each A_n is an O-group.*

The proof of this claim splits into two cases. In RCA_0, we cannot tell which case holds, but we know that one of them must hold. If $f(i) = n$ or if $n \notin \text{range}(f) \cup \text{range}(g)$ then $P(A_n) = \{ca_n + db_n \mid c > 0 \vee (c = 0 \wedge d \geq 0)\}$. If $g(i) = n$ then $P(A_n) = \{ca_n + db_n \mid c > 0 \vee (c = 0 \wedge d \leq 0)\}$. In each case it is easy to verify that the set given is the positive cone of a full order. This shows that $RCA_0 \vdash \forall n(A_n$ is an O-group$)$. By assumption, there is a uniform order on the A_n. Let P be the uniform positive cone. That is, P_n is the positive cone of a full order on A_n. Define S by

$$S = \{n \mid a_n \in P_n \longleftrightarrow b_n \in P_n\}.$$

S is the desired separating set since if n is in the range of f then $a_n \in P_n \longleftrightarrow b_n \in P_n$ while if n is in the range of g then $a_n \in P_n \longleftrightarrow -b_n \in P_n$. $\square$

Corollary 4.9 *There is a uniform sequence of computably fully orderable computable groups G_i, $i \in \omega$, such that $\Pi_{i\in\omega} G_i$ is a computable group with no computable full order. $\Pi_{i\in\omega} G_i$ does have a full order of low Turing degree.*

5 *The center*

In this section we show that the existence of the center is equivalent to ACA_0 and that this result holds even for 2 step nilpotent groups, which are intuitively the simplest nonabelian groups.

Definition 5.1 G is *n step nilpotent*, for $n > 1$, if $\zeta_n G = G$. G is *properly n step nilpotent* if G is n step nilpotent and $\zeta_{n-1} G \neq G$.

According to the definition, G is properly 2 step nilpotent if $C(G) \neq G$ and $G/C(G)$ is abelian. These groups can also be defined in terms of the lower central series. The following lemma states the essential property of this alternate definition.

Lemma 5.2 *G is 2 step nilpotent if and only if each commutator $[x, y]$ commutes with all the elements of the group.*

Lemma 5.2 can be used to establish the following identity for 2 step nilpotent groups.

$$[x^{-1}, y] = xy^{-1}x^{-1}y = xy^{-1}x^{-1}yxx^{-1}$$

$$= x \cdot [y, x] \cdot x^{-1} = [y, x]$$

Similarly, we have $[x, y^{-1}] = [y, x]$, $[x^{-1}, y^{-1}] = [x, y]$, and $[x, y]^{-1} = [y, x]$.

Let G be a free 2 step nilpotent group on the generators a_i, $i \in \mathbb{N}$. That is, G is presented by the generators a_i and subject to the relations $[[g, h], k] = 1_G$ for all $g, h, k \in G$. We have the following identity:

$$a_i a_j = a_j a_i a_i^{-1} a_j^{-1} a_i a_j = a_j a_i \cdot [a_i, a_j].$$

Using the identities above and performing similar calculations, we get

$$a_i^{-1} a_j = a_j a_i^{-1} \cdot [a_j, a_i]$$

$$a_i a_j^{-1} = a_j^{-1} a_i \cdot [a_j, a_i]$$

$$a_i^{-1} a_j^{-1} = a_j^{-1} a_i^{-1} \cdot [a_i, a_j].$$

Because these identities allow us to commute any pair of generators modulo a commutator of generators, we can write any element of G as

$$a_{j_0}^{k_0} a_{j_1}^{k_1} \cdots a_{j_l}^{k_l} \cdot c$$

where $j_0 < j_1 < \cdots < j_l$, $k_i \in \mathbb{Z} \setminus \{0\}$ and c is a product of commutators. Furthermore, we can write c as a product of powers of commutators of the form $[a_i, a_j]$ or $[a_i, a_j]^{-1}$ with $i < j$. To get a unique normal form for each element, we arrange these commutators so that a power of $[a_i, a_j]$ occurs to the left of a power of $[a_k, a_l]$ if and only if $i < k$ or $i = k$ and $j < l$.

These normal forms give us a computable presentation of the free 2 step nilpotent group. Furthermore, since we can write down a description of the normal form using only bounded quantifiers, we can define the free 2 step nilpotent group on generators a_i, $i \in \omega$, in RCA_0. Because an element is in the center if and only if it is a product of commutators, RCA_0 suffices to prove that there is a nilpotent code for this group.

Theorem 5.3 (RCA_0) *The following are equivalent:*

1. ACA_0
2. *For every group G the center of G, $C(G)$, exists.*

Proof:

Case 1: (1) $\Longrightarrow$ (2)

The center of G is defined by a Π^0_1 formula, so ACA_0 suffices to prove its existence.

Case 2: (2) $\Longrightarrow$ (1)

By Theorem 1.4, it suffices to prove the existence of the range of an arbitrary $1-1$ function f. Let G be the free 2 step nilpotent group on generators a_i and b_i for $i \in \mathbb{N}$ with the following extra relations

$$a_i a_j = a_j a_i \text{ for all } i, j \in \mathbb{N}$$

$$b_i b_j = b_j b_i \text{ for all } i, j \in \mathbb{N}$$

$$a_i b_j = b_j a_i \iff \forall k \leq i \, (f(k) \neq j).$$

Formally, elements of G have unique normal forms $a_{i_1}^{n_1} \cdots a_{i_k}^{n_k} b_{j_1}^{m_1} \cdots b_{j_l}^{m_l} \cdot c$ where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$, $n_p \neq 0$ for $1 \leq p \leq k$, $m_q \neq 0$ for $1 \leq q \leq l$, and c is a product of commutators with those which match the added relations removed. By the comments above, G exists as a group in RCA_0. However, as we are about to see, RCA_0 is not strong enough to prove that there is a code for G as a nilpotent group.

Let $C(G)$ be the center of G. To define the range of f we use the following equivalences:

$$\begin{aligned} b_j \in C(G) &\iff \forall i\,(a_i b_j = b_j a_i) \\ &\iff \forall i\, \forall k \leq i\,(f(k) \neq j) \\ &\iff \forall k\,(f(k) \neq j). \end{aligned}$$

Therefore, $b_j \in C(G)$ if and only if j is not in the range of f. This equivalence allows us to give a Σ_0^0 definition of the range of f.

$$\text{range}(f) = \{ j \mid b_j \notin C(G) \}$$

□

Corollary 5.4 *There is a computably fully orderable computable 2 step nilpotent group G such that $C(G) \equiv_T 0'$.*

Proof: Consider the group G constructed in the theorem when f is a computable $1-1$ function enumerating $0'$. G is clearly a computable 2 step nilpotent group. Since we can define the range of f from $C(G)$, we have $0' \leq_T C(G)$. However, because $C(G)$ has a Π_1^0 definition from G and G is computable, we know that $C(G) \leq_T 0'$.

It remains to show that G is computably fully orderable. Let H be the subgroup generated by the commutators. H is normal because G is 2 step nilpotent and H is computable because we can tell if an element is the product of commutators by looking at the normal form. H is generated by commutators of the form $[a_i, b_j]$ for which $\exists k \leq i\,(f(k) = j)$. There are no relations between these commutators, so H is a torsion free abelian group which can be computably fully ordered lexicographically from its generators. Since G is 2 step nilpotent, the elements of H commute with all elements of G. Therefore, any full order on H is a full G-order. G/H is the abelianization of G, so it is the free abelian group generated by a_i and b_j for $i, j \in \omega$. Again, there are no extra relations between these elements in G/H, so G/H can be computably fully ordered from its generators. Using Theorem 2.18, the orders on H and G/H can be combined into a computable full order on G. □

The use of infinitely many generators in the proof of Theorem 5.3 is unavoidable due to the following result.

Theorem 5.5 (Baumslag et al.) *The center of a finitely generated nilpotent group is computable.*

6 Semigroup conditions In addition to examining which group conditions imply full orderability, algebraists have also looked for semigroup conditions which imply full orderability. That is, given a group G, state conditions in terms of subsemigroups of G which imply the full orderability of G. In this section, we study three theorems giving such semigroup conditions. The versions stated in Kokorin and Kopytov [10] are given below. In these theorems, $S(a_1, \ldots, a_n)$ denotes the normal semigroup generated by $a_1, \ldots, a_n$. Recall that a semigroup is normal if it is closed under inner automorphisms.

Theorem 6.1 (Fuchs [5]) *A partial order on G with positive cone P can be extended to a full order if and only if for any finite sequence of nonidentity elements, $a_1, \ldots, a_n \in G$, there is a sequence $\epsilon_1, \ldots, \epsilon_n$ with $\epsilon_i = \pm 1$ such that*

$$P \cap S(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) = \varnothing.$$

Theorem 6.2 (Łos [12], Ohnishi [14]) *G is an O-group if and only if for any finite sequence of nonidentity elements $a_1, \ldots, a_n$ there exists a sequence $\epsilon_1, \ldots, \epsilon_n$ with each $\epsilon_i = \pm 1$ such that*

$$1_G \notin S(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}).$$

Theorem 6.3 (Lorenzen [11]) *G is an O-group if and only if for any finite sequence of nonidentity elements $a_1, \ldots, a_n$*

$$\bigcap S(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) = \varnothing$$

where the intersection extends over all sequences $\epsilon_1, \ldots, \epsilon_n$ with $\epsilon_i = \pm 1$.

The first step in studying these theorems in reverse mathematics is to translate the semigroup conditions into the language of second-order arithmetic. If A is a code for a finite sequence of elements of G, let $S(A)$ denote the normal semigroup generated by A. Think of $S(A)$ as built in stages with $S_0(A) = A$ and $S_{n+1}(A)$ containing all the elements that can be formed by conjugating a member of $S_n(A)$ or by multiplying two members of $S_n(A)$. Formally, we define a function s such that $x \in S_n(A)$ if and only if $s(A, n, m, x) = 1$ for some m. Define s by recursion on n with A and m as parameters.

$$s(A, 0, m, x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$s(A, n+1, m, x) = \begin{cases} 1 & \text{if } s(A, n, m, x) = 1 \text{ or} \\ & \exists a, g \leq m\,(s(A, n, m, a) = 1 \wedge x = gag^{-1}) \text{ or} \\ & \exists a, b \leq m\,(s(A, n, m, a) = s(A, n, m, b) = 1 \wedge \\ & \qquad \wedge\, x = ab) \\ 0 & \text{otherwise} \end{cases}$$

Definition 6.4 (RCA_0) If A is a code for a finite sequence of elements of G, let A^{-1} be the code for the finite sequence defined by $A^{-1}(k) = A(k)^{-1}$ for $0 \leq k < \mathrm{lh}(A)$.

Lemma 6.5 (RCA_0) *If A is a code for a finite sequence of elements of G and $s(A, n, m, x) = 1$, then $\exists p\,(s(A^{-1}, n, p, x^{-1}) = 1)$.*

Proof: The proof is by Σ^0_1 induction on n. For the base case, assume that $s(A,0,m,x) = 1$ and so $x \in A$. By definition, $x^{-1} \in A^{-1}$ and $s(A^{-1},0,m,x^{-1}) = 1$. For the induction case, assume $s(A,n+1,m,x) = 1$ and split into three subcases. First, if $s(A,n,m,x) = 1$ then the induction hypothesis implies there is a p such that $s(A^{-1},n,p,x^{-1}) = 1$ and hence $s(A^{-1},n+1,p,x^{-1}) = 1$. Second, if there are $g,a \le m$ with $s(A,n,m,a) = 1$ and $x = gag^{-1}$, then by the induction hypothesis there is a p with $s(A^{-1},n,p,a^{-1}) = 1$. Since $x^{-1} = ga^{-1}g^{-1}$, taking $\tilde{p}$ to be the largest of p, g and a^{-1} gives $s(A^{-1},n+1,\tilde{p},x^{-1}) = 1$. Third, if there are $a,b \le m$ with $s(A,n,m,a) = s(A,n,m,b) = 1$ and $x = ab$, then by the induction hypothesis there are p_1, p_2 such that $s(A^{-1},n,p_1,a^{-1}) = 1$ and $s(A^{-1},n,p_2,b^{-1}) = 1$. Let p be the largest of p_1, p_2, a^{-1} and b^{-1}. Since $x^{-1} = b^{-1}a^{-1}$, if follows that $s(A^{-1},n+1,p,x^{-1}) = 1$. □

Lemma 6.6 (RCA_0) *Let P be the positive cone of a full order on G and A be a code for a finite sequence of nonidentity elements of P. If $s(A,n,m,x) = 1$ then $x > 1_G$.*

Proof: The proof is by induction on n. For the base case, assume that $s(A,0,m,x) = 1$. Since $A \subset P$ and $1_G \notin A$, $x > 1_G$. For the induction case, use the same three subcases as in Lemma 6.5. □

The next step is to write the semigroup conditions using $s(A,n,m,x)$. Let $\mathrm{Fin}_{\pm 1}$ denote the set of codes for finite sequences of 1's and -1's. If $A \in \mathrm{Fin}_G$, $\sigma \in \mathrm{Fin}_{\pm 1}$, and $\mathrm{lh}(A) = \mathrm{lh}(\sigma)$, then let $A_\sigma \in \mathrm{Fin}_G$ be defined by

$$\mathrm{lh}(A_\sigma) = \mathrm{lh}(A)$$

$$\forall k < \mathrm{lh}(A_\sigma)\ (A_\sigma(k) = A(k)^{\sigma(k)}).$$

For example, if $A = \langle 1_G, a\rangle$ and $\sigma = \langle +1, -1\rangle$ then $A_\sigma = \langle 1_G, a^{-1}\rangle$.

In the remaining equations in this section, it is assumed that A ranges over $\mathrm{Fin}_{G \setminus 1_G}$ and σ ranges over $\mathrm{Fin}_{\pm 1}$. Theorems 6.1, 6.2 and 6.3 can now be stated in the language of second-order arithmetic. Notice that since $\exists \sigma \in \mathrm{Fin}_{\pm 1}$ with $\mathrm{lh}(\sigma) = \mathrm{lh}(A)$ is a bounded quantifier, each of the semigroup conditions is Π^0_1.

Theorem 6.7 (WKL_0) *A partial order on G with positive cone P can be extended to a full order if and only if*

$$\forall A\, \exists \sigma\, \forall x,n,m\, \big(\mathrm{lh}(A) = \mathrm{lh}(\sigma) \wedge (s(A_\sigma,n,m,x) = 0 \vee x \notin P)\big). \tag{2}$$

Theorem 6.8 (WKL_0) *G is an O-group if and only if*

$$\forall A\, \exists \sigma\, \forall n,m\, \big(\mathrm{lh}(A) = \mathrm{lh}(\sigma) \wedge s(A_\sigma,n,m,1_G) = 0\big). \tag{3}$$

Theorem 6.9 (WKL_0) *G is an O-group if and only if*

$$\forall A\, \forall x\, \exists \sigma\, \forall m,n\, \big(\mathrm{lh}(A) = \mathrm{lh}(\sigma) \wedge s(A_\sigma,n,m,x) = 0\big). \tag{4}$$

There are several connections between these theorems. G is an O-group if and only if the trivial partial order with positive cone $P = \{1_G\}$ can be extended to a full order. By Theorem 6.7, this condition is equivalent to:

$$\forall A \exists \sigma \forall x, n, m \left(\mathrm{lh}(A) = \mathrm{lh}(\sigma) \wedge (s(A_\sigma, n, m, x) = 0 \vee x \neq 1_G)\right)$$

which in turn is equivalent to equation 3. Hence, RCA_0 proves that Theorem 6.8 is a special case of Theorem 6.7. Furthermore, setting $x = 1_G$ shows that equation 4 implies equation 3.

Showing that equation 3 implies equation 4 requires more work. For $\sigma \in \mathrm{Fin}_{\pm 1}$, let σ^{-1} have the same length as σ with $\sigma^{-1}(k) = -\sigma(k)$. Notice that $A_{\sigma^{-1}} = A_\sigma^{-1}$ and $(A_{\sigma^{-1}})^{-1} = A_\sigma$. For a contradiction, suppose that equation 3 holds and equation 4 does not. Because equation 4 fails, there are A and x such that

$$\forall \sigma \in \mathrm{Fin}_{\pm 1} \exists m, n \left(\mathrm{lh}(\sigma) = \mathrm{lh}(A) \rightarrow s(A_\sigma, n, m, x) = 1\right). \tag{5}$$

Fix A and x. Because equation 3 holds, there is a σ such that $\mathrm{lh}(\sigma) = \mathrm{lh}(A)$ and

$$\forall n, m \left(s(A_\sigma, n, m, 1_G) = 0\right). \tag{6}$$

Fix σ. Applying equation 5 with σ^{-1}, we have $s(A_{\sigma^{-1}}, n, m, x) = 1$ for some m, n and hence by Lemma 6.5, $s(A_\sigma, n, p, x^{-1}) = 1$ for some p. Applying equation 5 with σ we have $s(A_\sigma, \tilde{n}, \tilde{m}, x) = 1$ for some $\tilde{m}, \tilde{n}$. Without loss of generality, assume $n \geq \tilde{n}$. By definition, $s(A_\sigma, n, \tilde{m}, x) = 1$ and so if k is larger than n, $\tilde{m}$ and p, then $s(A_\sigma, n, k, 1_G) = 1$. This fact contradicts equation 6.

Theorem 6.10 (RCA_0) *The following are equivalent:*

1. WKL_0
2. *Theorem 6.7*
3. *Theorem 6.8*
4. *Theorem 6.9*

By the comments above, we know that statement 2 implies statement 3 and that statement 3 and statement 4 are equivalent. It remains to show that statement 1 implies statement 2 and that statement 3 implies statement 1.

Lemma 6.11 (RCA_0) *If a partial order on G with positive cone P can be extended to a full order, then equation 2 holds for P.*

Proof: Assume Q is the positive cone of a full order extending P. Given any $A \in \mathrm{Fin}_{G \setminus 1_G}$, let $\sigma \in \mathrm{Fin}_{\pm 1}$ be such that $\mathrm{lh}(\sigma) = \mathrm{lh}(A)$ and for every $k < \mathrm{lh}(\sigma)$, $A(k)^{-\sigma(k)} \in Q$. For a contradiction, assume for some x, n, m we have

$$s(A_\sigma, n, m, x) = 1 \wedge x \in P.$$

Because $P \subseteq Q$, we have that $x \in Q$. Applying Lemma 6.5 to $s(A_\sigma, n, m, x) = 1$, we have $s(A_{\sigma^{-1}}, n, p, x^{-1}) = 1$ for some p. However by our choice of σ, $A_{\sigma^{-1}}$ must be contained in $Q \setminus 1_G$ and hence $x^{-1} > 1_G$ by Lemma 6.6. Thus $x, x^{-1} \in Q$ and so $x = 1_G$. This conclusion contradicts $x^{-1} > 1_G$. □

Proposition 6.12 (WKL_0) *If $P \subset G$ and equation 2 holds for P then P can be extended to the positive cone of a full order on G.*

Proof: This proof is similar to the proof of Theorem 4.6. Without loss of generality assume that the domain of G is $\mathbb{N}$ and that 0 represents the identity. We sometimes use g_i instead of i to indicate that we are thinking of i as an element of G. We build a binary branching tree T which codes the positive cone of a full order along every path. Equation 2 will imply that T is infinite and so WKL_0 guarantees that it has a path. To simplify the notation we construct $T \subseteq \mathrm{Fin}_{\pm 1}$ instead of $T \subseteq \mathrm{Fin}_{\{0,1\}}$. For each $\sigma \in T$ with $\mathrm{lh}(\sigma) = k$, let $Q_\sigma \in \mathrm{Fin}_{G \setminus 1_G}$ be

$$Q_\sigma = \langle g_1^{\sigma(1)}, \dots, g_{k-1}^{\sigma(k-1)} \rangle.$$

For example, if $\sigma = \langle +1, -1, -1 \rangle$ then $Q_\sigma = \langle g_1^{-1}, g_2^{-1} \rangle$. The reason for not including g_0 in Q_σ is so that $1_G \notin Q_\sigma$. Q_σ represents σ's guess at a subset of a strict positive cone extending P. T_k denotes the nodes of T at the end of stage k.

Construction

Stage 0: Set $T_0 = \{\langle\rangle\}$ and $Q_{\langle\rangle} = \langle\rangle$.

Stage 1: Set $T_1 = \{\langle\rangle, \langle -1 \rangle\}$ and $Q_{\langle -1 \rangle} = \langle\rangle$. The purpose of this stage is to code 1_G into every path without coding it into any Q_σ

Stage s = k+1: For each $\sigma \in T_k$ check if equation 2 has been violated with witnesses below k:

$$\exists x, n, m \leq k \left(s(Q_\sigma, n, m, x) = 1 \wedge x \in P \right).$$

If equation 2 has been violated, then do not put either $\sigma * -1$ or $\sigma * +1$ into T_{k+1}. Otherwise, extend σ by putting both $\sigma * -1$ and $\sigma * +1$ into T_{k+1}.

End of Construction

We need to verify various properties of the construction. Let $[k] = \langle g_1, \dots, g_{k-1} \rangle$ and $[k]_\sigma = \langle g_1^{\sigma(1)}, \dots, g_{k-1}^{\sigma(k-1)} \rangle$.

Lemma 6.13 (RCA_0) *T is infinite.*

Proof: It suffices to show that for each k there is an element of T of length k. Fix $k > 0$. Since P satisfies equation 2, there is a $\sigma \in \mathrm{Fin}_{\pm 1}$ with $\mathrm{lh}(\sigma) = k$ and

$$\forall x, n, m \left(s([k]_\sigma, n, m, x) = 0 \vee x \notin P \right).$$

In particular, this condition holds if we bound the quantifiers by k. From the definition of T, it follows that for all $i \leq k$, $\langle \sigma(0), \dots, \sigma(i-1) \rangle \in T$ and hence $\sigma \in T$. □

By Weak König's Lemma there is a path h through T. Let

$$h[n] = \langle h(0), \dots, h(n-1) \rangle \in \mathrm{Fin}_{\pm 1}$$

$$\hat{h}[n] = [n]_{h[n]} = \langle g_1^{h(1)}, \dots, g_{n-1}^{h(n-1)} \rangle \in \mathrm{Fin}_G.$$

Lemma 6.14 (RCA_0) *For any $x \in G \setminus 1_G$, $h(x) = 1 \longleftrightarrow h(x^{-1}) = -1$.*

Proof: Suppose $h(x) = h(x^{-1}) = 1$, $x^{-1} = g_j$, and k is the maximum of j and x. By definition, $x, x^{-1} \in \hat{h}[k+1]$ and since $s(\hat{h}[k+1], 0, 0, x) = 1$ and $s(\hat{h}[k+1], 0, 0, x^{-1}) = 1$, it follows that $s(\hat{h}[k+1], 1, k, 1_G) = 1$. But, $1_G \in P$ and so by the construction of T, $h[k+1]$ has no extensions. This statement contradicts the choice of h as a path. The case for $h(x) = h(x^{-1}) = -1$ is similar. □

We are now in a position to define Q and verify that it is a full order extending P.

$$g_i \in Q \longleftrightarrow h(i) = -1$$

Q exists by Δ^0_1 comprehension. It contains 1_G because $\sigma(0) = -1$ for every $\sigma \in T$ and it is both full and pure by Lemma 6.14. To simplify the notation, we write $h(g_i)$, or $h(a)$ if $a = g_i$ instead of $h(i)$.

Claim 6.15 $P \subset Q$

For a contradiction suppose $g_i \in P \setminus 1_G$ and $h(g_i) = 1$. By definition, $g_i \in \hat{h}[i+1]$ and so $s(\hat{h}[i+1], 0, 0, g_i) = 1$. As in Lemma 6.14, $s(\hat{h}[i+1], 0, 0, g_i) = 1$ and $g_i \in P$ contradicts the fact that h is a path.

Claim 6.16 *Q is closed under multiplication.*

Suppose that $a, b \in Q$ and $ab \notin Q$. From Lemma 6.14 and the definition of Q, it follows that $h(a^{-1}) = 1$, $h(b^{-1}) = 1$ and $h(ab) = 1$. For a large enough k, we have $a^{-1}, b^{-1}, ab \in \hat{h}[k]$ and hence if m is the maximum of a^{-1}, b^{-1} and ab, then $s(\hat{h}[k], 2, m, 1_G) = 1$. Since $1_G \in P$, this statement contradicts the fact that h is a path.

Claim 6.17 *Q is normal.*

Suppose $q \in Q$, $g \in G$ and $gqg^{-1} \notin Q$. As above, $h(q^{-1}) = 1$, $h(gqg^{-1}) = 1$ and there is a k with $q^{-1}, gqg^{-1} \in \hat{h}[k]$. There is an m such that $s(\hat{h}[k], 2, m, 1_G)$ since the definition of s yields the normal semigroup. As above, $h[k]$ cannot be on a path. This claim completes the proof that Q is a full order extending P. □

Together Lemma 6.11 and Proposition 6.12 show (1) implies (2) in Theorem 6.10. The last step is to show that (3) implies (1) in Theorem 6.10.

Proposition 6.18 (RCA_0) *For an abelian group G, equation 3 holds if and only if G is torsion free.*

Proof:

Case 1: Equation 3 holds $\Longrightarrow$ G is torsion free.

For a contradiction assume that equation 3 holds and $a \neq 1_G$ is a torsion element of G.

Claim 6.19 *For all $k \geq 1$, $\exists p[s(\langle a\rangle, k-1, p, a^k) = 1]$.*

The claim is proved by Σ^0_1 induction on k. If $k = 1$, then $a \in \langle a\rangle$ implies $s(\langle a\rangle, 0, 0, a) = 1$. For $k+1$, the induction hypothesis states that there are p and

p' such that $s(\langle a\rangle, k-1, p, a^k) = 1$ and $s(\langle a\rangle, k-1, p', a) = 1$. If p'' is the largest of p, p' and a, then $s(\langle a\rangle, k, p'', a^{k+1}) = 1$, which proves the claim.

If a is a torsion element then for some k, $a^k = (a^{-1})^k = 1_G$. Equation 3 for the sequence $\langle a\rangle$ says that either

$$\forall n, m\ \big(s(\langle a\rangle, n, m, 1_G) = 0\big)$$

$$\text{or} \qquad \forall n, m\ \big(s(\langle a^{-1}\rangle, n, m, 1_G) = 0\big).$$

But the claim implies there is a p such that

$$s(\langle a\rangle, k-1, p, 1_G) = s(\langle a\rangle, k-1, p, a^k) = 1$$

$$\text{and} \qquad s(\langle a^{-1}\rangle, k-1, p, 1_G) = s(\langle a^{-1}\rangle, k-1, p, (a^{-1})^k) = 1.$$

Case 2: G is torsion free $\Longrightarrow$ equation 3 holds.

The first step is to show that for an abelian group G the normal semigroup generated by $A \in \mathrm{Fin}_{G\setminus 1_G}$ is the same as the semigroup generated by A. That is, if $A = \langle a_1, \dots a_n\rangle$ then any element of $S(A)$ can be written as $a_1^{k_1} \cdots a_n^{k_n}$ for some choice of $k_1, \dots k_n \in \mathbb{N}$ with at least one $k_i > 0$. Informally this statement is clear because any subset of an abelian group is normal. To prove this fact in RCA_0, we use the function $\mathrm{prod}(A, \sigma)$ from $\mathrm{Fin}_G \times \mathrm{Fin}_{\mathbb{N}}$ to G that takes $A = \langle a_1, \dots, a_n\rangle$ and $\sigma = \langle \sigma_1, \dots, \sigma_n\rangle$ to $a_1^{\sigma_1} \cdots a_n^{\sigma_n}$. Formally, $\mathrm{prod}(A, \sigma)$ is defined by recursion on $\mathrm{lh}(A)$. The next two lemmas follow by straightforward induction proofs.

Lemma 6.20 (RCA_0) *If* $A \in \mathrm{Fin}_{G\setminus 1_G}$, $\sigma, \tau \in \mathrm{Fin}_{\mathbb{N}}$ *and* $\mathrm{lh}(A) = \mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$ *then* $prod(A, \sigma) \cdot prod(A, \tau) = prod(A, \sigma + \tau)$ *where* $\sigma + \tau \in \mathrm{Fin}_{\mathbb{N}}$ *is defined by* $(\sigma + \tau)(k) = \sigma(k) + \tau(k)$.

Proof: This lemma is proved by induction on $\mathrm{lh}(A)$. □

Lemma 6.21 (RCA_0) *If* $A \in \mathrm{Fin}_{G\setminus 1_G}$, $n \in \mathbb{N}$, $x \in G$ *and* $\exists m\ [s(A, n, m, x) = 1]$ *then there is a* $\sigma \in \mathrm{Fin}_{\mathbb{N}}$ *with* $\mathrm{lh}(\sigma) = \mathrm{lh}(A)$ *and at least one* $k < \mathrm{lh}(\sigma)$ *with* $\sigma(k) > 0$ *such that* $x = prod(A, \sigma)$.

Proof: This lemma is proved by induction on n. □

We can now prove that if G is torsion free abelian then equation 3 holds by Π^0_1 induction on $\mathrm{lh}(A)$. For the base case, we need to show that for each $a \in G \setminus 1_G$ either

$$\forall n, m\ \big(s(\langle a\rangle, n, m, 1_G) = 0\big)$$

$$\text{or} \qquad \forall n, m\ \big(s(\langle a^{-1}\rangle, n, m, 1_G) = 0\big).$$

Suppose that neither equation holds and that $s(\langle a\rangle, n, m, 1_G) = 1$. By Lemma 6.21, $1_G = \mathrm{prod}(\langle a\rangle, \sigma)$ for some σ with $\sigma(0) > 0$, and so $1_G = a^{\sigma(0)}$ by the definition of prod. Therefore a is a torsion element which contradicts the fact that G is torsion free.

The induction step will be presented less formally to avoid an undue amount of notational baggage. Assume equation 3 holds for $\langle a_1, \dots, a_n\rangle$ and fails for

$\langle a_1, \dots, a_n, b\rangle$. Let $\langle \epsilon_1, \dots, \epsilon_n\rangle$ be the exponents in equation 3 for $\langle a_1, \dots, a_n\rangle$. By assumption, there are n_1, m_1, n_2, m_2 such that

$$s(\langle a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n}, b\rangle, n_1, m_1, 1_G) = 1$$

$$s(\langle a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n}, b^{-1}\rangle, n_2, m_2, 1_G) = 1.$$

By Lemma 6.21, there are $k_1, \dots, k_{n+1}$ and $l_1, \dots, l_{n+1}$ such that

$$a_1^{\epsilon_1 k_1} \cdots a_n^{\epsilon_n k_n} b^{k_{n+1}} = 1_G \qquad \text{and} \qquad a_1^{\epsilon_1 l_1} \cdots a_n^{\epsilon_n l_n} b^{-l_{n+1}} = 1_G$$

which gives $a_1^{\epsilon_1(k_1 l_{n+1} + k_{n+1} l_1)} \cdots a_n^{\epsilon_n(k_n l_{n+1} + k_{n+1} l_n)} = 1_G$. This equation contradicts equation 3 for $\langle a_1, \dots, a_n\rangle$. □

Proposition 6.18 shows that statement (3) in Theorem 6.10 implies that every torsion free abelian group is an O-group. By Theorem 3.3, this statement implies WKL_0. We have now completed the proof of Theorem 6.10.

7 Hölder's Theorem An early conjecture about ordered groups was that the number of full orders of a given O-group was always a power of 2. This conjecture included the statement that a group could not have a countable number of orders. Buttsworth [2] constructed a group with a countably infinite number of orders and Kargapolov et al. [9] showed the conjecture was false for groups with a finite number of orders. A more difficult problem is to classify all possible full orders for a given class of O-groups. One of the few classes for which this problem has been solved is the class of torsion free abelian groups of finite rank. These results can be found in several places, including Teh [18]. The key ingredient in each of these results about counting or classifying full orders is Hölder's Theorem. For a more in-depth discussion, see either [10] or Mura and Rhemtulla [13]. In this section, we will show that Hölder's Theorem is provable in RCA_0. It remains open whether the classification theorems mentioned above can also be proved in RCA_0, or whether they require somewhat stronger set existence axioms.

Definition 7.1 (RCA_0) If G is an f.o. group, then the *absolute value* of $x \in G$ is given by

$$|x| = \begin{cases} x & \text{if } \; x \in P(G) \\ x^{-1} & \text{if } \; x \notin P(G). \end{cases}$$

Definition 7.2 (RCA_0) If G is an f.o. group, then $a \in G$ is *Archimedean less than* $b \in G$, denoted $a \ll b$, if $|a^n| < |b|$ for all $n \in \mathbb{N}$. If there are $n, m \in \mathbb{N}$ such that $|a^n| \geq |b|$ and $|b^m| \geq |a|$, then a and b are *Archimedean equivalent*, denoted $a \approx b$. The notation $a \lessapprox b$ means $a \approx b \vee a \ll b$. G is an *Archimedean fully ordered group* if G is fully ordered and for all $a, b \neq 1_G$, $a \approx b$.

It is not hard to check that $\approx$ is an equivalence relation and that $\ll$ is transitive, antireflexive, and antisymmetric. The next lemma lists several other straightforward properties of $\approx$ and $\ll$. For proofs, see [6].

Lemma 7.3 (RCA_0) *If G is a f.o. group, then the following conditions hold for all $a, b, c \in G$.*

1. *Exactly one of the following holds:* $a \ll b$, $b \ll a$, *or* $a \approx b$.
2. $a \ll b$ *implies that* $xax^{-1} \ll xbx^{-1}$ *for all* $x \in G$.
3. $a \ll b$ *and* $a \approx c$ *imply that* $c \ll b$.
4. $a \ll b$ *and* $b \approx c$ *imply that* $a \ll c$.

Hölder's Theorem states that every f.o. Archimedean group can be embedded in the naturally ordered additive group of the reals. In this section we show that Hölder's Theorem is provable in RCA_0. Recall that real numbers in second-order arithmetic are given by functions from $\mathbb{N}$ to $\mathbb{Q}$ with appropriate convergence properties, so the first step towards proving Hölder's theorem is to decide what a subgroup of the real numbers should be in second-order arithmetic.

Definition 7.4 (RCA_0) A *real number* is a function $f : \mathbb{N} \to \mathbb{Q}$, usually denoted by $\langle q_k \mid k \in \mathbb{N} \rangle$, such that for all k and i, $|q_k - q_{k+i}| \leq 2^{-k}$. Two real numbers $x = \langle q_k \mid k \in \mathbb{N} \rangle$ and $y = \langle q'_k \mid k \in \mathbb{N} \rangle$ are *equal* if for all k, $|q_k - q'_k| \leq 2^{-k+1}$. The *sum* $x + y$ is the real number $\langle q_{k+1} + q'_{k+1} \mid k \in \mathbb{N} \rangle$.

Definition 7.5 (RCA_0) A *nontrivial subgroup of the additive real numbers* $(\mathbb{R}, +_{\mathbb{R}})$ is a sequence of reals $A = \langle r_n \mid n \in \mathbb{N} \rangle$ together with a function $+_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and a distinguished number $i \in \mathbb{N}$ such that

1. $r_i = 0_{\mathbb{R}}$
2. $n +_A m = p$ if and only if $r_n +_{\mathbb{R}} r_m = r_p$
3. $(\mathbb{N}, +_A)$ satisfies the group axioms with i as the identity element.

Combining these definitions, we see that A is a double indexed sequence of rationals $A = \langle q_{n,m} \mid n, m \in \mathbb{N} \rangle$ where $r_n = \langle q_{n,m} \mid m \in \mathbb{N} \rangle$.

Let $(G, \leq)$ be an Archimedean fully ordered group. Because G must be abelian (see Lemma 7.6), we use additive notation for G. The idea of the proof of Hölder's theorem is to pick an element $a \in P(G)$, $a \neq 1_A$, and define a subgroup of $(\mathbb{R}, +)$ using a to approximate the other elements of G. For now, assume that 2^n divides a in G for all n. That is, assume there exists $c \in G$ such that $2^n c = a$. To construct the real corresponding to $g \neq 1_G$, we first find $p_0 \in \mathbb{Z}$ such that $p_0\, a \leq g < (p_0 + 1)\, a$. Such a p_0 exists because G is Archimedean. Next we find p_1 such that $p_1(a/2) \leq g < (p_1 + 1)(a/2)$ and continue to find p_i such that $p_i(a/2^i) \leq g < (p_i + 1)(a/2^i)$. The real corresponding to g will be $\langle p_i/2^i | i \in \mathbb{N} \rangle$. Because the elements $a/2^i$ may not exist, we achieve the same effect by choosing p_i such that $p_i\, a \leq 2^i\, g < (p_i + 1)\, a$. The standard proofs of Hölder's Theorem are similar but use Dedekind cuts instead of Cauchy sequences. The motivation for using Cauchy sequences here is they are simpler to use in the context of second-order arithmetic.

Lemma 7.6 (RCA_0) *Every Archimedean fully ordered group is abelian.*

Proof: The standard proof goes through in RCA_0. For the details, see [10]. $\square$

Theorem 7.7 (Hölder's Theorem) (RCA_0) *Every nontrivial Archimedean f.o. group is order isomorphic to a nontrivial subgroup of the naturally ordered additive group* $(\mathbb{R}, +)$.

Proof: Let $(G, \leq)$ be an Archimedean f.o. group and $g_0, g_1, \ldots$ be an enumeration of G with no repetitions such that $g_0 = 1_G$ and $g_1 \in P(G)$. We construct a subgroup A of $(\mathbb{R}, +)$ by constructing $r_n = \langle q_{n,m} \mid m \in \mathbb{N} \rangle$ uniformly in n from g_n and g_1. For simplicity of notation, let $a = g_1$. The first two elements of A are $r_0 = \langle 0 \mid m \in \mathbb{N} \rangle$ and $r_1 = \langle 1 \mid m \in \mathbb{N} \rangle$. To construct r_n for $n > 1$, define $p_{n,m} \in \mathbb{N}$ and $q_{n,m} \in \mathbb{Q}$ by

$$p_{n,m}\, a \leq 2^m g_n < (p_{n,m} + 1)\, a \qquad \text{and} \qquad q_{n,m} = \frac{p_{n,m}}{2^m}.$$

Because G is Archimedean, the $p_{n,m}$ exist and are uniquely determined. Define $r_n = \langle q_{n,m} \mid m \in \mathbb{N} \rangle$. It remains to show that $A = \langle r_n \mid n \in \mathbb{N} \rangle$ is a subgroup of $(\mathbb{R}, +)$ and that the map from G to A sending g_n to r_n is an order preserving isomorphism.

Claim 7.8 *Each r_n is a real number.*

To prove this claim we must verify that $|\, q_{n,m} - q_{n,m+k} \,| \leq 2^{-m}$ for all m and k. It follows from $p_{n,m}\, a \leq 2^m g_n < (p_{n,m} + 1)\, a$ that $2 p_{n,m}\, a \leq 2^{m+1} g_n < (2 p_{m,n} + 2)\, a$. Hence either $p_{n,m+1} = 2 p_{n,m}$ (and $q_{n,m+1} = q_{n,m}$) or $p_{n,m+1} = 2 p_{n,m} + 1$ (and $q_{n,m+1} = q_{n,m} + 1/2^{m+1}$). Thus,

$$|\, q_{n,m} - q_{n,m+k} \,| \;\leq\; \sum_{i=1}^{i=k} \frac{1}{2^{m+i}} \;<\; \frac{1}{2^m}.$$

Claim 7.9 *If $g_n + g_m = g_k$ then $r_n + r_m = r_k$.*

As above, this claim reduces to checking the convergence rates. By definition, $r_n + r_m = \langle q_{n,i+1} + q_{m,i+1} \mid i \in \mathbb{N} \rangle$. To prove $r_n + r_m = r_k$ we need to show that $|\, q_{n,i+1} + q_{m,i+1} - q_{k,i} \,| < 2^{-i+1}$ for every $i \in \mathbb{N}$. Adding the equations defining $p_{n,i+1}$ and $p_{m,i+1}$ yields

$$(p_{n,i+1} + p_{m,i+1})\, a \leq 2^{i+1} g_k < (p_{n,i+1} + p_{m,i+1} + 2)\, a.$$

Thus $p_{k,i+1}$ is either $p_{n,i+1} + p_{m,i+1}$ or $p_{n,i+1} + p_{m,i+1} + 1$. In either case, $q_{k,i+1} - q_{n,i+1} - q_{m,i+1} \leq 2^{-i-1}$ and we have the following inequalities.

$$\begin{aligned} |\, q_{n,i+1} + q_{m,i+1} - q_{k,i} \,| &\leq |\, q_{n,i+1} + q_{m,i+1} - q_{k,i+1} \,| + |\, q_{k,i+1} - q_{k,i} \,| \\ &\leq 2^{-i-1} + 2^{-i} \\ &< 2^{-i+1} \end{aligned}$$

The map sending g_n to r_n is onto by definition and the following claim shows it is 1-1.

Claim 7.10 *If $n \neq m$, then $r_n \neq r_m$.*

To establish $r_n \neq r_m$, we need to find an i such that $|\, q_{n,i} - q_{m,i} \,| > 2^{-i+1}$ or equivalently, $|\, p_{n,i} - p_{m,i} \,| > 2$. Because $n \neq m$ implies $g_n \neq g_m$ assume that $g_n < g_m$. There are four cases to consider.

Case 1: $g_n < 1_G \leq g_m$

Because G is Archimedean there is an i such that $2^i g_n < -3a < 1_G \leq 2^i g_m$. It follows that $p_{n,i} < -3$ and $p_{m,i} \geq 0$. Hence $|\, p_{n,i} - p_{m,i} \,| \geq 3$.

Case 2: $g_n = 1_G < g_m$

There is an i such that $1_G < 3a < 2^i g_m$. It follows that $p_{n,i} = 0$ while $p_{m,i} \geq 3$.

Case 3: $1_G < g_n < g_m$

Since $1_G < g_m - g_n$, there is an i which yields the following equations:

$$1_G < a < 2^i (g_m - g_n) = 2^i g_m - 2^i g_n$$

$$2^i g_n < a + 2^i g_n < 2^i g_m$$

$$2^{i+2} g_n < 4a + 2^{i+2} g_n < 2^{i+2} g_m.$$

There is a k such that $ka \leq 2^{i+2} g_n < (k+1)a$. Combining these equations yields $ka \leq 2^{i+1} g_n < (k+4)a \leq 4a + 2^{i+2} g_n < 2^{i+2} g_m$. It follows that $p_{n,i+2} = k$ and $p_{m,i+2} \geq k+4$.

Case 4: $g_n < g_m < 1_G$

In this case, $1_G < g_m - g_n$ and so the previous argument works. This case completes the proof of the claim and shows the map is 1-1.

The claims show A is a subgroup of $(\mathbb{R}, +)$ and is isomorphic to G. Finally, to show that $g_n < g_m$ implies $r_n < r_m$, notice that if $g_n < g_m$ then $q_{n,i} \leq q_{m,i}$ for every i. Thus, $r_n \leq r_m$. But, since $g_n \neq g_m$ implies $r_n \neq r_m$, we have $r_n < r_m$. □

8 Strong divisible closures The algebraic closure of a field, the real closure of an ordered field and the divisible closure of an abelian group are three naturally occurring notions of closure in algebra. Every abelian group has a unique divisible closure up to isomorphism and is isomorphic to a subgroup of its divisible closure. Similar results hold for the other notions of closure. From the perspective of reverse mathematics, it is useful to separate three aspects of these closure operations and examine each individually. That is, we ask the following questions about the divisible closure. How hard is it to prove that each abelian group has a divisible closure? How hard is it to prove that this divisible closure is unique up to isomorphism? How hard is it to prove that each abelian group has a divisible closure for which it is isomorphic to a subgroup of that divisible closure? Similar questions can be asked about the other notions of closure, and the questions can easily be reworded to reflect concerns about computable mathematics instead of about reverse mathematics.

Friedman et al. proved several results about these closures, including the following theorem which illustrates that the answers to these questions need not be the same.

Theorem 8.1 (Friedman, Simpson, and Smith) (RCA_0)

1. *Every field has an algebraic closure.*
2. WKL_0 *is equivalent to the statement that every field has a unique algebraic closure.*
3. ACA_0 *is equivalent to the statement that every field has an algebraic closure such that the original field is isomorphic to a subfield of the algebraic closure.*

Friedman, Simpson and Smith [4] give the following definitions.

Definition 8.2 (RCA_0) Let D be an abelian group. D is *divisible* if for all $d \in D$ and all $n \geq 1$ there exists a $c \in D$ such that $nc = d$.

Definition 8.3 (RCA_0) Let A be an abelian group. A *divisible closure* of A is a divisible group D together with a monomorphism $h : A \to D$ such that for all $d \in D, d \neq 1_D$, there exists $n \in \mathbb{N}$ with $nd = h(a)$ for some $a \in A, a \neq 1_A$.

Smith [16] proved that every computable abelian group has a computable divisible closure and that this divisible closure is unique if and only if there is a uniform algorithm which for each prime p decides if an arbitrary element of the original group is divisible by p. Friedman, Simpson, and Smith [4] proved the following theorem.

Theorem 8.4 (Friedman, Simpson, and Smith) (RCA_0)

1. *Every abelian group has a divisible closure.*
2. ACA_0 *is equivalent to the statement that every abelian group has a unique divisible closure.*

In this section, we extend these results to strong divisible closures. Downey and Kurtz [3] considered another possible extension. They proved that every computably fully ordered computable abelian group has a computably unique divisible closure. An examination of their proof shows that RCA_0 suffices to prove the uniqueness of the divisible closure for fully ordered abelian groups.

Theorem 8.5 (Downey and Kurtz) (RCA_0) *Every fully ordered abelian group G has a f.o. divisible closure $h : G \to D$ such that h is order preserving. This divisible closure is unique up to order preserving isomorphism.*

Definition 8.6 (RCA_0) Let A be an abelian group. A *strong divisible closure* of A is a divisible closure $h : A \to D$ such that h is an isomorphism of A onto a subgroup of D. If A is a f.o. group, D is fully ordered and h is order preserving, then we call $h : A \to D$ an *f.o. strong divisible closure*.

Because RCA_0 suffices to prove the uniqueness of the divisible closure for f.o. abelian groups, but ACA_0 is required to prove the uniqueness for abelian groups in general, it is reasonable to hope that proving the existence of a strong divisible closure would be easier for f.o. abelian groups than for abelian groups. The next theorem shows this is not the case.

Theorem 8.7 (RCA_0) *The following are equivalent:*

1. ACA_0
2. *Every abelian group has a strong divisible closure.*
3. *Every fully ordered Archimedean group has an f.o. strong divisible closure.*

The idea of proving (3) implies (1) is fairly simple. Let p_k be an enumeration of the primes in increasing order. Given a 1-1 function f, let G be the subgroup of $\mathbb{Q}$ generated by 1 and p^{-k} for each k in the range of f. This group has an Archimedean full order and the range of f can be recovered from the strong divisible closure by

$$\text{range}(f) = \{k \mid \frac{h(1)}{p_k} \in h(G)\}.$$

Lemma 8.8 (RCA_0) *Let p_k enumerate the primes in increasing order. If $k \in \mathbb{Z}$, $j \in \mathbb{N}$ and $\forall i \leq j$ $(0 \leq m_i < p_i)$, then $\sum_{i \leq j} m_i / p_i = k$ implies that $k = 0$ and $m_i = 0$ for all $i \leq j$.*

Proof: Let $\hat{p}$ be the product $p_0 \cdots p_j$ and let $\hat{p}_l$ be $\hat{p}/p_l$. If we multiply the sum by $\hat{p}$ we obtain

$$\sum_{i \leq j} m_i \hat{p}_i = k\hat{p}.$$

This equation must hold modulo p_l for all $l \leq j$. However, if $u \neq l$, then $(m_u \hat{p}_u = 0) \bmod p_l$ because p_l divides $\hat{p}_u$. Therefore, we have

$$\left(\sum_{i \leq j} m_i \hat{p}_i = m_l \hat{p}_l \right) \bmod p_l.$$

Also, $(k\hat{p} = 0) \bmod p_l$ and so we have $\big(m_l \hat{p}_l = 0\big) \bmod p_l$. It follows that p_l divides m_l. Because $0 \leq m_l < p_l$, m_l must be 0. $\square$

Using Lemma 8.8, we can give a proof of Theorem 8.7.

Proof:

Case 1: (1) $\Longrightarrow$ (2)

ACA_0 suffices to prove that the image of the embedding h exists.

Case 2: (2) $\Longrightarrow$ (3)

The following full order on D makes h order preserving.

$$\begin{aligned} P(D) &= \{d \in D \mid \exists n > 0 \exists g \in P(G)(nd = h(g))\} \\ &= \{d \in D \mid \forall n > 0 \forall g \in G(nd = h(g) \to g \in P(G))\} \end{aligned}$$

Because $P(D)$ has a Δ^0_1 definition, RCA_0 suffices to prove it exists and to verify that it is a full order on D.

Case 3: (3) $\Longrightarrow$ (1)

Let f be a 1-1 function and let p_k be an enumeration of the primes in increasing order. It suffices to show that the range of f exists. Let G be the group with generators a and x_i for $i \in \mathbb{N}$, and relations $p_{f(i)} x_i = a$. The intuition is that G is isomorphic to a subgroup of $\mathbb{Q}$ with $a \mapsto 1$ and $x_i \mapsto p^{-1}_{f(i)}$. In RCA_0 we represent the elements of G by finite sums $ka + \sum_{i \leq j} m_i x_i$ where $k \in \mathbb{Z}$, $0 \leq m_i < p_{f(i)}$ and $m_j \neq 0$. Using the relation equations, any element of G can be reduced to one of these finite sums. We need to show that no two of these finite sums represent the same element of G.

Claim 8.9 *If $ka + \sum_{i \leq j} m_i x_i = \tilde{k}a + \sum_{i \leq \tilde{j}} \tilde{m}_i x_i$ then $k = \tilde{k}$, $j = \tilde{j}$ and for all $i \leq j$, $(m_i = \tilde{m}_i)$.*

First notice that 1_G has a unique representation as the finite sum $0a$. Indeed, if $ka + \sum_{i \leq j} m_i x_i = 1_G = 0 \cdot a$ then using the relations, we obtain $\sum_{i \leq j} m_i (a/p_{f(i)}) = -ka$. Because G is torsion free, this equation implies $\sum_{i \leq j} m_i / p_{f(i)} = -k$. By

Lemma 8.8, $k = 0$ and $m_i = 0$. To show that j must equal $\tilde{j}$ in the claim, suppose that $j < \tilde{j}$ and

$$ka + \sum_{i \le j} m_i x_i = \tilde{k}a + \sum_{i \le \tilde{j}} \tilde{m}_i x_i.$$

Reducing $(k - \tilde{k})\, a + \sum_{i \le j}(m_i - \tilde{m}_i)x_i + \sum_{j<i<\tilde{j}} \tilde{m}_i x_i$, we obtain

$$k'a + \sum_{i \le j'} m'_i x_i + \sum_{j<i<\tilde{j}} \tilde{m}_i x_i = 1_G$$

for some k' and m'_i. Because 1_G has a unique normal form, $\tilde{m}_{\tilde{j}} = 0$ which gives the desired contradiction and shows $j = \tilde{j}$.

A similar argument shows that $m_i = \tilde{m}_i$ for all $i \le j$. Suppose there is an $i \le j$ such that $m_i \neq \tilde{m}_i$. Since we can always subtract off equal terms, we can assume without loss of generality that $m_j < \tilde{m}_j$. If

$$(\tilde{k} - k)a + \sum_{i \le j-1} (\tilde{m}_i - m_i)x_i$$

reduces to the normal form $k'a + \sum_{i \le j'} m'_i x_i$ for some j' and m'_i, then $(\tilde{k} - k)a + \sum_{i \le j}(\tilde{m}_i - m_i)x_i$ reduces to the normal form

$$k'a + \sum_{i \le j'} m'_i x_i + (\tilde{m}_j - m_j)x_j = 0_G.$$

By the uniqueness of the normal form for 0_G, we have that $\tilde{m}_j - m_j = 0$, which is a contradiction. Therefore, $\tilde{m}_i = m_i$ for all $i \le j$. Our equation reduces to $ka = \tilde{k}a$ which implies that $k = \tilde{k}$.

Claim 8.10 *G is fully orderable.*

Define the positive cone $P(G)$ by

$$ka + \sum_{i \le j} m_i x_i \in P(G) \longleftrightarrow k + \sum_{i \le j} \frac{m_i}{p_{f(i)}} \ge 0.$$

$P(G)$ is normal because G is abelian. To verify the other properties, notice that if two finite sums $ka + \sum_{i \le j} m_i x_i$ and $\tilde{k}a + \sum_{i \le \tilde{j}} \tilde{m}_i x_i$, not necessarily in normal form, are equivalent under the group relations then

$$k + \sum_{i \le j} \frac{m_i}{p_{f(i)}} = \tilde{k} + \sum_{i \le \tilde{j}} \frac{\tilde{m}_i}{p_{f(i)}}.$$

This property is proved by induction on the number of applications of relation equations it takes to transform one sum into the other. This property immediately yields that $P(G)$ is a pure, full semigroup with identity. Furthermore, it shows that G is Archimedean under this order because $\mathbb{Q}$ is Archimedean.

Applying condition (3) from the theorem, we have a divisible closure $h : G \to D$ and the image $h(G)$ exists.

$$X = \{k \,|\, \frac{h(a)}{p_k} \in h(G)\}$$

$$\begin{aligned}\frac{h(a)}{p_k} \in h(G) &\longleftrightarrow p_k \text{ divides } a \text{ in } G\\ &\longleftrightarrow \exists i\,(p_k x_i = a)\\ &\longleftrightarrow \exists i\,(f(i) = k)\end{aligned}$$

Thus X is the range of f. □

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