Notre Dame Journal of Formal Logic
Volume 39, Number 4, Fall 1998

A Note on the Modal and Temporal Logics for N-Dimensional Spacetime

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Abstract We generalize an observation made by Goldblatt in "Diodorean modality in Minkowski spacetime" by proving that each n-dimensional integral spacetime frame equipped with Robb's irreflexive 'after' relation determines a unique temporal logic. Our main result is that, unlike n-dimensional spacetime where, as Goldblatt has shown, the Diodorean modal logic is the same for each frame $(\mathbb{R}^n, \leq)$, in the case of n-dimensional *integral* spacetime, the frame $(\mathbb{Z}^n, \leq)$ determines a unique Diodorean modal logic.

1 Introduction N-dimensional spacetime is the frame $(\mathbb{R}^n, \preceq)$ where $\mathbb{R}^n (n \geq 2)$ is the set of all n-tuples of real numbers and $\preceq$ is a binary relation. The relation $\preceq$ for $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$ where $x, y \in \mathbb{R}^n$ is defined by

$$x \preceq y \quad \text{iff} \quad \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq (y_n - x_n)^2 \quad \text{and} \quad x_n \leq y_n$$

Intuitively, $x \preceq y$ means that a luminal signal can be sent from x to y and hence that y is in the 'causal future' of x. The relation $\preceq$ determines the *future light cone* of x which is just $\{y \in \mathbb{R}^n : x \preceq y\}$. Note that $\mathbb{R}^4$ is Minkowski spacetime, the mathematical model of spacetime which underlies Einstein's Special Theory of Relativity (see Taylor and Wheeler [5] for an accessible explanation of the theory). Evidently, for any $n \geq 2$, $(\mathbb{R}^n, \preceq)$ is isomorphic to $(\mathbb{R}^n, \leq)$ where the isomorphism is just the 45-degree rotation and for $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$:

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \quad \text{iff} \quad x_i \leq y_i \quad \text{for each } i.$$

The main result of Goldblatt [1] is that the Diodorean modal logic of the frames $(\mathbb{R}^n, \leq)$ for $n \geq 2$ is the well-known system **S4.2**. Goldblatt also considers frames that have an irreflexive relation α where

$$x \alpha y \quad \text{iff} \quad x \leq y \quad \text{and} \quad x \neq y.$$

Received July 9, 1998; revised May 3, 1999

The relation α is the 'after' relation axiomatized by Robb in [4]. In [1], the problem of axiomatizing temporal logics for the frames $(\mathbb{R}^n, \leq)$ with $n \geq 2$ is left open as are the corresponding problems for n-dimensional *integral* spacetime, in particular for the frames $(\mathbb{Z}^n, \leq)$ and $(\mathbb{Z}^n, \alpha)$ with $n \geq 2$ (where $\mathbb{Z}^n$ is the set of all n-tuples of integers).

An interesting feature of the frames $(\mathbb{R}^n, \alpha)$ pointed out by Goldblatt is that the temporal logic of $(\mathbb{R}^2, \alpha)$ differs from that of the frame $(\mathbb{R}^3, \alpha)$. We shall show how Goldblatt's observation can be generalized to prove that for each frame $(\mathbb{Z}^n, \alpha)$ with $n \geq 2$, the frame $(\mathbb{Z}^n, \alpha)$ determines a unique temporal logic. An easy corollary of this result is that for each frame $(\mathbb{R}^n, \alpha)$ with $n \geq 2$, the frame $(\mathbb{R}^n, \alpha)$ determines a unique temporal logic. A more surprising result is that, unlike the case involving n-dimensional spacetime where the Diodorean modal logic is the same for each frame $(\mathbb{R}^n, \leq)$, in the case of n-dimensional *integral* spacetime (for every $n \geq 2$) the frame $(\mathbb{Z}^n, \leq)$ determines a unique Diodorean modal logic.

2 Preliminaries A pair (W, R) is a *frame* just in case W is a nonempty set and R is a binary relation on W. The language $\mathcal{L}$ consists of a countable set of atomic sentences or *atoms* p_i where $i = 0, 1, 2, \ldots$ along with the Boolean connectives $\neg$ and $\wedge$ and the modal operator $\Diamond$. The set of $\mathcal{L}$-formulas is constructed in the usual way from the atoms using the Boolean connectives $\neg$ and $\wedge$ and the modal operator $\Diamond$. We write p, q, r, and so on, for arbitrary formulas. Introduction of the abbreviations $\top$ (constant true), $\bot$ (constant false), $\vee$, and $\rightarrow$ is done in the usual way. Additionally, we introduce the abbreviation $\Box$, where $\Box p$ abbreviates $\neg\Diamond\neg p$.

The language $\mathcal{L}^*$ is just like $\mathcal{L}$ except $\mathcal{L}^*$ contains the temporal operators F and P instead of the modal operator $\Diamond$. The set of $\mathcal{L}^*$-formulas is likewise constructed in the usual way. Note that we follow custom and abbreviate $\neg F \neg p$ as Gp and $\neg P \neg p$ as Hp.

A structure or *model* (with respect to $\mathcal{L}$ or $\mathcal{L}^*$) is a triple $M = (W, R, V)$ where (W, R) is a frame and V is a function assigning each p_i a subset of W. We generally refer to such a function as a *valuation*. Truth in a model is defined recursively in the usual way (consult van Benthem [6] and Hughes and Cresswell [3] for the details).

A formula p is *$\mathcal{L}$-valid* (or *$\mathcal{L}^*$-valid*) in a frame $\mathcal{F}$ if and only if p is true in $(\mathcal{F}, V)$ at w for all $w \in \mathcal{F}$. We shall follow the practice of using the expression 'valid', relying on the context to make the meaning of the expression clear. For a frame (W, R) we write $ML(W, R)(TL(W, R))$ to denote the modal logic (temporal logic) of (W, R), that is, the set of formulas in the language $\mathcal{L}(\mathcal{L}^*)$ that are valid on (W, R). We assume the reader is familiar with the notion of a p-morphism (e.g., see [6] or Goldblatt [2] for discussion).

3 Logics for frames with Robb's 'after' relation We begin with a result for n-dimensional integral spacetime with respect to temporal formulas.

Theorem 3.1 *For any $n \geq 2$, $TL(\mathbb{Z}^{n+1}, \alpha) \neq TL(\mathbb{Z}^m, \alpha)$ where $2 \leq m \leq n$.*

Proof: In order to prove the theorem, it suffices to show that for any n-dimensional frame with $n \geq 2$, there is a formula φ valid on the n-dimensional frame and every m-dimensional frame where $2 \leq m \leq n$ such that φ is invalid on the $(n+1)$-dimensional frame. Fix an n-dimensional frame $(\mathbb{Z}^n, \alpha)$, with $n \geq 2$. For ease of exposition, we adopt the following abbreviations:

$$\begin{array}{rcllr}
Q_1 & = & (p_1 \rightarrow (\neg p_2 \wedge G\neg p_2)\wedge & \cdots & \wedge(\neg p_n \wedge G\neg p_n) \wedge (\neg p_{n+1} \wedge G\neg p_{n+1})) \\
Q_2 & = & (p_2 \rightarrow (\neg p_1 \wedge G\neg p_1)\wedge & \cdots & \wedge(\neg p_n \wedge G\neg p_n) \wedge (\neg p_{n+1} \wedge G\neg p_{n+1})) \\
 & & & \vdots & \\
Q_{n+1} & = & (p_{n+1} \rightarrow (\neg p_1 \wedge G\neg p_1) & \wedge & (\neg p_2 \wedge G\neg p_2) \wedge \cdots \wedge (\neg p_n \wedge G\neg p_n))
\end{array}$$

The formula represented by the following schema must be valid on $(\mathbb{Z}^n, \alpha)$:

$$\text{Rob}^n\colon Fp_1 \wedge \cdots \wedge Fp_n \wedge Fp_{n+1} \wedge GQ_1 \wedge \cdots \wedge GQ_{n+1} \rightarrow \vee_{1\leq i<j\leq n+1} F(Fp_i \wedge Fp_j)$$

In order to prove that the formula denoted by the above schema is valid on $(\mathbb{Z}^n, \alpha)$, we assume the antecedent of Rob^n holds at some point $w \in \mathbb{Z}^n$. Without loss of generality, we may suppose that $w = (0, \ldots, 0)$ where w contains n 0s. We thus have

(1) There is a point c_1 such that $w \neq c_1$ where $w\alpha c_1$ and p_1 and $(\neg p_2 \wedge G\neg p_2) \wedge \cdots \wedge (\neg p_n \wedge G\neg p_n) \wedge (\neg p_{n+1} \wedge G\neg p_{n+1})$ is true at c_1

$$\vdots$$

(n+1) There is a point c_{n+1} such that $w \neq c_{n+1}$ where $w\alpha c_{n+1}$ and p_{n+1} and $(\neg p_1 \wedge G\neg p_1) \wedge (\neg p_2 \wedge G\neg p_2) \wedge \cdots \wedge (\neg p_n \wedge G\neg p_n)$ is true at c_{n+1}

where

$$\begin{array}{l}
c_1 = (a_1^1, \ldots, a_1^n) \\
c_2 = (a_2^1, \ldots, a_2^n) \\
c_3 = (a_3^1, \ldots, a_3^n) \\
\qquad\vdots \\
c_{n+1} = (a_{n+1}^1, \ldots, a_{n+1}^n).
\end{array}$$

Given (1) – $(n+1)$, it is easy to see that these points must be mutually noncomparable. We call the points $a_1^1, a_2^1, \ldots, a_{n+1}^1$ the *first column of coordinates* of $c_1, c_2, \ldots, c_n, c_{n+1}$ and extend this notion in the obvious way (i.e., $a_1^2, a_2^2, \ldots, a_{n+1}^2$ is the second column of coordinates, etc.). Now suppose there are no points c_j, c_k such that the m^{th} coordinate (where $1 \leq m \leq n+1$) of both c_j and c_k is greater than the m^{th} coordinate of some point c_g. It follows from our supposition that in each column of coordinates, either every point c_i has the same value, or there are $x, y \in \mathbb{Z}$ such that $x < y$ and one point c_i has the value y in the column while every other point c_h has the value x in that column. Note that in order for two points c_h, c_i to be distinct and non-comparable they must differ on at least two coordinates; in particular, it must be the case that c_h has a lower value than c_i on one coordinate and a higher value on some other coordinate. In the situation at hand, we know that at least n of the coordinates in each column must be identical. Thus, for any column k there must be $(n+1) - k$ points that are identical on all coordinates in the first k columns. So for the $(n-1)$st column there are two such points c_h,c_i; but then c_h, c_i differ only in the last column (the nth). It follows that either c_h sees c_i or c_i sees c_h, a contradiction. □

We have now established that there are points c_j, c_k such that the mth coordinate (where $1 \leq m \leq n+1$) of both c_j and c_k is greater than the mth coordinate of some point c_g. Where $c_j = a_j^1, \ldots, a_j^n$ and $c_k = (a_k^1, \ldots, a_k^n)$, let $d =$

$(\min(a_j^1, a_k^1), \ldots, \min(a_j^n, a_k^n))$. Since the mth coordinate of d must be greater than 0, $d \neq w$. But then, since $w \,\alpha\, d$, it follows that $F(Fp_j \wedge Fp_k)$ is true at w and hence that Rob^n is valid on $(\mathbb{Z}^n, \alpha)$.

We note that, given the fact that there is a p-morphism from $\mathbb{Z}^{n+1}$ onto $\mathbb{Z}^n$ for $n \geq 2$ (by deleting the first coordinate), it follows by the p-morphism theorem that Rob^n must be valid on each of $(\mathbb{Z}^2, \alpha), \ldots, (\mathbb{Z}^n, \alpha)$. Therefore, all that remains is to show that Rob^n fails on the $(n+1) - D$ frame. Let p_1 be false everywhere except at $b_1 = (1, 0, \ldots, 0)$ (where b_1 has $n+1$ coordinates), p_2 be false everywhere except $b_2 = (0, 1, \ldots, 0), \ldots$, and p_{n+1} be false everywhere except $b_{n+1} = (0, 0, \ldots, 1)$. We leave it to the reader to check that Rob^n is false at $b_0 = (0, 0, \ldots, 0)$ on this valuation.

The proof of Theorem 3.1 also establishes the following corollary.

Corollary 3.2 *For any $n \geq 2$, $ML(\mathbb{R}^{n+1}, \alpha) \neq ML(\mathbb{R}^m, \alpha)$ where $2 \leq m \leq n$.*

4 N-dimensional integral spacetime

We now establish a result concerning Diodorean modal logics for n-dimensional integral spacetime.

Theorem 4.1 *For any $n \geq 2$, $ML(\mathbb{Z}^n, \leq) \neq ML(\mathbb{Z}^m, \leq)$ where $m \geq 2$ and $m \neq n$.*

Proof: We note that the formula Zip^2 is valid on the $2 - D$ frame but fails on every $n - D$ frame for $n > 2$.

$$\begin{aligned}
\text{Zip}^2 : &\neg p \wedge \neg q \wedge \neg r \wedge \\
&\Diamond(\Box p \wedge \neg q \wedge \neg r) \wedge \\
&\Diamond(\Box q \wedge \neg p \wedge \neg r) \wedge \\
&\Diamond(\Box r \wedge \neg p \wedge \neg q) \rightarrow \\
&\Diamond[\Diamond(\Box p \wedge \neg q \wedge \neg r) \wedge \Diamond(\Box q \wedge \neg p \wedge \neg r) \wedge \neg\Diamond(\Box r \wedge \neg p \wedge \neg q)] \vee \\
&\Diamond[\Diamond(\Box p \wedge \neg q \wedge \neg r) \wedge \Diamond(\Box r \wedge \neg p \wedge \neg q) \wedge \neg\Diamond(\Box q \wedge \neg p \wedge \neg r)].
\end{aligned}$$

In order to prove the theorem, it suffices to show that for any $n - D$ frame with $n > 2$ there is a formula valid on the $n - D$ frame that is invalid on every $(n+m) - D$ frame such that $m \geq 1$. Fix an $n - D$ frame $(\mathbb{Z}^n, \leq)$, with $n > 2$. For sake of clarity, we adopt the following abbreviations (with respect to the atoms $p_1, p_2, \ldots, p_n, p_{n+1}$).

$$\begin{aligned}
\text{ALL} &= (p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge p_{n+1}). \\
\text{NONE} &= (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}). \\
*p_i &= (p_i \wedge \wedge_{j \neq i} \neg p_j).
\end{aligned}$$

The formula represented by the following schema must be valid on $(\mathbb{Z}^n, \leq)$.

$$\begin{aligned}
\text{Zip}^n : \text{NONE} \wedge \Box[\text{NONE} \rightarrow \Box(\text{ALL} \vee \text{NONE} \vee *p_1 \vee \cdots \vee *p_{n+1})] \wedge \\
\Box[\text{NONE} \rightarrow \Diamond(*p_1) \wedge \cdots \wedge \Diamond(*p_{n+1})] \wedge \\
\Box[p_1 \rightarrow \Box(*p_1 \vee \text{ ALL})] \wedge \\
\vdots \\
\Box[p_{n+1} \rightarrow \Box(*p_{n+1} \vee \text{ALL})] \rightarrow \vee_{i \neq j} \Diamond[\Diamond * p_i \wedge \Diamond * p_j \wedge \ \neg \wedge \Diamond_{k \neq i, k \neq j} * p_k)]
\end{aligned}$$

In order to prove that the formula denoted by the above schema is valid on $(\mathbb{Z}^n, \leq)$, we may assume without loss of generality that the antecedent of Zip^n holds and the consequent fails at $w = (0, \ldots, 0)$ where w contains n 0s. We thus have

(1) There is a point c_1 such that $w \neq c_1$ where $w \leq c_1$ and $\Box p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}$ is true at c_1

$\vdots$

(n+1) There is a point c_{n+1} such that $w \neq c_{n+1}$ where $w \leq c_{n+1}$ and $\Box p_{n+1} \wedge \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ is true at c_{n+1}

where

$$
\begin{aligned}
c_1 &= (a_1^1, \ldots, a_1^n) \\
c_2 &= (a_2^1, \ldots, a_2^n) \\
c_3 &= (a_3^1, \ldots, a_3^n) \\
&\vdots \\
c_{n+1} &= (a_{n+1}^1, \ldots, a_{n+1}^n).
\end{aligned}
$$

Now consider the points

$$
\begin{gathered}
(\min(a_1^1, a_2^1), \ldots, \min(a_1^n, a_2^n)) \\
\vdots \\
(\min(a_{n+1}^1, a_1^1), \ldots, \min(a_1^n, a_{n+1}^n)) \\
\vdots \\
(\min(a_2^1, a_3^1), \ldots, \min(a_2^n, a_3^n)) \\
\vdots \\
(\min(a_2^1, a_{n+1}^1), \ldots, \min(a_2^n, a_{n+1}^n)) \\
\vdots \\
(\min(a_n^1, a_{n+1}^1), \ldots, \min(a_n^n, a_{n+1}^n))
\end{gathered}
$$

The number of these points is determined by the equation:

$$k = \sum_{1 \leq i \leq n} i$$

Let $d_1, \ldots, d_k$ denote these points. Suppose each of $d_1, \ldots, d_k$ is equal to w. As before, we call the points $a_1^1, a_2^1, \ldots, a_{n+1}^1$ the *first column of coordinates* of $c_1, c_2, \ldots, c_n, c_{n+1}$ and extend this notion in the obvious way. We observe that there are $n+1$ coordinates in each column of coordinates.

We know that at least n of the coordinates in each column must be equal to 0; for suppose there are two coordinates α, β that do not equal 0. By assumption, $\min(\alpha, \beta) = 0$ which is impossible, given that neither of α, β equals 0.

From the fact that at least n of the coordinates in each column must be equal to 0, it follows that $n * n$ of the $(n+1) * n$ total coordinates for $c_1, \ldots, c_{n+1}$ must be equal to 0. But since none of $c_1, \ldots, c_{n+1}$ equals w there must be $n+1$ coordinates which are not equal to 0. This is impossible, however, given that (the number of total coordinates) $= (n+1) * n = (n * n) + n$ and this is less than the sum of the number of 0 coordinates and the number of non-0 coordinates.

We have now shown that at least one of $d_1, \ldots, d_k$ is not equal to w. Let d_i be such a point, where $d_i = (\min(a_j^1, a_k^1), \ldots, \min(a_j^n, a_k^n))$, that is, $d_i = \min(c_j, c_k)$. So d_i sees c_j and c_k where c_j and c_k are noncomparable. Thus $\Diamond * p_j \wedge \Diamond * p_k$ with $j \neq k$ is true at d_i. Since, by hypothesis, the consequent of Zip^n is false at w, we know $\Diamond * p_1 \wedge \cdots \wedge \Diamond * p_{n+1}$ is true at d_i

We know c_j and c_k must differ on two coordinates where one coordinate is greater in c_j and the other is greater in c_k. Without loss of generality, we may assume they differ on the first two coordinates and suppose c_j is higher on the first coordinate and c_k is higher on the second. We now prove there must be a point x seen by d_i such that x sees points y and z where x and y differ by exactly one on the mth coordinate and are identical on all others, and x and z differ by one on precisely one coordinate but are identical on the mth and all others. In addition, we want NONE to be true at x, and for each of y and z either $*p_i$ or ALL is true. We call a point x of the sort we have described *suitable*. Assume d_i sees no suitable point. Let

$$\begin{aligned} c_j &= (a_1, a_2, a_3, \ldots, a_n), \\ c_k &= (b_1, b_2, b_3, \ldots, b_n), \\ d_i &= (b_1, a_2, c_3, \ldots, c_n) \\ &\quad \text{and} \\ c &= \max(c_j, c_k) = (a_1, b_2, d_3, \ldots, d_n). \end{aligned}$$

Consider the array of points determined by the following method. The base of the last column in the array is the point c_k. Let b_h be the highest coordinate of c_k where $b_h \neq d_h$ (so $b_h < d_h$). Where $g = d_h - b_h$ the next g points in the column are

$$\begin{array}{c} (b_1, b_2, \ldots, b_{h+g}, \ldots, b_n) \\ \vdots \\ (b_1, b_2, \ldots, b_{h+1}, \ldots, b_n) \\ (b_1, b_2, \ldots, b_h, \ldots, b_n) \end{array}$$

We follow this procedure for the next highest coordinate $b_f (f < h)$ of c_k where $b_f \neq d_f$ and continue until every such coordinate has been handled. Where $y = b_2 - a_2$, the next y columns (proceeding to the left) are just like the last column except for their second coordinates.

$$\begin{array}{ccccc} (a_1, a_2, d_3, \ldots, d_n) & \ldots & (a_1, b_{2-1}, d_3, \ldots, d_n) & & (a_1, b_2, d_3, \ldots, d_n) \\ \vdots & & \vdots & & \vdots \\ (b_1, a_2, \ldots, b_{h+g}, \ldots, b_n) & \ldots & (b_1, b_{2-1}, \ldots, b_{h+g}, \ldots, b_n) & \ldots & (b_1, b_2, \ldots, b_{h+g}, \ldots, b_n) \\ \vdots & & \vdots & & \vdots \\ (b_1, a_2, \ldots, b_{h+1}, \ldots, b_n) & \ldots & (b_1, b_{2-1}, \ldots, b_{h+1}, \ldots, b_n) & \ldots & (b_1, b_2, \ldots, b_{h+1}, \ldots, b_n) \\ (b_1, a_2, \ldots, b_h, \ldots, b_n) & \ldots & (b_1, b_{2-1}, \ldots, b_h, \ldots, b_n) & \ldots & (b_1, b_2, \ldots, b_h, \ldots, b_n) \end{array}$$

Now consider the highest coordinate b_h in c_k where the hth coordinate c_h in d_i is such that $c_h \neq b_h$. The next column in the array is just like the last except each point in the column contains b_{h-1} in the hth coordinate. We continue as before until we have a column with c_h as the hth coordinate in every point. We repeat this procedure for each pair of nonidentical coordinates in d_i, c_k in descending order until we have the first column with d_i at the base and c_j at the top.

As an illustration of the method, we consider a simple example. Suppose $n = 4$, $c_j = (3, 0, 5, 5)$, and $c_k = (0, 3, 3, 7)$. Then $d_i = (0, 0, 3, 5)$ and $c = (3, 3, 5, 7)$. The array in this case is depicted below.

(3, 0, 5, 5)	(3, 0, 5, 6)	(3, 0, 5, 7)	(3, 1, 5, 7)	(3, 2, 5, 7)	(3, 3, 5, 7)
(2, 0, 5, 5)	(2, 0, 5, 6)	(2, 0, 5, 7)	(2, 1, 5, 7)	(2, 2, 5, 7)	(2, 3, 5, 7)
(1, 0, 5, 5)	(1, 0, 5, 6)	(1, 0, 5, 7)	(1, 1, 5, 7)	(1, 2, 5, 7)	(1, 3, 5, 7)
(0, 0, 5, 5)	(0, 0, 5, 6)	(0, 0, 5, 7)	(0, 1, 5, 7)	(0, 2, 5, 7)	(0, 3, 5, 7)
(0, 0, 4, 5)	(0, 0, 4, 6)	(0, 0, 4, 7)	(0, 1, 4, 7)	(0, 2, 4, 7)	(0, 3, 4, 7)
(0, 0, 3, 5)	(0, 0, 3, 6)	(0, 0, 3, 7)	(0, 1, 3, 7)	(0, 2, 3, 7)	(0, 3, 3, 7)

Observe that the array of points determined by this method is such that for any point x that is not in the top row or in the last column where NONE is true at x, x is suitable if for both the point y immediately above x and the point z immediately to the right of x either $*p_i$ or ALL is true.

Returning now to the general case, assume that no points in the array are suitable. The basic situation is depicted below.

$$\begin{array}{cccccc}
c_j & . & . & . & t & c \\
. & . & . & . & s & u \\
. & . & . & . & . & . \\
d_i & . & . & . & . & c_k
\end{array}$$

Given that c_j sees t and c_k sees u it follows that either $*p_j$ or ALL is true at t and either $*p_k$ or ALL is true at u. But then, since s sees both t and u, and s is not suitable, we know that either $*p_j$ or ALL or $*p_k$ is true at s. It is not hard to see that continuing to argue in this fashion forces either $*p_j$ or ALL or $*p_k$ to be true at d_i. This is impossible since then $\Box p_j$ or $\Box p_k$ is true at d_i, but d_i sees c_j and c_k. Thus, we have established that one of the points in the array must be suitable.

Let the points x, y, z be as described earlier, where x is suitable. Since NONE is true at x, $\Diamond * p_1 \wedge \cdots \wedge \Diamond * p_{n+1}$ must true at x. Thus, x must see at least $(n+1) - 2$ points $q_1, \ldots, q_{n-1}$ noncomparable with y, z. For example, if $*p_1$ is true at both y and z then there must be $n + 1$ points noncomparable with y and z; if $*p_1$ is true at y and $*p_3$ is true at z, then there are $(n + 1) - 2$, that is, $n - 1$, points noncomparable with y and z, and so on.

Given that either $*p_i$ or ALL is true at each of y, z it follows that each of $q_1, \ldots, q_{n-1}$ must have identical fth and gth coordinates where x and y differ by one on the fth coordinate and x and z differ by one on the gth coordinate. This is because if, say, q_i differs from q_{i+1} on the fth coordinate, then one of q_i, q_{i+1} must be higher on the fth coordinate (note that both must have fth coordinates greater than or equal to the fth coordinate of x). Without loss of generality we may suppose q_i is higher on the fth coordinate. But then either (1) $q_i = y$ or (2) y sees q_i, since y differs from x only by being one higher on the fth coordinate. Both (1) and (2) are impossible given that q_i and y are noncomparable.

We know that there are either $n-1, n,$ or $n+1$ points noncomparable with y and z. Now assume that for any $s \geq m$ where $3 < m \leq n$ there is a suitable point x (and the points y, z in virtue of which x is suitable) where x, y, z are seen by w and

(i) there are $s, s-1,$ or $s+1$ points $q_1, \ldots, q_j$ seen by x where $q_1, \ldots, q_j$ are noncomparable with y and z and one another and

(ii) where one of $*p_1, \ldots, *p_{n+1}$ is true at each of $q_1, \ldots, q_j$ and

(iii) $q_1, \ldots, q_j$ and x, y, z are identical on $(n-s)+1$ coordinates.

So, for example, in case of n (as we have shown) we have a suitable x where there are either $n, n-1,$ or $n+1$ points seen by x and noncomparable with y and z and one another where one of

$$(p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}), \ldots, (p_{n+1} \wedge \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

is true at each of these noncomparable points and where these points and x, y, z are identical on $((n-n)+1) * 2 = 2$ coordinates. We now prove that for $m-1$ there is a suitable x with x, y, z seen by w and

(i*) there are $m-2, m-1,$ or m points seen by x and noncomparable with $y, z,$ and one another and

(ii*) one of $*p_1, \ldots, *p_{n+1}$ or ALL is true at each of $q_1, \ldots, q_j$ and

(iii*) $q_1, \ldots, q_j$ and x, y, z are identical on $(n-(m-1))+1$ coordinates.

By hypothesis, we have x_m, y_m, z_m with x_m suitable where

(1) w sees x_m, y_m, z_m and

(2) x_m sees either $m-1, m,$ or $m+1$ points $q_1, \ldots, q_j$ seen by x_m and

(3) $q_1, \ldots, q_j$ are noncomparable with y_m, z_m, and one another and

(4) one of $*p_1, \ldots, *p_{n+1}$ is true at each of $q_1, \ldots, q_j$ and

(5) $q_1, \ldots, q_j$ and x_m, y_m, z_m are identical on $(n-m)+1$ coordinates.

Since $m > 3$, we know we can choose two of these points that are noncomparable with y_m and z_m and one another. We may designate these points c_j and c_k and argue as before that there must be some suitable x with x, y, z seen by w with $m-2, m-1,$ or m points $q_1, \ldots, q_j$ seen by x and noncomparable with y and z and one another where one of $*p_1, \ldots, *p_{n+1}$ is true at each of $q_1, \ldots, q_j$, and $q_1, \ldots, q_j$ and x, y, z are identical on $(n-(m-1))+1$ coordinates.

From the above, it follows that there must be a suitable point x such that x, y, z are seen by w where there are two, three, or four points seen by x and noncomparable with y and z and one another. Moreover, one of $*p_1, \ldots, *p_{n+1}$ must be true at each of these noncomparable points, and these points and x, y, z must be identical on all but two coordinates. Note that by the definition of suitability, x and y and x and z differ only by one on one coordinate. Let s and t be two of the points (or the only two points) noncomparable with y and z and one another.

We know that $x, y, z, s,$ and t agree on all but two coordinates. Assume that (listing only the two relevant coordinates) $x = (a, b), y = (a+1, b), z = (a, b+1)$, $s = (g, h)$, and $t = (i, j)$. Since s and t cannot be seen by either y or z it follows that $s = (a, b)$ and $t = (a, b)$. This is impossible, since s and t are noncomparable.

All that remains is to show that Zip^n fails on every $(n+m) - D$ frame $m \geq 1$. For any $m - D$ frame $m > (n+1)$ let p_1 be true everywhere except at $b_0 = (0, 0, \ldots, 0)$ (where b_0 has m coordinates) and $b_2 = (0, 1, \ldots, 0), \ldots, b_{n+1} = (0, 0, \ldots, 1, 0, \ldots, 0)$ (where b_{n+1} contains $m - (n+1)$ 0s after the 1), p_2 be true everywhere except b_0, b_1, and $b_3, \ldots, b_{n+1}, \ldots$, and p_{n+1} be true everywhere except $b_0, b_1, \ldots, b_n$. We leave it to the reader to check that Zip^n fails on this valuation. □

5 Remarks The contrast between the case of the frames $(\mathbb{R}^n, \leq)$ which are indiscernible from one another with respect to the validity of formulas in $\mathcal{L}$ and the case of the frames $(\mathbb{Z}^n, \leq)$, which are all distinguishable from one another in this sense, is quite striking. Since the logics of the frames $(\mathbb{Z}^n, \alpha)$ with $n \geq 2$ (and the frames $(\mathbb{R}^n, \alpha)$ where $n \geq 2$) are all distinct, it is natural to consider the case involving the other 'standard' irreflexive relation R which is defined by

$$xRy \quad \text{iff} \quad \sum_{i=1}^{n-1}(y_i - x_i)^2 < (y_n - x_n)^2 \text{ and } x_n < y_n.$$

Given our inability to discover dimension-dependent formulas for frames equipped with the relation R, we are tempted to conjecture that the frames $(\mathbb{R}^n, R)$ with $n \geq 2$ (and the frames $(\mathbb{Z}^n, R)$ where $n \geq 2$) are indiscernible with respect to formulas in both $\mathcal{L}$ and $\mathcal{L}^*$. In any event, it seems worthwhile to point out that the question remains open. (Byrd has discovered that the frames $(\mathbb{Z}^n, R)$ where $n \geq 2$ are discernible in the above sense.)

Acknowledgments I am grateful to Mike Byrd for a number of helpful comments on this paper.

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