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TOPOLOGICAL CHARACTERISTICS OF SOLUTION SETS FOR FRACTIONAL EVOLUTION EQUATIONS AND APPLICATIONS TO CONTROL SYSTEMS

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Abstract. This paper explores an abstract Riemann–Liouville fractional evolution model with a weighted delay initial condition. We develop the resolvent technique, a generalization of semigroup method, to formulate an appropriate notion of mild solutions to this abstract system and present the topological characteristics of the corresponding solution set in a weighted space. Furthermore, in view of the topological characteristics, we analyze the approximate controllability of the abstract system without Lipschitz assumption. We end up addressing an infinite dimensional fractional delay diffusion control system and a finite dimensional fractional ordinary differential control system by utilizing our theoretical findings.

1. Introduction

Fractional differential systems have in recent years been active research topics because of their broad applicability in describing many physical problems with memory features and genetic properties. Many fruitful findings on fractional equations have been reported in the literature (see [23], [29], [33], [36]).

Du and Wang [8] demonstrated that Riemann–Liouville fractional systems are more suitable to describe some practical applications in viscoelastic materials

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than equations with Caputo derivatives. Furthermore, authors in [31] pointed out that when the Caputo-type derivative is employed in the definition of "state space description" in practical applications, the system memory no longer exists at time $s = 0$, but appears at $s > 0$, namely the system is not physically consistent. Thereby, it is necessary to focus on Riemann–Liouville fractional equations.

On the other hand, Agarwal et al. [1] handled a Riemann–Liouville evolution system with a delay initial condition satisfying $\phi(0) = 0$. However, it is well-known that the solutions of Riemann–Liouville fractional systems admit singularity at zero, except that $\phi(0) = 0$. If $\phi(0) \neq 0$, the solutions of the delay evolution system in [1] may not be well-defined. Hence, if $\phi(0) \neq 0$, it is necessary to introduce a new suitable initial condition.

To investigate fractional evolution models, the initial step is a question how to formulate a suitable notion of mild solutions. The concept of a mild solution to a Caputo fractional evolution system was first presented by applying the idea of probability density functions and C_0-semigroups (see [9]). In [30], by introducing a solution operator, Prüss analyzed a Volterra equation and investigated the well-posedness of this equation. It should be pointed out that, with the help of the solution operator approach in [30], the definition of mild solutions to a Caputo type fractional system in [11] was also formulated.

Emphasis here is that the solution operator method, a generalization of the semigroup technique, is convenient and efficient in investigating fractional systems. However, the solution operator method from [30] and [11] can not be used to solve Riemann–Liouville fractional systems since these systems admit singularity at zero. New approaches must be proposed. In [22], by introducing a β-order fractional resolvent, the notion of solutions to a Riemann–Liouville fractional homogeneous problem was constructed. The results in [22] were later generalized to an inhomogeneous linear system (see [10]) and a fractional semilinear system (see [39], [40]) by the resolvent method. In this paper, we will develop the resolvent technique to treat the fractional delay diffusion control system (2.1).

When the uniqueness of solutions cannot be ensured, a natural question is to conduct some investigations on the topological characteristics of the solution set. Moreover, the topological characteristics are powerful tools to study periodic problems (see [3], [7]). Thereby, an increasing research interest has been devoted to analyzing the topological structure problems (refer to [2], [6], [15], [20], [34]). But it is a pity that few related results have been proposed for Riemann–Liouville evolution systems and this fact is another motivation of the current research.

In addition, the approximate controllability problems of evolution systems have drawn tremendous attention because of their extensive applications in many fields, such as engineering practice, control theory, electrical circuit and technical science, etc. Under the Lipschitz assumption of the nonlinear terms, many investigators analyzed the problems by using the range condition proposed by Naito [27], such as [21] and [25]. For example, Kumar and Sukavanam [21] recently investigated the approximate controllability of Caputo type fractional delay evolution systems in Hilbert spaces with the help of the range condition and the Lipschitz assumption of the nonlinear term f. It should be mentioned here that the authors in above literature only verified the condition (H_c) (see Section 4).

Motivated by the above two aspects, we are interested in studying the topological structure of solution set to a Riemann–Liouville fractional delay semilinear system and displaying its application to approximate controllability problems by utilizing the resolvent theory and the condition (H_c), when the Lipschitz continuity of f is lacked.

The novelties of the current article are highlighted as follows:

(1) Considering that the solutions of Riemann–Liouville fractional delay evolution system (2.2) have singularity at zero, we introduce a weighted delay initial condition for this system. Furthermore, by the resolvent approach, we propose an appropriate notion of solutions in a weighted space.

(2) We combine the topological structure of solution set and control problem organically. Moreover, employing the resolvent and the topological characteristics, we overcome the difficulty of the lack of Lipschitz assumption without imposing any additional conditions, when addressing the approximate controllability problem. In addition, we apply the theoretical findings to the fractional delay diffusion control system (2.1) and a finite dimensional fractional ordinary differential control system.

The arrangement of the present article is as follows. We come up with our problem and collect some preliminaries required in Section 2. Section 3 is devoted to formulating an appropriate definition of mild solutions and displaying the topological characteristics of the corresponding solution set by employing the resolvent technique. Section 4 contains the approximate controllability results of the abstract model. We end the article with addressing an infinite dimensional fractional delay diffusion control system and a finite dimensional fractional ordinary differential control system in Section 5.

2. Model statement and preliminaries

This paper copes with the approximate controllability of an abstract fractional delay evolution model which can describe an infinite dimensional fractional diffusion system by employing a resolvent approach and the topological characteristics of the solution set.

2.1. A fractional delay diffusion system. We tackle the following fractional delay diffusion control system with a β-order Riemann–Liouville fractional derivative:

$$\begin{cases} D^\beta y(t,x) = \Delta y(t,x) + f(t, \widetilde{y}_t(x)) + (Bu)(t,x), & t \in J',\ x \in \Omega, \\ y(t,x) = 0, & t \in J',\ x \in \partial\Omega, \\ \widetilde{y}_0(t,x) = \phi(t,x), & t \in [-r,0],\ x \in \Omega. \end{cases} \tag{2.1}$$

Here $\beta \in (0,1)$, $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a C^2-boundary $\partial\Omega$, $J' = (0,b]$, B is a bounded linear map. Moreover, $\widetilde{y}(t,x) = \Gamma(\beta)t^{1-\beta}y(t,x)$ for $t \in J := [0,b]$, $\widetilde{y}(0,x) = \lim_{t\to 0^+} \widetilde{y}(t,x)$,

$$\widetilde{y}_t(\theta,x) = \widetilde{y}(t+\theta,x) = \begin{cases} \Gamma(\beta)(t+\theta)^{1-\beta}y(t+\theta,x), & t+\theta \in [0,b], \\ \phi(t+\theta,x), & t+\theta \in [-r,0], \end{cases}$$

for $t \in J$ and $\theta \in [-r,0]$, ϕ is continuous and f is a nonlinear function without Lipschitz condition.

If $\beta = 1$, system (2.1) is reduced to the classical parabolic system, which can serve as models for describing various physical phenomena in many fields, such as heat conduction, diffusion and seepage.

For $0 < \beta < 1$, by employing Fourier-Laplace techniques, Hilfer [14] investigated the existence of the solutions to the following Riemann–Liouville fractional diffusion equation:

$$\begin{cases} D^\beta y(t,x) = C_\beta \Delta y(t,x), & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ J_t^{1-\beta} y(0,x) = y_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where C_β is a diffusion constant.

To analyze the fractional delay diffusion control system (2.1), we first address the following abstract control system with a β-order Riemann–Liouville fractional derivative:

$$\begin{cases} D^\beta y(t) = Ay(t) + f(t,\widetilde{y}_t) + (Bu)(t), & t \in J' = (0,b], \\ \widetilde{y}_0(t) = \phi(t), & t \in [-r,0] \end{cases} \tag{2.2}$$

in a Banach space V, where A generates a β-order resolvent $\{R_\beta(t)\}_{t>0}$, ϕ is continuous on $[-r,0]$, $\widetilde{y}(t) = \Gamma(\beta)t^{1-\beta}y(t)$ for $t \in J$, $\widetilde{y}(0) = \lim_{t\to 0^+} \widetilde{y}(t)$, and

$$\widetilde{y}_t(\theta) = \widetilde{y}(t+\theta) = \begin{cases} \Gamma(\beta)(t+\theta)^{1-\beta}y(t+\theta), & t+\theta \in [0,b], \\ \phi(t+\theta), & t+\theta \in [-r,0], \end{cases}$$

for $t \in J$ and $\theta \in [-r,0]$.

For our subsequent investigations, we collect here some preliminaries. Let X, Y and Z be three metric spaces and let $(V, \|\cdot\|)$ and $(U, \|\cdot\|_U)$ be two Banach spaces. The symbol $\mathscr{L}(U;V)$ denotes the set consisting of linear and bounded operators from U to V and $\mathscr{L}(V)$ represents $\mathscr{L}(V;V)$. Moreover, the notation $C([c,e];V)$ means the collection of V-valued functions which are continuous and normed by $\|y\|_{[c,e]} = \sup\limits_{s\in[c,e]} \|y(s)\|$ for $y \in C([c,e];V)$. Let

$$C_{1-\beta}(J;V) = \big\{y \in C(J';V) : \widetilde{y}(\,\cdot\,) = \Gamma(\beta)(\,\cdot\,)^{1-\beta} y(\,\cdot\,),\ \widetilde{y} \in C(J;V)\big\} \tag{2.3}$$

be normed by $\|y\|_{C_{1-\beta}} = \sup\limits_{s\in J} \|\Gamma(\beta)s^{1-\beta}y(s)\|$, where $\widetilde{y}(0) = \lim\limits_{s\to 0^+} \widetilde{y}(s)$. Then $C_{1-\beta}(J;V)$ is a Banach space. Furthermore, we put $P(X) = \{D \subseteq X : D \neq \emptyset\}$ and employ the symbol $*$ to denote the convolution of functions, i.e.

$$(g*h)(s) = \int_0^s g(s-\tau)h(\tau)\,d\tau, \quad s > 0.$$

We begin with some definitions and facts from multi-valued analysis.

Definition 2.1 ([16]). A multi-valued operator $h\colon X \to P(Y)$ is said to be quasicompact if $h(D)$ is precompact for any compact set $D \subseteq X$.

Lemma 2.2 ([19]). *Let $h\colon X \to P(Y)$ be quasicompact and closed. Then h is upper semi-continuous* (*briefly, u.s.c.*).

Definition 2.3 ([4]). X is called:

(a) an absolute retract (shortly, $X \in \mathrm{AR}$) if for any metric space Z and any closed set $D \subset Z$, every continuous operator $\mu\colon D \to X$ admits a continuous extension $\overline{\mu}\colon Z \to X$,

(b) an absolute neighbourhood retract ($X \in \mathrm{ANR}$, in short) provided that for any metric space Z and any closed set $D \subset Z$, every continuous operator $\mu\colon D \to X$ possesses a continuous extension $\overline{\mu}\colon \mathcal{U} \to X$, where $\mathcal{U}$ is a neighbourhood of D.

Definition 2.4 ([7]). $D \in P(X)$ is called a contractible set provided that there exists a continuous homotopy $\nu\colon D \times [0,1] \to D$ and a point $y_0 \in D$ to ensure that for any $y \in D$, $\nu(y,1) = y_0$ and $\nu(y,0) = y$.

Definition 2.5 ([17]). $D \in P(X)$ is called an R_δ-set provided that $D = \bigcap\limits_{m=1}^{\infty} D_m$, where $\{D_m\}$ is a decreasing sequence of nonempty, compact and contractible sets.

Definition 2.6 ([7]). $h\colon X \to P(Y)$ is called an R_δ-map provided that h is u.s.c. and $h(y)$ is an R_δ-set for any $y \in X$.

Lemma 2.7 ([5]). *Let V be a Banach space and $\omega\colon X \to V$ a continuous operator. Suppose that ω is proper* (*for each compact set $K \subseteq V$, $\omega^{-1}(K)$ is compact*). *If, in addition, there exists a sequence $\{\omega_m\}$ with $\omega_m\colon X \to V$ satisfying*

(a) *ω_m is proper and $\lim_{m\to\infty}\omega_m=\omega$, uniformly on X;*
(b) *for all $z\in\mathcal{U}(z_0)$, the equation $\omega_m(x)=z$ possesses exactly one solution, where z_0 is a given point and $\mathcal{U}(z_0)$ is a neighbourhood of z_0 in V,*

then $\omega^{-1}(z_0)$ is an R_δ-set.

Lemma 2.8 ([13]). *Suppose that $\varphi\colon X\to P(X)$ can be factorized by*

$$\varphi=\varphi_m\circ\ldots\circ\varphi_1.$$

Here $\varphi_i\colon X_{i-1}\to P(X_i)$, $i=1,\ldots,m$, are R_δ-maps, $X_0=X_m=X\in\mathrm{AR}$ and $X_i\in\mathrm{ANR}$, $i=1,\ldots,m-1$. If, in addition, there is a compact set K to ensure that $\varphi(X)\subseteq K\subseteq X$, then $\mathrm{Fix}(\varphi)$, the fixed point set of φ, is nonempty.

Additionally, reviewing the concept of resolvent, we propose some properties.

Definition 2.9 ([22]). Let $\beta\in(0,1)$. By a β-order fractional resolvent (resolvent, for short), we understand a family $\{R_\beta(s)\}_{s>0}\subseteq\mathscr{L}(V)$ satisfying

(a) $R_\beta(\,\cdot\,)y\in C((0,\infty);V)$ and $\lim_{s\to0^+}\Gamma(\beta)s^{1-\beta}R_\beta(s)y=y$ for any $y\in V$;
(b) $R_\beta(\tau)R_\beta(s)=R_\beta(s)R_\beta(\tau)$ for $s,\tau>0$;
(c) $R_\beta(\tau)J_s^\beta R_\beta(s)-J_\tau^\beta R_\beta(\tau)R_\beta(s)=g_\beta(\tau)J_s^\beta R_\beta(s)-g_\beta(s)J_\tau^\beta R_\beta(\tau)$, for $s,\tau>0$,

where $g_\beta(s)=s^{\beta-1}/\Gamma(\beta)$, $s>0$ and the symbol J_s^β means the β-order fractional integral operator.

The generator $A\colon D(A)\subseteq V\to V$ of the resolvent $\{R_\beta(s)\}_{s>0}$ is

$$Ay=\Gamma(2\beta)\lim_{s\to0^+}\frac{s^{1-\beta}R_\beta(s)y-y/\Gamma(\beta)}{s^\beta},$$

where

$$D(A)=\bigg\{y\in V:\lim_{s\to0^+}\frac{s^{1-\beta}R_\beta(s)y-y/\Gamma(\beta)}{s^\beta}\ \text{exists}\bigg\}.$$

Lemma 2.10. *Assume that A generates a resolvent $\{R_\beta(s)\}_{s>0}$. Then*

$$M:=\sup_{s\in J}\big\|\Gamma(\beta)s^{1-\beta}R_\beta(s)\big\|<\infty, \tag{2.4}$$

where $\Gamma(\beta)s^{1-\beta}R_\beta(s)|_{s=0}:=\lim_{s\to0^+}\Gamma(\beta)s^{1-\beta}R_\beta(s)$.

Proof. For $y\in V$, due to Definition 2.9 and

$$\Gamma(\beta)s^{1-\beta}R_\beta(s)y\big|_{s=0}=\lim_{s\to0^+}\big(\Gamma(\beta)s^{1-\beta}R_\beta(s)y\big),$$

we have $\sup_{s\in J}\big\|\Gamma(\beta)s^{1-\beta}R_\beta(s)y\big\|<\infty$. Thus, in view of $\Gamma(\beta)s^{1-\beta}R_\beta(s)\in\mathscr{L}(V)$ and the uniform boundedness principle, we derive $M<\infty$. $\square$

Lemma 2.11 ([22]). *Let $\{R_\beta(s)\}_{s>0}$ be a resolvent with generator A. Then*

(a) *$R_\beta(s)D(A)\subseteq D(A)$, $s>0$;*

(b) $\overline{D(A)} = V$;

(c) $R_\beta(s)y = g_\beta(s)y + A(g_\beta * R_\beta)(s)y$ *for any* $y \in V$ *and* $s > 0$.

Hereafter, we always suppose that

(HA) $\{s^{1-\beta}R_\beta(s)\}_{s>0}$ is compact and there exists a constant $C > 0$ to ensure that $\|d\left(s^{1-\beta}R_\beta(s)\right)/ds\| \le C/s$, $s \in J'$.

REMARK 2.12. This assumption comes from the practical applications. Furthermore, based on Lemma 3.8 in [11], if $\{s^{1-\beta}R_\beta(s)\}_{s>0}$ is a compact and analytic operator family of analyticity type (ω_0, θ_0), (HA) is automatically satisfied.

LEMMA 2.13. *Suppose that condition* (HA) *holds. Then*

$$\lim_{\tau\to 0} \left\|(s+\tau)^{1-\beta}R_\beta(s+\tau) - s^{1-\beta}R_\beta(s)\right\| = 0, \quad s \in J'.$$

PROOF. Let $s \in J'$, $y \in V$ with $\|y\| \le 1$, and $|\tau| < s$ with $s + \tau \in J'$. It follows, upon employing (HA), that

$$\begin{aligned} &\left\|(s+\tau)^{1-\beta}R_\beta(s+\tau)y - s^{1-\beta}R_\beta(s)y\right\| \\ &\qquad \le \left\| \int_s^{s+\tau} \frac{d(\sigma^{1-\beta}R_\beta(\sigma))}{d\sigma} y\, d\sigma \right\| \le C\|y\| \left| \int_s^{s+\tau} \sigma^{-1}\, d\sigma \right| \\ &\qquad\qquad \le C|\ln(s+\tau) - \ln s| \to 0, \quad \tau \to 0. \end{aligned}$$

This indicates that $\lim_{\tau\to 0} \left\|(s+\tau)^{1-\beta}R_\beta(s+\tau) - s^{1-\beta}R_\beta(s)\right\| = 0$, for $s \in J'$. □

Due to Lemma 2.13, we can derive the following result which is similar to the semigroup property by following the procedure in Lemmas 3.4 and 3.5 of [11].

LEMMA 2.14. *Let assumption* (HA) *be fulfilled. Then, for* $s \in J'$,

(a) $\lim_{\tau\to 0^+} \left\|\Gamma(\beta)(s+\tau)^{1-\beta}R_\beta(s+\tau) - \left(\Gamma(\beta)\tau^{1-\beta}R_\beta(\tau)\right)\left(\Gamma(\beta)s^{1-\beta}R_\beta(s)\right)\right\| = 0$;

(b) $\lim_{\tau\to 0^+} \left\|\Gamma(\beta)s^{1-\beta}R_\beta(s) - \left(\Gamma(\beta)\tau^{1-\beta}R_\beta(\tau)\right)\left(\Gamma(\beta)(s-\tau)^{1-\beta}R_\beta(s-\tau)\right)\right\| = 0$.

3. Topological characteristics of solution set

The target of this section is to formulate an appropriate concept of mild solutions to system (2.2) and display the topological characteristics of the corresponding solution set by utilizing the resolvent approach. To this end, we assume that

(Hf) $f\colon J \times C([-r,0];V) \to V$ satisfies:

(a) $\upsilon \to f(s,\upsilon)$ is continuous for almost every $s \in J$;

(b) $s \to f(s,\upsilon)$ is measurable for each $\upsilon \in C([-r,0];V)$;

(c) for all $\upsilon \in C([-r,0];V)$ and almost every $s \in J$,

$$\|f(s,\upsilon)\| \le a(s) + k\|\upsilon\|_{[-r,0]}$$

with $k \in [0, \beta/(Mb))$ and $a \in L^p(J; \mathbb{R}_+)$, $p > 1/\beta$, where M is defined by formula (2.4).

(HB) $B\colon L^p(J;U) \to L^p(J;V)$ is a linear and bounded operator.

We start with the following important result which is helpful in proposing a suitable concept of mild solutions to (2.2).

Lemma 3.1. *Suppose that condition* (HA) *is fulfilled and* $p > 1/\beta$. *Then*

(a) $R_\beta * g \in C(J;V)$, *where* $g \in L^p(J;V)$;

(b) *the map* $\Lambda\colon L^p(J;V) \to C_{1-\beta}(J;V)$, *defined by* $(\Lambda g)(\,\cdot\,) = (R_\beta * g)(\,\cdot\,)$, *is compact.*

Proof. Following the proofs of Lemmas 3.1 and 4.2 in [40], we can easily check the results of this lemma. □

To formulate the definition of mild solutions to (2.2) by the resolvent technique, for convenience, we first deal with the following system:

$$
\begin{cases} D^\beta y(t) = Ay(t) + f(t), & t \in J' = (0,b], \\ \widetilde{y}_0(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{3.1}
$$

where $f \in L^p(J;V)$, ϕ is continuous on $[-r,0]$.

Let y satisfy (3.1). Due to $\lim_{t\to 0^+} \widetilde{y}(t) = \widetilde{y}(0) = \phi(0)$ and the dominated convergence theorem, we can easily derive $J_t^{1-\beta} y(t)\big|_{t=0} = \phi(0)$. Thus, for $t \in J'$, by employing the operator J_t^β on both sides of (3.1), we obtain

$$
y(t) = g_\beta(t)\phi(0) + A(g_\beta * y)(t) + (g_\beta * f)(t).
$$

Employing (c) of Lemma 2.11 yields

$$
\begin{aligned} g_\beta * y &= (R_\beta - A(g_\beta * R_\beta)) * y = R_\beta * (y - A(g_\beta * y)) \\ &= R_\beta * (\phi(0) g_\beta + g_\beta * f) = g_\beta * (\phi(0) R_\beta + R_\beta * f)\,, \end{aligned}
$$

which indicates that

$$
y(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-s) f(s)\,ds, \quad t \in J'.
$$

Remark 3.2. It is worth mentioning that for the Riemann–Liouville delay evolution system in [1], to ensure the continuity of $y \in C([-r,b];V)$, the condition of ϕ, $\phi(0) = 0$, is necessary. Now, we make full use of the properties of resolvent to introduce a weighted delay initial condition in system (3.1). According to (a) of Definition 2.9 and the definition of $\widetilde{y}$ on $[-r,b]$, that is,

$$
\widetilde{y}(t) = \begin{cases} \Gamma(\beta) t^{1-\beta} y(t), & t \in [0,b], \\ \phi(t), & t \in [-r,0], \end{cases}
$$

$\widetilde{y}$ is continuous in $C([-r,b];V)$ for any $\phi \in C([-r,0];V)$.

As such, we can present the following notion of mild solutions to (2.2).

DEFINITION 3.3. For given $u \in L^p(J;U)$ with $p > 1/\beta$, by a mild solution to system (2.2) related to u, we understand the function $\widetilde{y} \in C([-r,b];V)$ which satisfies $y|_{J'} \in C_{1-\beta}(J;V)$,

$$y(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)(f(\tau,\widetilde{y}_\tau) + (Bu)(\tau))\,d\tau, \quad t \in J'$$

and

$$\widetilde{y}(t) = \phi(t), \quad t \in [-r,0].$$

REMARK 3.4. For $y \in C_{1-\beta}(J;V)$, let

$$\widetilde{y}[\phi](t) = \begin{cases} \Gamma(\beta)t^{1-\beta}y(t), & t \in J, \\ \phi(t), & t \in [-r,0]. \end{cases}$$

Then based on Definition 2.9 and Lemma 3.1, $\widetilde{y}[\phi] \in C([-r,b];V)$ is a mild solution of (2.2) related to u if and only if $y \in C_{1-\beta}(J;V)$ satisfies

$$y(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)\big(f(\tau,\widetilde{y}[\phi]_\tau) + (Bu)(\tau)\big)\,d\tau, \quad t \in J'. \tag{3.2}$$

For simplicity, set $S(u) = \big\{y \in C_{1-\beta}(J;V) : y \text{ satisfies } (3.2)\big\}$. Moreover, we abbreviate the notation $\widetilde{y}[\phi]$ to $\widetilde{y}$. We then propose a priori estimate for $S(u)$ which is useful in the later analysis.

LEMMA 3.5. *Let hypotheses* (HA), (Hf) *and* (HB) *hold. Then, for any* $y \in S(u)$,

$$\|y\|_{C_{1-\beta}} \le \overline{\lambda} := E_\beta\left(Mbk\Gamma(\beta)\right)\bigg(\frac{Mkb\|\phi\|_{[-r,0]}}{\beta} + M\|\phi(0)\| + M\bigg(b\frac{p-1}{\beta p-1}\bigg)^{1-1/p}\left(\|a\|_{L^p} + \|Bu\|_{L^p}\right)\bigg),$$

where M is defined by formula (2.4).

PROOF. Let $y \in S(u)$. Then, for $\tau \in [0,t]$, $t \in J'$, we derive

$$\begin{aligned}\|\widetilde{y}_\tau\|_{[-r,0]} &= \sup_{\theta\in[-r,0]}\|\widetilde{y}(\tau+\theta)\| \\ &\le \sup_{s\in[-r,0]}\|\widetilde{y}(s)\| + \sup_{s\in[0,\tau]}\|\widetilde{y}(s)\| \le \|\phi\|_{[-r,0]} + \sup_{s\in[0,\tau]}\Gamma(\beta)s^{1-\beta}\|y(s)\|.\end{aligned}$$

Thus, for $t \in J'$,

$$\begin{aligned}&\Gamma(\beta)t^{1-\beta}\|y(t)\| \\ &\quad\le M\|\phi(0)\| + Mb^{1-\beta}\int_0^t (t-\tau)^{\beta-1}\big(a(\tau) + k\|\widetilde{y}_\tau\|_{[-r,0]} + \|(Bu)(\tau)\|\big)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&\leq M\bigg(\|\phi(0)\| + \Big(b\frac{p-1}{\beta p-1}\Big)^{1-1/p}(\|a\|_{L^p} + \|Bu\|_{L^p}) + \frac{kb\|\phi\|_{[-r,0]}}{\beta}\bigg) \\
&\quad + Mkb^{1-\beta}\int_0^t (t-\tau)^{\beta-1} \sup_{s\in[0,\tau]} \Gamma(\beta)s^{1-\beta}\|y(s)\|\,d\tau.
\end{aligned}
$$

Let

$$
g(t) = \int_0^t (t-\tau)^{\beta-1} \sup_{s\in[0,\tau]} s^{1-\beta}\|y(s)\|\,d\tau.
$$

Then we can easily see that g is a monotonously increasing function. In fact, we have

$$
g(t) = \int_0^t \theta^{\beta-1} \sup_{s\in[0,t-\theta]} s^{1-\beta}\|y(s)\|\,d\theta,
$$

which means that, for $0 < t_1 < t_2$,

$$
\begin{aligned}
g(t_2) - g(t_1) &\leq \int_0^{t_2} \theta^{\beta-1} \sup_{s\in[0,t_2-\theta]} s^{1-\beta}\|y(s)\|\,d\theta \\
&\quad - \int_0^{t_1} \theta^{\beta-1} \sup_{s\in[0,t_1-\theta]} s^{1-\beta}\|y(s)\|\,d\theta \\
&\leq \int_0^{t_1} \theta^{\beta-1}\bigg(\sup_{s\in[0,t_2-\theta]} s^{1-\beta}\|y(s)\| - \sup_{s\in[0,t_1-\theta])} s^{1-\beta}\|y(s)\|\bigg)\,d\theta \\
&\quad + \int_{t_1}^{t_2} \theta^{\beta-1} \sup_{s\in[0,t_2-\theta]} s^{1-\beta}\|y(s)\|\,d\theta \geq 0,
\end{aligned}
$$

that is, g is monotonously increasing. Thus,

$$
\begin{aligned}
\Gamma(\beta)t^{1-\beta}\|y(t)\| &\leq \sup_{\eta\in[0,t]} \Gamma(\beta)\eta^{1-\beta}\|y(\eta)\| \\
&\leq M\bigg(\|\phi(0)\| + \Big(b\frac{p-1}{\beta p-1}\Big)^{1-1/p}(\|a\|_{L^p} + \|Bu\|_{L^p}) + \frac{kb\|\phi\|_{[-r,0]}}{\beta}\bigg) \\
&\quad + Mkb^{1-\beta}\int_0^t (t-\tau)^{\beta-1} \sup_{s\in[0,\tau]} \Gamma(\beta)s^{1-\beta}\|y(s)\|\,d\tau.
\end{aligned}
$$

Hence, the Gronwall inequality of singular version [37] tells us

$$
\begin{aligned}
\Gamma(\beta)t^{1-\beta}\|y(t)\| \leq E_\beta\,(Mbk\Gamma(\beta))\bigg(&\frac{Mkb\|\phi\|_{[-r,0]}}{\beta} + M\|\phi(0)\| \\
&+ M\Big(b\frac{p-1}{\beta p-1}\Big)^{1-1/p}(\|a\|_{L^p} + \|Bu\|_{L^p})\bigg).
\end{aligned}
$$

Therefore, $\|y\|_{C_{1-\beta}} \leq \overline{\lambda}$. □

With the aid of Lemmas 3.1 and 3.5, we now focus on the topological characteristics of $S(u)$, including compactness and R_δ-property.

Theorem 3.6. *Assume that conditions* (HA), (HB) *and* (Hf) *hold. Then, for fixed $u \in L^p(J;U)$, $p > 1/\beta$, $S(u)$ is nonempty and compact.*

Proof. Due to Lemma 3.1, we can define a map $\Phi\colon C_{1-\beta}(J;V)\to C_{1-\beta}(J;V)$ by

$$(\Phi y)(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)\big(f(\tau,\widetilde{y}_\tau) + (Bu)(\tau)\big)\,d\tau, \quad t\in J'.$$

Since $y\in S(u)$ is equivalent to $y\in \mathrm{Fix}(\Phi)$, our problem reduces to checking that $\mathrm{Fix}(\Phi)$ is nonempty and compact. For clarity, we split the verification into the following procedures.

Step 1. Set $B_\lambda = \{y\in C_{1-\beta}(J;V) : \|y\|_{C_{1-\beta}} \le \lambda\}$, where

$$\lambda \ge \max\bigg\{\frac{M}{\beta - Mkb}\bigg(\beta\|\phi(0)\| + \beta\bigg(b\frac{p-1}{\beta p-1}\bigg)^{1-1/p}\big(\|a\|_{L^p} + \|Bu\|_{L^p}\big) + kb\|\phi\|_{[-r,0]}\bigg), \overline{\lambda}\bigg\}.$$

We shall demonstrate that $\Phi(B_\lambda)\subseteq B_\lambda$. Let $y\in B_\lambda$. Then, for $\tau\in[0,t]$, $t\in J'$, we have

$$\|\widetilde{y}_\tau\|_{[-r,0]} \le \|\phi\|_{[-r,0]} + \|y\|_{C_{1-\beta}}.$$

Thus, for $t\in J'$, we derive

$$\begin{aligned}
&\Gamma(\beta)t^{1-\beta}\|(\Phi y)(t)\| \\
&\le M\|\phi(0)\| + Mb^{1-\beta}\int_0^t (t-\tau)^{\beta-1}\big(a(\tau) + k\|\widetilde{y}_\tau\|_{[-r,0]} + \|(Bu)(\tau)\|\big)\,d\tau \\
&\le M\|\phi(0)\| + Mb^{1-\beta}\int_0^t (t-\tau)^{\beta-1}\big(k\lambda + k\|\phi\|_{[-r,0]} + a(\tau) + \|(Bu)(\tau)\|\big)\,d\tau \\
&\le M\|\phi(0)\| + \frac{Mkb(\lambda + \|\phi\|_{[-r,0]})}{\beta} + M\bigg(b\frac{p-1}{\beta p-1}\bigg)^{1-1/p}\big(\|a\|_{L^p} + \|Bu\|_{L^p}\big) \le \lambda.
\end{aligned}$$

Hence, $\Phi(B_\lambda)\subseteq B_\lambda$.

Step 2. We verify the continuity of Φ on B_λ. Let $\{y_m\}_{m\ge1}\subseteq B_\lambda$ with $\lim_{m\to\infty} y_m = y$ in B_λ. Then, for $t\in J'$, we get $\lim_{m\to\infty}\widetilde{y_m}(t) = \widetilde{y}(t)$. Moreover, for $t\in[-r,0]$, according to the definition of $\widetilde{y}$, we have $\lim_{m\to\infty}\widetilde{y_m}(t) = \phi(t) = \widetilde{y}(t)$. Furthermore, we derive $\|y_m\|_{C_{1-\beta}}\le\lambda$ and $\|y\|_{C_{1-\beta}}\le\lambda$. As such, for $\tau\in[0,t]$, $t\in J'$, we have

$$\lim_{m\to\infty}\widetilde{y_m}_\tau = \widetilde{y}_\tau, \quad \|\widetilde{y_m}_\tau\|_{[-r,0]} \le \lambda + \|\phi\|_{[-r,0]}, \quad \|\widetilde{y}_\tau\|_{[-r,0]} \le \lambda + \|\phi\|_{[-r,0]}.$$

Hence, according to (Hf), for $\tau\in(0,t)$, $t\in J'$, we derive

$$\|f(\tau,\widetilde{y_m}_\tau) - f(\tau,\widetilde{y}_\tau)\| \le 2(a(\tau) + k\lambda + k\|\phi\|_{[-r,0]})$$

and

$$\|f(\tau,\widetilde{y_m}_\tau) - f(\tau,\widetilde{y}_\tau)\|^p \to 0, \quad m\to+\infty.$$

Consequently, for $t \in J'$, by virtue of Hölder's inequality and the dominated convergence theorem, we obtain

$$\begin{aligned}\sup_{t\in J}&\{t^{1-\beta}\,\Gamma(\beta)\|(\Phi y_m)(t)-(\Phi y)(t)\|\}\\&\leq b^{1-\beta}M\sup_{t\in J}\bigg\|\int_0^t(t-\tau)^{\beta-1}(f(\tau,\widetilde{y_m}_\tau)-f(\tau,\widetilde{y}_\tau)\,d\tau\bigg\|\\&\leq M\bigg(b\,\frac{p-1}{\beta p-1}\bigg)^{1-1/p}\bigg(\int_0^b\|f(\tau,\widetilde{y_m}_\tau)-f(\tau,\widetilde{y}_\tau)\|^p\,d\tau\bigg)^{1/p}\to 0,\end{aligned}$$

as $m\to+\infty$. Thus, we can infer that Φ is continuous on B_λ.

Step 3. We investigate the compactness of the map Φ on B_λ.

Due to (Hf) and (HB), one derives $\big(f(\,\cdot\,,\widetilde{y}_{(\,\cdot\,)})+(Bu)(\,\cdot\,)\big)\in L^p(J;V)$, for $p>1/\beta$. Thus, Lemma 3.1 leads to the compactness of the operator Φ on B_λ. Therefore, by applying Schauder's fixed point theory, we achieve $\mathrm{Fix}(\Phi)\neq\emptyset$.

Step 4. We study the compactness of the set $\mathrm{Fix}(\Phi)$. By virtue of Lemma 3.5, we acquire the boundedness of $\mathrm{Fix}(\Phi)$ and $\mathrm{Fix}(\Phi)\subseteq B_{\overline{\lambda}}\subseteq B_\lambda$. Thus, the compactness of the map Φ on B_λ signifies that $\Phi(\mathrm{Fix}(\Phi))$ is relatively compact. Moreover, due to the continuity of Φ, we can easily check the closeness of $\mathrm{Fix}(\Phi)$. Hence, from $\mathrm{Fix}(\Phi)\subseteq\Phi(\mathrm{Fix}(\Phi))$, we conclude that $\mathrm{Fix}(\Phi)$ is compact, i.e. $S(u)$ is compact. □

Theorem 3.7. *Let conditions* (HA), (HB) *and* (Hf) *be fulfilled. Then for fixed $u\in L^p(J;U)$, $p>1/\beta$, $S(u)$ is an R_δ-set.*

Proof. Based on (Hf) and the well-known Lasota-Yorke approximation theorem (see Theorem 17.6 in [12]), we can find a sequence $\{f_m\}$ with $f_m\colon J\times C([-r,0];V)\to V$ to ensure that $f_m(\tau,\,\cdot\,)$ is locally Lipschitz continuous and

$$\|f_m(\tau,\upsilon)-f(\tau,\upsilon)\|<\varepsilon_m,$$

for almost every $\tau\in J$ and $\upsilon\in C([-r,0];V)$, where $\varepsilon_m\in(0,1)$ and $\lim_{m\to\infty}\varepsilon_m=0$. Thus, (Hf) implies

$$\|f_m(\tau,\upsilon)\|\leq a(\tau)+k\|\upsilon\|_{[-r,0]}+1. \tag{3.3}$$

Now, we introduce the map $\Phi_m\colon C_{1-\beta}(J;V)\to C_{1-\beta}(J;V)$ of the form

$$(\Phi_m y)(t)=R_\beta(t)\phi(0)+\int_0^t R_\beta(t-\tau)(f_m(\tau,\widetilde{y}_\tau)+(Bu)(\tau))\,d\tau,\quad t\in J'.$$

Due to (3.3) and Lemma 3.1, the map Φ_m is well-defined. Let $y\in S(u)$. Then, for $t\in J'$,

$$\begin{aligned}t^{1-\beta}\Gamma(\beta)&\big\|\big((I-\Phi_m)y\big)(t)-\big((I-\Phi)y\big)(t)\big\|\\&\leq Mb^{1-\beta}\bigg\|\int_0^t(t-\tau)^{\beta-1}\left(f_m(\tau,\widetilde{y}_\tau)-f(\tau,\widetilde{y}_\tau)\right)d\tau\bigg\|\leq\frac{Mb\varepsilon_m}{\beta},\end{aligned}\tag{3.4}$$

where $I\colon C_{1-\beta}(J;V) \to C_{1-\beta}(J;V)$ is an identity map. Thus, in view of (3.4), we derive that $\lim\limits_{m\to\infty}(I-\Phi_m) = I-\Phi$, uniformly on $C_{1-\beta}(J;V)$. Moreover, similar to the verification of Theorem 3.6, for any $z \in C_{1-\beta}(J;V)$, it can be deduced that the equation $(I-\Phi_m)(y) = z$, that is,

$$y(t) = z(t) + R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)\big(f_m(\tau,\widetilde{y}_\tau) + (Bu)(\tau)\big)\,d\tau, \quad t \in J' \tag{3.5}$$

possesses solutions. Furthermore, with the help of the locally Lipschitz assumption on $f_m(\tau,\,\cdot\,)$, an argument similar to the one utilized in Theorem 3.6 in [35] shows that the solution to (3.5) is unique.

Next, we shall demonstrate that the map $I-\Phi_m$ is proper. On account of Step 2 of Theorem 3.6, we can obtain the continuity of $I-\Phi_m$. Now, for each compact set K in $C_{1-\beta}(J;V)$, we come to check the compactness of the set $(I-\Phi_m)^{-1}(K)$. For convenience, put $(I-\Phi_m)^{-1}(K) = D$, i.e. $(I-\Phi_m)(D) = K$. Then, for a sequence $\{y_n\} \subseteq D$, we can choose $\{z_n\} \subseteq K$ satisfying $(I-\Phi_m)(y_n) = z_n$. As such, we have

$$y_n(t) = z_n(t) + R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)\big(f_m(\tau,\widetilde{y_n}_\tau) + (Bu)(\tau)\big)\,d\tau, \quad t \in J'.$$

The combination of (3.3), the boundedness of $\{z_n\}$ and Lemma 3.5 indicates that $\{y_n\}$ is bounded in $C_{1-\beta}(J;V)$. Thus, due to the compactness of K and following the proof employed in Step 3 of Theorem 3.6, we can derive the relative compactness of $\{y_n\}$ in $C_{1-\beta}(J;V)$. Moreover, by the closeness of K and the continuity of $I-\Phi_m$, D is a closed set. Hence, D is compact. Consequently, $I-\Phi_m$ is proper. Similarly, $I-\Phi$ is proper. Therefore, by utilizing Lemma 2.7, one can infer that $S(u) = (I-\Phi)^{-1}(0)$ is an R_δ-set. □

Remark 3.8. By introducing the weighted delay initial condition and utilizing the resolvent approach, we have displayed the topological characteristics of solution sets of Riemann–Liouville fractional delay evolution systems in a weighted space, which extends some results in recent literature on topological properties of solution sets of evolution systems.

4. Approximate controllability

In this section, with the help of the topological characteristics of solution set to system (2.2) and the resolvent method, we exhibit the approximate controllability results for this system without the Lipschitz assumption of the nonlinear terms.

Definition 4.1. By the reachable set of (2.2), we mean the set $K_b(f) = \{y(b,u,f) : y(\,\cdot\,,u,f) \in S(u)\}$. Furthermore, if $\overline{K_b(f)} = V$, system (2.2) is said to be approximately controllable on J.

For convenience, define a linear map $\Psi\colon L^p(J;V)\to V$ by

$$\Psi(g)=\int_0^b R_\beta(b-\tau)g(\tau)\,d\tau,\quad g\in L^p(J;V),\ p>\frac{1}{\beta}.$$

To display the approximate controllability results, we need the following additional hypothesis.

(H_c) For any $g\in L^p(J;V)$, there exists $u\in L^p(J;U)$ such that $\Psi(Bu)=\Psi g$.

By virtue of (H_c) and Lemma 2 in [32], we can choose a continuous operator $G\colon L^p(J;V)\to L^p(J;U)$ satisfying that, for any $g\in L^p(J;V)$,

$$\Psi(B(Gg))+\Psi g=0, \tag{4.1}$$

$$\|Gg\|_{L^p(J;U)}\le d\|g\|_{L^p(J;V)}, \tag{4.2}$$

where $d>0$ is a constant.

We first explore the approximate controllability for the linear system of (2.2).

Theorem 4.2. *Let conditions* (HA), (HB) *and* (H_c) *hold. Then* $\overline{K_b(0)}=V$.

Proof. Let $\xi\in D(A)$. For arbitrary $\varepsilon>0$, in view of $\overline{D(A)}=V$, we can take $\zeta\in D(A)$ satisfying $\|\zeta-\phi(0)\|<b^{1-\beta}\Gamma(\beta)\varepsilon/(2M)$.

On the other hand, by virtue of Lemma 2.10 and (a) of Definition 2.9, we can choose some $\overline{b}\in J'$ to satisfy

$$\begin{aligned}&\big\|\big(\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})\big)^2(\xi-R_\beta(b)\zeta)-(\xi-R_\beta(b)\zeta)\big\|\\&\qquad\le\big\|\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})+I\big\|\big\|\big(\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})-I\big)(\xi-R_\beta(b)\zeta)\big\|<\frac{\varepsilon}{2}.\end{aligned}$$

According to Lemma 2.11, we have $\xi-R_\beta(b)\zeta\in D(A)$. Thus, we can take

$$g(\tau)=\begin{cases}\dfrac{-(b-\tau)^{1-\beta}\Gamma^2(\beta)}{\overline{b}}\bigg[-(b-\tau)^{1-\beta}R_\beta(b-\tau)\\ \quad+2(b-\tau)\dfrac{d\big((b-\tau)^{1-\beta}R_\beta(b-\tau)\big)}{d\tau}\bigg](\xi-R_\beta(b)\zeta), & t\in[b-\overline{b},b],\\ 0, & t\in[0,b-\overline{b}],\end{cases}$$

to ensure that $\Psi g=\big(\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})\big)^2(\xi-R_\beta(b)\zeta)$. Additionally, because of condition (HA) and Lemma 2.13, we get $g\in L^p(J;V)$. In addition, due to (H_c), we can choose a function $u\in L^p(J;U)$ to guarantee that

$$\Psi(Bu)=\Psi g=\big(\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})\big)^2(\xi-R_\beta(b)\zeta).$$

Hence, we get

$$\begin{aligned}&\|\xi-(R_\beta(b)\phi(0)+\Psi(Bu))\|\\&\quad\le\|R_\beta(b)\phi(0)-R_\beta(b)\zeta\|+\big\|(\xi-R_\beta(b)\zeta)-\big(\Gamma(\beta)\overline{b}^{1-\beta}R_\beta(\overline{b})\big)^2(\xi-R_\beta(b)\zeta)\big\|\\&\quad\le b^{\beta-1}\big\|(b^{1-\beta}R_\beta(b))(\phi(0)-\zeta)\big\|+\frac{\varepsilon}{2}\le\frac{b^{\beta-1}M}{\Gamma(\beta)}\|\phi(0)-\zeta\|+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

Thus, we arrive at $D(A) \subset \overline{K_b(0)}$, which results in $\overline{K_b(0)} = V$. $\square$

Now, with the help of the topological characteristics of solution set of system (2.2) and Theorem 4.2, we analyze the approximate controllability of (2.2).

THEOREM 4.3. *Under conditions* (HA), (HB), (Hf) *and* (H_c), *system* (2.2) *is approximately controllable if*

$$Mkbd\left(\frac{p-1}{\beta p-1}\right)^{1-1/p}\|B\|E_\beta(Mkb\Gamma(\beta)) < 1, \tag{4.3}$$

where $d > 0$ *satisfies* (4.2) *and* M *is defined by formula* (2.4).

PROOF. To make our verification more transparent, we analyze this problem in the following steps.

Step 1. We show that the map $S\colon L^p(J;U) \to P(C_{1-\beta}(J;V))$ is an R_δ-map. In view of Theorem 3.7 and Definition 2.6, we just need to check that S is u.s.c.

First, we demonstrate the quasicompactness of S. Let D be a bounded set in $L^p(J;U)$. For a sequence $\{y_n\} \subseteq S(D)$, by the analogous approach as employed in Theorem 3.6, we can acquire the relative compactness of $\{y_n\}$. Hence, we can infer that S is quasicompact.

Next, we handle the closeness of S. Let $\lim_{m\to\infty} u_m = u$ in $L^p(J;U)$ and $y_m \in S(u_m)$ with $\lim_{m\to\infty} y_m = y$ in $C_{1-\beta}(J;V)$. According to $y_m \in S(u_m)$, one derives

$$y_m(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)(f(\tau,\widetilde{y_{m\tau}}) + (Bu_m(\tau))\,d\tau, \quad t \in J'. \tag{4.4}$$

Then we obtain through Lemma 3.5 that $\{y_m\}$ is bounded in $C_{1-\beta}(J;V)$. Thus, similar to the verification of Step 2 of Theorem 3.6, for $t \in J'$, we can easily infer that

$$\int_0^t R_\beta(t-\tau)f(\tau,\widetilde{y_{m\tau}})\mathrm{d}\tau \to \int_0^t R_\beta(t-\tau)f(\tau,\widetilde{y}_\tau)\,d\tau, \quad m\to\infty.$$

Moreover, for $t \in J'$, Hölder's inequality implies that

$$\begin{aligned}\bigg\| \int_0^t & R_\beta(t-\tau)((Bu_m)(\tau) - (Bu)(\tau))\,d\tau \bigg\| \\ &\le \frac{M}{\Gamma(\beta)}\, b^{\beta-1/p}\left(\frac{p-1}{\beta p-1}\right)^{1-1/p}\|Bu_m - Bu\|_{L^p} \to 0.\end{aligned}$$

As such, taking the limit $m\to\infty$ to both sides of (4.4), for $t \in J'$, we derive

$$y(t) = R_\beta(t)\phi(0) + \int_0^t R_\beta(t-\tau)(f(\tau,\widetilde{y}_\tau) + (Bu)(\tau))\,d\tau. \tag{4.5}$$

Hence, we achieve that $y \in S(u)$. Therefore, the closeness of S is acquired. Consequently, Lemma 2.2 guarantees that S is u.s.c.

Step 2. Define a multi-map $Q\colon L^p(J;U) \to P(L^p(J;U))$ by

$$Q(u) = G \circ \mathscr{F} \circ S(u_0 + u), \quad u \in L^p(J;U),$$

for any $u_0 \in L^p(J;U)$. Here the operator $\mathscr{F}\colon C_{1-\beta}(J;V) \to L^p(J;V)$ is defined by

$$(\mathscr{F}y)(\tau) = f(\tau, \widetilde{y}_\tau) \quad \text{for } y \in C_{1-\beta}(J;V) \text{ and } \tau \in J,$$

and G satisfies (4.1) and (4.2). We shall show that $\mathrm{Fix}(Q) \neq \emptyset$ by employing Lemma 2.8. Obviously, Q is well-defined. Moreover, since the operators G and $\mathscr{F}$ are single-valued and continuous, they are R_δ-maps. Additionally, one can easily verify that $L^p(J;U),\ C_{1-\beta}(J;V) \in AR$.

Now, let $u \in L^p(J;U)$ and $\overline{u} \in Q(u)$. For $y \in S(u_0+u)$, due to (4.2), we obtain

$$\begin{aligned}\|\overline{u}\|_{L^p(J;U)} &\leq d\|\mathscr{F}(y)\|_{L^p(J;V)} \leq d\bigg(\int_0^b \|f(t,\widetilde{y}_t)\|^p\,dt\bigg)^{1/p}\\ &\leq d\big(\|a\|_{L^p} + kb^{1/p}(\|y\|_{C_{1-\beta}} + \|\phi\|_{[-r,0]})\big).\end{aligned}$$

which together with Lemma 3.5 implies that

$$\limsup_{\|u\|_{L^p(J;U)}\to\infty} \frac{\|\overline{u}\|_{L^p(J;U)}}{\|u\|_{L^p(J;U)}} \leq Mkbd\bigg(\frac{p-1}{\beta p-1}\bigg)^{1-1/p}\|B\|E_\beta(Mkb\Gamma(\beta)) < 1.$$

Thus, there exists $\overline{r} > 0$ to ensure that $QB_{\overline{r}} \subset B_{\overline{r}}$, where

$$B_{\overline{r}} = \big\{u \in L^p(J;U) : \|u\|_{L^p(J;U)} \leq \overline{r}\big\}.$$

Furthermore, by the similar approach as utilized in Theorem 3.6, we can derive the compactness of $S(u_0 + B_{\overline{r}})$. Thus, by means of the continuity of G and $\mathscr{F}$, we derive the compactness of the set $K := Q(B_{\overline{r}})$. Therefore, Lemma 2.8 yields that $\mathrm{Fix}(Q) \neq \emptyset$.

Step 3. We shall check that $\overline{K_b(f)} = V$. Owing to Theorem 4.2, we only need to investigate the relation between $K_b(0)$ and $K_b(f)$. For $y^b \in K_b(0)$, we can pick $u_0 \in L^p(J;U)$ satisfying that

$$y^b = R_\beta(b)\phi(0) + \Psi(Bu_0).$$

For above $u_0 \in L^p(J;U)$, let $\overline{u} \in \mathrm{Fix}(Q)$. Then we can take $y \in S(u_0 + \overline{u})$ such that $\overline{u} = G \circ \mathscr{F}(y)$. Thus, according to (4.1), we have

$$\begin{aligned}y(b, u_0+\overline{u}, f) &= R_\beta(b)\phi(0) + \Psi(B(u_0+\overline{u})) + \Psi \circ \mathscr{F}(y)\\ &= R_\beta(b)\phi(0) + \Psi(Bu_0) + \Psi(B(\overline{u}) + \mathscr{F}(y))\\ &= y^b + \Psi(B \circ G \circ \mathscr{F}(y) + \mathscr{F}(y)) = y^b,\end{aligned}$$

which indicates that $K_b(0) \subseteq K_b(f)$. Hence, $\overline{K_b(f)} = V$. Therefore, the approximate controllability of (2.2) is achieved. $\square$

Remark 4.4. In most of the existing results on the approximate controllability problems (see [21], [25], [27]), with the help of the Lipschitz assumption of the nonlinear terms and the range condition proposed by Naito [27], many investigators handle these problems in Hilbert spaces by utilizing a space decomposition method. In the present paper, we explore the problems in general Banach spaces by employing the topological structure of the solution set and the resolvent method without the Lipschitz condition. Hence, our conclusion extends and generalizes some recent results on this topic.

5. Applications

Up to now, we have established our theoretical results by utilizing the resolvent theory and the topological characteristics of the solution set. In what follows, we firstly address the fractional delay diffusion control system (2.1) by our theoretical findings and provide an example about approximate controllability of the system (2.1). Then, we explore a finite dimensional fractional ordinary differential control system.

5.1. Infinite dimensional delay diffusion control systems. For simplicity, we only study the case when $N=1$, $\Omega=(0,1)$ and $J'=(0,1]$ in system (2.1). By an analogous technique, we can handle more general cases.

From now on, we always suppose that $V=U=L^2(0,1)$, $1/2<\beta<1$ and $A=\partial^2/\partial x^2$ with domain

$$D(A)=\{\nu\in V:\nu',\nu''\in V,\ \nu(0)=\nu(1)=0\}.$$

It follows that the eigenvalues of operator A are $-m^2\pi^2$, $m\in\mathbb{N}_+$, with the corresponding normalized eigenvectors $e_m(x)=\sqrt{2}\sin(m\pi x)$, $m\in\mathbb{N}_+$, $x\in(0,1)$. Then the resolvent $R_\beta(t)$ generated by A (see [22]) is

$$R_\beta(t)g(x)=\sum_{m=1}^{\infty}t^{\beta-1}E_{\beta,\beta}(-m^2\pi^2t^\beta)\langle g,e_m\rangle e_m(x),\quad t>0,\ g\in V. \tag{5.1}$$

In addition, due to [28], A also generates a compact and analytic semigroup $\{T(t)\}_{t>0}$ with

$$T(t)g(x)=\sum_{m=1}^{\infty}e^{-m^2\pi^2t}\langle g,e_m\rangle e_m(x).$$

Thus, we can deduce, by employing Laplace transformations and probability density functions [24], that for any $g\in V$,

$$t^{1-\beta}R_\beta(t)g(x)=\beta\int_0^\infty\tau\xi_\beta(\tau)T(t^\beta\tau)g(x)\,d\tau,\ t>0,$$

which indicates that

$$t^{1-\beta}R_\beta(t)=\beta\int_0^\infty\tau\xi_\beta(\tau)T(t^\beta\tau)\,d\tau,$$

where

$$\xi_\beta(\tau) = \frac{1}{\beta}\tau^{-1-1/\beta}\varpi_\beta(\tau^{-1/\beta}),$$

$$\varpi_\beta(\tau) = \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\tau^{-(m+1)\beta-1}\frac{\Gamma((m+1)\beta+1)}{(m+1)!}\sin((m+1)\pi\beta), \quad \tau \in \mathbb{R}_+.$$

Moreover, based on the compactness of $\{T(t)\}_{t>0}$, we derive that $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ is compact (see Lemma 3.4 in [38]). Additionally, since $\{T(t)\}_{t>0}$ satisfies $T'(t) = AT(t)$, $t\|AT(t)\| \le C_1$, $0 < t \le 1$ and $\|T(t)\| \le 1$, $t \ge 0$ (see [28]), by the dominated convergence theorem, we can easily acquire that

$$\frac{d(t^{1-\beta}R_\beta(t))}{dt} = \beta^2 t^{\beta-1}\int_0^\infty \tau^2\xi_\beta(\tau)AT(t^\beta\tau)\,d\tau, \quad 0 < t \le 1,$$

$$\left\|\frac{d(t^{1-\beta}R_\beta(t))}{dt}\right\| \le \frac{\beta C_1}{t\Gamma(\beta)}, \quad 0 < t \le 1$$

and

$$\|\Gamma(\beta)t^{1-\beta}R_\beta(t)\| \le 1, \quad 0 \le t \le 1,$$

where C_1 is a constant. For convenience, denote $M = 1$. Hence, (HA) is satisfied.

On the other hand, for any $u \in L^2(J;U)$, one has

$$u(t) = \sum_{m=1}^{\infty} u_m(t)e_m, \qquad u_m(t) = \langle u(t), e_m\rangle.$$

Inspired by [24], we now introduce the operator $B\colon L^2(J;U) \to L^2(J;V)$ as follows:

$$(Bu)(t) = \sum_{m=1}^{\infty} \overline{u}_m(t)e_m, \tag{5.2}$$

where

$$\overline{u}_m(t) = \begin{cases} 0, & 0 \le t < 1 - m^{-2/\beta}, \\ u_m(t), & 1 - m^{-2/\beta} \le t < 1, \\ 0, & t = 1. \end{cases}$$

Obviously, B is bounded. Let

$$y(t)(x) = y(t,x) = \sum_{m=1}^{\infty} y_m(t)e_m(x),$$

$$\widetilde{y}(t)(x) = \widetilde{y}(t,x) = \sum_{m=1}^{\infty} \widetilde{y}_m(t)e_m(x),$$

and

$$\phi(t)(x) = \phi(t,x) = \sum_{m=1}^{\infty} \phi_m(t)e_m(x).$$

Then the corresponding linear system of (2.1) can be transformed into the form:

$$
\begin{cases} D^\beta y_m(t) = -m^2\pi^2 y_m(t) + \overline{u}_m(t), & 0 < t \le 1, \\ \widetilde{y}_m(t) = \phi_m(t), & t \in [-r, 0]. \end{cases} \tag{5.3}
$$

LEMMA 5.1. *Let $R_\beta(t)$ be an operator defined in* (5.1) *and $B\colon L^2(J;U) \to L^2(J;V)$ an operator given in* (5.2)*. Then, for any $q \in L^2(J;V)$, there exists $u \in L^2(J;U)$ to ensure that*

$$
\int_0^1 R_\beta(1-\tau)q(\tau)\,d\tau = \int_0^1 R_\beta(1-\tau)(Bu)(\tau)\,d\tau.
$$

PROOF. Denote

$$
\begin{aligned}
\mathcal{W} &= \int_0^1 \big(1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\big)^2\,d\tau, \\
\widehat{\mathcal{W}} &= \int_{1-m^{-2/\beta}}^1 \big((1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\big)^2\,d\tau.
\end{aligned}
$$

In view of Theorem 1.6 in [29], we derive

$$
|E_{\beta,\beta}(-m^2\pi^2(1-\tau)^\beta)| \le \frac{C}{1+m^2\pi^2(1-\tau)^\beta}, \quad \tau \in J, \tag{5.4}
$$

where C is a positive constant, not depending on m. Moreover, by Cauchy–Schwarz inequality, it follows that

$$
\mathcal{W} \ge \bigg(\int_0^1 (1-\tau)^{\beta-1} E_{\beta,\beta}(-m^2\pi^2(1-\tau)^\beta)\,d\tau\bigg)^2 \ge E^2_{\beta,\beta+1}\big(-m^2\pi^2\big).
$$

Thus, we get

$$
0 < E^2_{\beta,\beta+1}\big(-m^2\pi^2\big) \le \mathcal{W} \le \frac{C^2}{2\beta-1}.
$$

Similarly, we have

$$
0 < m^{(2/\beta)-4} E^2_{\beta,\beta+1}\big(-\pi^2\big) \le \widehat{\mathcal{W}} \le \frac{C^2 m^{(2/\beta)-4}}{2\beta-1}.
$$

Furthermore, due to (5.4), we obtain

$$
\begin{aligned}
\frac{\mathcal{W}}{\widehat{\mathcal{W}}} &\le 1 + \frac{\displaystyle\int_0^{1-m^{-2/\beta}} \big((1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\big)^2\,d\tau}{\widehat{\mathcal{W}}} \\
&\le 1 + \frac{\displaystyle\int_0^{1-m^{-2/\beta}} (1-\tau)^{2\beta-2}\bigg(\frac{C}{1+m^2\pi^2(1-\tau)^\beta}\bigg)^2\,d\tau}{m^{(2/\beta)-4} E^2_{\beta,\beta+1}(-\pi^2)} \\
&\le 1 + \frac{C^2}{\pi^4 E^2_{\beta,\beta+1}(-\pi^2)}.
\end{aligned} \tag{5.5}
$$

For any $q \in L^2(J;V)$, we set

$$q(\tau) = \sum_{m=1}^{\infty} q_m(\tau) e_m, \quad q_m(\tau) = \langle q(\tau), e_m \rangle$$

and

$$h = \int_0^1 R_\beta(1-\tau) q(\tau)\, d\tau = \sum_{m=1}^{\infty} h_m e_m, \quad h_m = \langle h, e_m \rangle.$$

Based upon (5.1), we have

$$\int_0^1 (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big) q_m(\tau)\, d\tau = h_m.$$

For linear system (5.3), by employing the minimum norm property (see Lemma 2.1 in [18]), we obtain

$$\int_0^1 |\overline{q}_m(\tau)|^2\, d\tau \le \int_0^1 |q_m(\tau)|^2\, d\tau, \tag{5.6}$$

where

$$\overline{q}_m(\tau) = (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big) \mathcal{W}^{-1} h_m. \tag{5.7}$$

Now, we take

$$\widehat{u}_m(\tau) = (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big) \widehat{\mathcal{W}}^{-1} h_m. \tag{5.8}$$

Then, we derive

$$\int_{1-m^{-2/\beta}}^1 (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big) \widehat{u}_m(\tau)\, d\tau = h_m. \tag{5.9}$$

In addition, we define $u(\tau) = \sum\limits_{m=1}^{\infty} u_m(\tau) e_m$, where

$$u_m(\tau) = \begin{cases} 0, & 0 \le \tau < 1 - m^{-2/\beta}, \\ \widehat{u}_m(\tau), & 1 - m^{-2/\beta} \le \tau < 1, \\ 0, & \tau = 1. \end{cases}$$

By virtue of (5.5)–(5.8), it follows that

$$\begin{aligned} \|u\|^2_{L^2(J;U)} &= \sum_{m=1}^{\infty} \int_{1-m^{-2/\beta}}^1 |\widehat{u}_m(\tau)|^2\, d\tau = \sum_{m=1}^{\infty} \widehat{\mathcal{W}}^{-1} h_m^2 \\ &= \sum_{m=1}^{\infty} \frac{\mathcal{W}}{\widehat{\mathcal{W}}} \int_0^1 |\overline{q}_m(\tau)|^2\, d\tau \le \bigg(1 + \frac{C^2}{\pi^4 E^2_{\beta,\beta+1}(-\pi^2)}\bigg) \|q\|_{L^2}, \end{aligned}$$

which indicates that $u \in L^2(J;U)$. Additionally, in view of the definition of the operator B, we derive

$$(Bu)(\tau) = \sum_{m=1}^{\infty} \overline{u}_m(\tau) e_m,$$

where

$$\overline{u}_m(\tau) = \begin{cases} 0, & 0 \le \tau < 1 - m^{-2/\beta}, \\ u_m(\tau), & 1 - m^{-2/\beta} \le \tau < 1, \\ 0, & \tau = 1, \end{cases}$$

$$= \begin{cases} 0, & 0 \le \tau < 1 - m^{-2/\beta}, \\ \widehat{u}_m(\tau), & 1 - m^{-2/\beta} \le \tau < 1 \\ 0, & \tau = 1. \end{cases}$$

Thus, by means of (5.1) and (5.9), we obtain

$$\begin{aligned} &\int_0^1 R_\beta(1-\tau)(Bu)(\tau)\,\tau \\ &\quad = \int_0^1 \sum_{m=1}^{\infty} (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\langle (Bu)(\tau), e_m\rangle e_m \, d\tau \\ &\quad = \sum_{m=1}^{\infty} \int_0^1 (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\overline{u}_m(\tau) e_m \, d\tau \\ &\quad = \sum_{m=1}^{\infty} \int_{1-m^{-2/\beta}}^1 (1-\tau)^{\beta-1} E_{\beta,\beta}\big(-m^2\pi^2(1-\tau)^\beta\big)\widehat{u}_m(\tau) \, d\tau e_m \\ &\quad = \sum_{m=1}^{\infty} h_m e_m = \int_0^1 R_\beta(1-\tau) q(\tau)\, d\tau, \end{aligned}$$

which means that system (2.1) satisfies (H_c). □

Hence, due to Lemma 5.1 and Theorem 4.3, we can easily derive the following result:

Theorem 5.2. *Let $1/2 < \beta < 1$, $B\colon L^2(J;U) \to L^2(J;V)$ be an operator defined in* (5.2) *and* (Hf) *hold. Then system* (2.1) *is approximately controllable if*

$$\frac{kd}{\sqrt{2\beta-1}} \|B\| E_\beta(k\Gamma(\beta)) < 1,$$

where $d > 0$ satisfies (4.2).

Example 5.3. Consider the following Riemann–Liouville fractional delay diffusion system:

$$\text{(5.10)} \qquad \begin{cases} D^{2/3} y(t,x) = \dfrac{\partial^2}{\partial x^2} y(t,x) \\ \qquad\qquad + \dfrac{e^{-t}}{e^t + e^{-t}} \sin \widetilde{y}_t(x) + (Bu)(t,x), & t, x \in (0,1], \\ y(t,0) = y(t,1) = 0, & t \in (0,1], \\ \widetilde{y}_0(t,x) = \phi(t,x), & t \in [-r, 0], \end{cases}$$

where $\widetilde{y}(t,x) = \Gamma(2/3)t^{1/3}y(t,x)$ for $t \in [0,1]$, $\widetilde{y}(0,x) = \lim_{t\to 0^+} \widetilde{y}(t,x)$, $\widetilde{y}_t(\theta,x) = \widetilde{y}(t+\theta,x)$ for $t \in [0,1]$ and $\theta \in [-r,0]$, ϕ is continuous.

Define the operator $f\colon [0,1]\times C([-r,0];V) \to V$ by

$$f(t,\widetilde{y}_t)(x) = \left(\frac{e^{-t}}{e^t + e^{-t}}\right)\sin \widetilde{y}_t(x).$$

Then f is bounded. Hence, we can take $k = 0$ in condition (Hf).

Let $B\colon L^2(J;U) \to L^2(J;V)$ be an operator defined in (5.2). Then, due to Theorem 5.2, system (5.10) is approximately controllable.

Remark 5.4. The Example 5.3 is adapted from [14]. In [14], Hilfer investigated the homogeneous fractional diffusion system without delay of (2.1) and explained the importance of this kind of system for the theoretical physics and chemistry. In particular, he (or she) pointed out that the significance of studying this system originates in the necessary to sharpen the concepts of equilibrium, stationary states, and time evolution in the long time limit.

Remark 5.5. In applications about approximate controllability of infinite dimensional diffusion systems, we can always give the concrete expressions of operator A, order β and function f. It is a pity that it is difficult to give a concrete expression of operator B directly. However, for finite dimensional ordinary differential control systems, we can give the expression of operator B directly. Next, we provide an example about approximate controllability of these systems.

5.2. Finite dimensional ordinary differential control systems.

Example 5.6. Consider the following fractional control system

$$\begin{cases} D^{2/3}y(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} y(t) + \sin(\widetilde{y}(t)) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), & t \in [0,t_1], \\ \lim_{t\to 0^+} \widetilde{y}(t) = y_0, \end{cases} \tag{5.11}$$

where $y(t) \in \mathbb{R}^2$, $\widetilde{y}(t) = \Gamma(2/3)t^{1/3}y(t)$, $u \in L^2([0,t_1];\mathbb{R})$ and $y_0 \in \mathbb{R}^2$.

Denote

$$\beta = \frac{2}{3}, \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f(t,\widetilde{y}(t)) = \sin(\widetilde{y}(t)).$$

By utilizing Laplace transformations, the solutions to equation (5.11) can be expressed by

$$y(t) = R_\beta(t)y_0 + \int_0^t R_\beta(t-\tau)\big(Bu(\tau) + f(\tau,\widetilde{y}(\tau))\big)\,d\tau,$$

where $R_\beta(t) = t^{\beta-1}E_{\beta,\beta}(At^\beta)$. By virtue of Definition 2.9, it is easy to check that $\{R_\beta(t)\}_{t>0}$ is a resolvent. Moreover, by applying Laplace transformations and probability density functions, we can derive

$$t^{1-\beta}R_\beta(t) = E_{\beta,\beta}(At^\beta) = \beta\int_0^\infty \tau\xi_\beta(\tau)e^{At^\beta\tau}\,d\tau.$$

Since all the eigenvalues of A are negative, e^{At} $(t > 0)$ is bounded and compact. Hence, $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ is bounded, compact and equicontinuous (see Lemma 3.4 in [38]).

In order to verify that system (5.11) satisfies condition (H_c), we first introduce the following lemma:

Lemma 5.7.

(a) *The linear system of* (5.11) *is controllable on* $[0, t_1]$ *(for any* y_0*,* $y_1 \in \mathbb{R}^2$*, there exists* $u \in L^2([0, t_1]; \mathbb{R})$ *to ensure that the solution* y *of the linear system satisfies* $y(t_1) = y_1$*) if and only if*

$$\operatorname{rank}[B, AB] = 2.$$

(b) *The linear system of* (5.11) *is controllable on* $[0, t_1]$ *if and only if the Gramian matrix*

$$\mathcal{W} = \int_0^{t_1} R_\beta(t_1-\tau)BB^TR_\beta^T(t_1-\tau)\,d\tau$$

is invertible. In such a case, for any $h \in \mathbb{R}^2$*, the control function*

$$u(t) = B^TR_\beta^T(t_1-t)\mathcal{W}^{-1}h$$

satisfies the minimum norm property, that is,

$$\|u\|_{L^2(J;\mathbb{R})} = \inf\bigg\{\|v\|_{L^2(J;\mathbb{R})} : \int_0^{t_1} R_\beta(t_1-\tau)Bv(\tau)\,d\tau = h\bigg\}.$$

Proof. Similar to the proofs of Theorem 3 in [26] as well as Lemma 2.1 in [18], we can easily check the results of this Lemma. $\square$

Now, we are ready to show that system (5.11) satisfies (H_c). Based on $\operatorname{rank}[B, AB] = 2$ and Lemma 5.7, we know that the linear system of (5.11) is controllable on $[0, t_1]$ and $\mathcal{W}$ is invertible. Thus, for any $g \in L^2([0, t_1]; \mathbb{R}^2)$, setting

$$h = \int_0^{t_1} R_\beta(t_1-\tau)g(\tau)\,d\tau,$$

we can take

$$u(\,\cdot\,) = B^TR_\beta^T(t_1-\,\cdot\,)\mathcal{W}^{-1}h \in L^2([0, t_1]; \mathbb{R})$$

to ensure that

$$\int_0^{t_1} R_\beta(t_1-\tau)g(\tau)\,d\tau = \int_0^{t_1} R_\beta(t_1-\tau)(\overline{B}u)(\tau)\,d\tau,$$

where $\overline{B}\colon L^2([0,t_1];\mathbb{R}) \to L^2([0,t_1];\mathbb{R}^2)$ is a linear and bounded operator defined by

$$(\overline{B}u)(\tau) = Bu(\tau).$$

Thus, (H_c) holds. Therefore, according to Theorem 4.3 and the boundedness of f, system (5.11) is approximately controllable.

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