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EXISTENCE OF POSITIVE GROUND SOLUTIONS FOR BIHARMONIC EQUATIONS VIA POHOŽAEV–NEHARI MANIFOLD

Liping Xu — Haibo Chen

Abstract. We investigate the following nonlinear biharmonic equations with pure power nonlinearities:

$$\begin{cases} \triangle^2 u - \triangle u + V(x)u = u^{p-1}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{for } u \in H^2(\mathbb{R}^N), \end{cases}$$

where $2 < p < 2^* = 2N/(N-4)$. Under some suitable assumptions on $V(x)$, we obtain the existence of ground state solutions. The proof relies on the Pohožaev–Nehari manifold, the monotonic trick and the global compactness lemma, which is possibly different to other papers on this problem. Some recent results are extended.

1. Introduction

This paper is to study the existence of positive ground state solutions of the following biharmonic equation with pure power nonlinearities:

$$\text{(EQ)} \qquad \begin{cases} \triangle^2 u - \triangle u + V(x)u = u^{p-1}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{for } u \in H^2(\mathbb{R}^N), \end{cases}$$

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where $\triangle^2 := \triangle(\triangle)$ is the biharmonic operator, $2 < p < 2^* = 2N/(N-4)$ and $N > 4$. The potential $V(\cdot)$ is continuous on $\mathbb{R}^N$, and satisfies the following hypotheses:

(v_1) $\langle \nabla V(x), x\rangle \in L^\infty(\mathbb{R}^N) \cup L^{2^*/(2^*-2)}(\mathbb{R}^N)$ and $4V(x) + \langle \nabla V(x), x\rangle \geq 0$ for almost every $x \in \mathbb{R}^N$.

(v_2) $0 \leq V(x) \leq V(\infty) := \liminf_{|x|\to\infty} V(x) < +\infty$ and $V(x) \not\equiv V(\infty)$, and the inequality is strict in a subset of positive measure.

The biharmonic equations arise in the study of traveling waves in suspension bridges (see [8], [13], [21]) and the study of the static deflection of an elastic plate in a fluid. In the last decades there are many results for biharmonic equations. We refer the reader to [1], [3], [10], [11], [24], [31] for the case of bounded domain, and [2], [6], [5], [15], [23], [26]–[29] for the case of unbounded domain. For example, on bounded domains, An and Liu [3] used the mountain pass theorem to get the existence results for the following problem

$$\begin{cases} \triangle^2 u + c\triangle u = g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a smooth bounded domain, c is a constant. The multiplicity result of sign-changing solutions for (1.1) has been proved in [31] by using the sign-changing critical theorems. Compared with the case of bounded domain, the case of unbounded domain seems to be more complicate. For example, under the following conditions:

(v_3) $V(x) \in C(\mathbb{R}^N)$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$,

(v_4) for each $b > 0$, $|\{x \in \mathbb{R}^N : V(x) \leq b\}| < +\infty$, where $|\cdot|$ is the Lebesgue measure,

Yin and Wu [11] obtained a sequence of high energy solutions of the following problem:

$$\begin{cases} \triangle^2 u - \triangle u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases} \tag{1.2}$$

Corresponding results for (1.2) were further improved by Y. Ye and C. Tang [27] and W. Zhang, X.Tang and J. Zhang [29]. The condition (v_4) was used in T. Bartsch and Z. Wang [4], which shows that $V(x)$ must be coercive. But there are a great number of functions not satisfying the coerciveness. Replacing (v_4) by a more general condition:

(v_4)$'$ There exists $b > 0, |\{x \in \mathbb{R}^N : V(x) \leq b\}| < +\infty$, where $|\cdot|$ is the Lebesgue measure.

J. Liu, S. Chen and X. Wu in [18] showed the existence and multiplicity results of equation (1.2), while a positive parameter needs to be added to the equation.

Recently, [16] obtained the existence and multiplicity of solutions of a class of biharmonic equations with critical nonlinearity in $\mathbb{R}^N$. Under (v_3) and (v_4), G. Che and H. Chen [7] proved that the problem (EQ) has at least energy nodal solution by Nehari manifold, and also obtained other nontrivial solutions under suitable conditions.

The similar hypotheses on $V(x)$ as above (v_1)–(v_2) are introduced in [14], [19], [20], [30] and have physical meaning. Under the conditions, L. Zhao, G. Li and Z. Liu in [30], [14], [19], [20] obtained the ground solutions of Schrödinger–Maxwell equations and Kirchhoff type equations, respectively. Moreover, there are indeed many functions satisfying (v_1)–(v_2). For instance, $V(x) = V_0 - 1/(\tau(|x|+1))$, where $V_0 > 1$ and $\tau > 4$ are positive constants.

Motivated by the works just described, the purpose of this paper is to establish the existence of positive ground state solutions for the problem (EQ) by using Pohožaev–Nehari manifold method, combined with the monotone trick of L. Jeanjean (see [12]), concentration-compactness (due to [17], [25]) and a global compactness Lemma (see Lemma 4.5 below). To the best of our knowledge, in the literature there are few results on the existence of positive ground state solutions for (EQ) by Pohožaev–Nehari manifold method (see Section 3 below).

The main results are the following theorems.

Theorem 1.1. *Assume that $V(x)$ is a positive constant, then the problem* (EQ) *has a positive ground state solution for any $3 \le p < 2^* - 1$ and $N > 4$.*

Remark 1.2. If $V(x)$ is a positive constant, motivated by [30], [14], [19], [20], we use the constrained minimization on a Pohožaev–Nehari manifold to prove Theorem 1.1. We prove that such a manifold has two perfect characteristics: it is a natural constraint for the reduced functional and it contains every solutions of the problem (EQ).

Theorem 1.3. *If $V(x)$ satisfies* (v_1)–(v_2)*, assume that $N > 4$ with $3 \le p < 2^* - 1$, then the problem* (EQ) *has a positive ground state solution.*

Remark 1.4. The similar conditions like (v_1)–(v_2) were introduced in [14], [19], [20], [30]. Theorem 1.3 extends the main results in [14], [19], [20], [30] to the biharmonic equations. And the hypothesis $V(x) \ge 0$ in (v_2) could also be replaced by: there exists a constant ν such that

$$\nu = \inf_{u \in H^2(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(x)|u|^2)\,dx}{\displaystyle\int_{\mathbb{R}^N} u^2\,dx} > 0,$$

which is proposed in [30].

Now we state our main ideas for the proof of the results. To prove Theorem 1.1, assuming that $V(x)$ is a positive constant, we look for a minimizer of the reduced functional restricted to a Pohožaev–Nehari manifold M (the definition of M can be found in Section 3 below), which is obtained by combining a Pohožaev type identity and the usual Nehari manifold. If $V(x)$ is not a constant, it is difficult to get the boundedness of any (PS) sequence. Inspired by [14], [19], [20], [30], we use Jeanjean's result [12] to construct a special bounded (PS) sequence. At last, in order to obtain the compactness of bounded (PS) sequence of energy functional, we use concentration-compactness principle of Lions [17] to establish a new global compactness lemma, which help us to complete the proof of Theorem 1.3. Moreover, this paper gives a unified method to deal with the existence of ground state solution to the problem (EQ) for all $3 \le p < 2^* - 1$.

The paper is as follows. In Section 2, we introduce a variational setting and present some preliminaries results. In Section 3, we consider the constant potential case and give the proof Theorem 1.1. In Section 4, we will establish a global compactness lemma and give the proof of Theorem 1.3.

Notations. Throughout this paper, we denote the norms of u in $D_0^{2,2}(\mathbb{R}^N)$ and $L^s(\mathbb{R}^N)(1 \le s \le \infty)$ by

$$\|u\|^2_{D_0^{2,2}} := \int_{\mathbb{R}^N} |\triangle u|^2\,dx \quad \text{and} \quad \|u\|_s := \bigg(\int_{\mathbb{R}^N} |u|^s\,dx\bigg)^{1/s},$$

respectively. Here $D_0^{2,2}(\mathbb{R}^N)$ is the completion of the space $C_0^\infty(\mathbb{R}^N)$ under the above norm. We also have to use the notations the best Sobolev constant

$$S := \inf_{u \in D_0^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\triangle u|^2 dx}{\bigg(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\,dx\bigg)^{2/2^*}}$$

and

$$B_R(z) := \{x \in \mathbb{R}^N : |x - z| \le R\}.$$

We use C_i $(i = 1, 2, \ldots)$ to denote various positive constants.

2. Preliminaries

Let

$$E = \bigg\{u \in H^{(2)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(x)|u|^2)\,dx < +\infty\bigg\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\triangle u \triangle v + \nabla u \nabla v + V(x)uv)\,dx, \qquad \|u\| = \langle u, u\rangle^{1/2}.$$

Weak solutions to (EQ) correspond to critical points of the following functional

$$I_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 dx + |\nabla u|^2 + V(x)|u|^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx. \tag{2.1}$$

This is a well-defined C^1-functional whose derivative is given by, for all $u, v \in E$,

$$\langle I_V'(u), v\rangle = \int_{\mathbb{R}^N} (\triangle u \triangle v + \nabla u \nabla v + V(x)uv)\,dx - \int_{\mathbb{R}^N} |u|^{p-1}uv\,dx. \tag{2.2}$$

Now we give some preliminary results, which plays a significant role in the proof of our results.

PROPOSITION 2.1 (see [12]). *Let $(X, \|\cdot\|)$ be a Banach space and $\mathrm{h} \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$I_\delta(u) = A(u) - \delta B(u), \quad \delta \in \mathrm{h}$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\delta = \inf_{\gamma\in\Gamma_\delta} \max_{t\in[0,1]} I_\delta(\gamma(t)) > \max\{I_\delta(v_1), I_\delta(v_2)\}, \quad \textit{for all } \delta \in \mathrm{h},$$

where $\Gamma_\delta = \{\gamma \in C([0,1], X) : \gamma(0) = v_1,\ \gamma(1) = v_2\}$. Then, for almost every $\delta \in \mathrm{h}$, there is a sequence $\{u_n\} \subset X$ such that

(a) *$\{u_n\}$ is bounded,*
(b) *$I_\delta(u_n) \to c_\delta$,*
(c) *$I_\delta'(u_n) \to 0$ in the dual X^{-1} of X.*

Moreover, the map $\delta \to c_\delta$ is continuous from the left.

We introduce a Pohožaev identity for the problem (EQ).

LEMMA 2.2. *Under the assumptions* (v_1)–(v_2)*, let u be a solution of* (EQ) *in $H^2(\mathbb{R}^N)$, then*

$$\begin{aligned}\frac{N-4}{2}\int_{\mathbb{R}^N}|\triangle u|^2\,dx + \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}V(x)|u|^2\,dx \\ + \frac{1}{2}\int_{\mathbb{R}^N}\langle\nabla V(x), x\rangle|u|^2\,dx - \frac{N}{P+1}\int_{\mathbb{R}^N}|u|^{P+1}\,dx = 0.\end{aligned} \tag{2.3}$$

The proof is standard, so we omit it (see [9]).

If $V(x)$ is a positive constant V, the Pohožaev identity can be rewritten as follows:

$$\begin{aligned}\frac{N-4}{2}\int_{\mathbb{R}^N}|\triangle u|^2\,dx + \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx \\ + \frac{N}{2}\int_{\mathbb{R}^N}V|u|^2\,dx - \frac{N}{P+1}\int_{\mathbb{R}^N}|u|^{P+1}\,dx = 0.\end{aligned} \tag{2.4}$$

The following concentration-compactness principle is due to P. Lions (see [17]).

LEMMA 2.3 ([17], [25]). *Let $R > 0$, $2 \le q < 2^* = 2N/(N-4)$ with $N > 4$. Assume $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u|^q\,dx = 0,$$

then $u_n \to 0$ *in* $L^s(\mathbb{R}^N)$ *for* $s \in (2, 2^*)$.

Lemma 2.4. *For all* $t \geq 0$, *define function*

$$g(t)\colon \mathbb{R}^+ \to \mathbb{R}, \qquad g(t) = C_1 t^{N-2} + C_2 t^N + C_3 t^{N+2} - C_4 t^{N+p+1},$$

where $C_1, \ldots, C_4$ *are positive constants, and* $N > 4$. *Then, we claim that* g *has a unique critical point which corresponds to its maximum.*

Proof. We borrow an idea from [22]. Indeed, since $N > 4$ and $2 < p < 2^* - 1$, $g(t) \to -\infty$ as $t \to +\infty$ and $g(t)$ is positive for $t > 0$ small. Consequently, g has a maximum. We now show that this is the only critical point of g. Consider some derivatives of g:

$$\begin{aligned} g'(t) = {} & C_1(N-2)t^{N-3} + C_2 N t^{N-1} + C_3(N+2)t^{N+1} - C_4(N+p+1)t^{N+p}, \\ g''(t) = {} & C_1(N-2)(N-3)t^{N-4} + C_2 N(N-1)t^{N-2} \\ & + C_3(N+2)(N+1)t^N - C_4(N+p+1)(N+p)t^{N+p-1}, \\ g'''(t) = {} & C_1(N-2)(N-3)(N-4)t^{N-5} \\ & + C_2 N(N-1)(N-2)t^{N-3} + C_3(N+2)(N+1)N t^{N-1} \\ & - C_4(N+p+1)(N+p)(N+P-1)t^{N+p-2}. \end{aligned}$$

Note that $g'''(t) \to -\infty$ as $t \to +\infty$ and is positive for $t > 0$ small since $N > 4$ and $2 < p < 2^* - 1$. Then, there exists $t_1 > 0$ such that $g'''(t_1) = 0$ and $g'''(t) > 0$ for $t < t_1$. Then, for $t < t_1$, $g''(t)$ is increasing. Since $g''(0) = 0$, there exists at least $0 < t < t_1$ such that $g''(t) > 0$ and $g''(t)$ decreases, tending to $-\infty$ for $t > t_1$. Consequently, there exists $t_2 > t_1$ such that $g''(t_2) = 0$ and $g''(t) > 0$ for $t < t_2$. Repeating the above argument, there exists $t_3 > t_2$ such that $g'(t_3) = 0$ and $g'(t) > 0$ for $t < t_3$. Therefore, t_3 is the unique critical point of $g(t)$, which implies the lemma. □

3. Constant potential case

In this section we will assume that $V(\,\cdot\,)$ is a positive constant V and give the proof of Theorem 1.1. That is to say that the purpose of this section is to show that the ground state solution can be obtained on a suitable manifold. In view of $V(x) = V$, the functional I_V is reduced to be

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V|u|^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx. \tag{3.1}$$

Next we try to use the constrained minimization on a suitable manifold to prove Theorem 1.1. To solve the difficulty is to prove the boundedness of the minimizing sequence on usual Nehari manifold. Inspired by [14], [22], we combine the Nehari manifold and the corresponding Pohožaev type identity. In fact, we introduce the following manifold

$$M := \{u \in W^{1,P}(\mathbb{R}^N) \setminus \{0\} : G(u) = 0\},$$

where

$$\begin{aligned}G(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{N+2}{2}\int_{\mathbb{R}^N} V|u|^p\,dx \\ - \frac{p+1+N}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx = \langle I'(u), u\rangle + P(u),\end{aligned}$$

here $P(u)$ is defined by (2.4).

Using an idea from [14], [19], [20], [22], [30], we can establish the following properties of M.

LEMMA 3.1.

(a) *Assume* $2 < p < 2^* - 1$, *for any* $u \in H^2(\mathbb{R}^N) \setminus \{0\}$, *then there exists a unique number* $t_0 = t_0(u) > 0$ *such that* $u_{t_0} = t_0 u(x/t_0) \in M$. *Moreover,* $I(u_{t_0}) = \max_{t\geq 0} I(u_t)$.

(b) $0 \notin \partial \mathrm{M}$ *and* $\inf I|_{\mathrm{M}} > 0$.

(c) *For any* $u \in \mathrm{M}$, $\mathrm{G}'(u) \neq 0$, *that is* M *is a* C^1 *manifold.*

(d) M *is a nature constraint of* I. *That is every critical point of* $I|_M$ *is a critical point of* I.

PROOF. (a) For any $u \in H^2(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$, set $u_t(x) = tu(x/t)$. Consider

$$\begin{aligned}\gamma(t) = I\bigg(tu\bigg(\frac{x}{t}\bigg)\bigg) = \frac{1}{2}t^{N-2}\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \frac{1}{2}\,t^N\int_{\mathbb{R}^N} |\nabla u|^2\,dx \\ + \frac{1}{2}\,t^{N+2}\int_{\mathbb{R}^N} V|u|^2\,dx - \frac{1}{p+1}t^{N+p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx.\end{aligned}$$

It follows from Lemma 2.4 that $\gamma(t)$ has a unique critical point $t_0 > 0$ corresponding to its maximum. Then $\gamma'(t_0) = 0$ and $\gamma(t_0) = \max_{t\geq 0} I(u_t)$. Thus

$$\begin{aligned}\frac{N-2}{2}\,t_0^{N-3}\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \frac{N}{2}\,t_0^{N-1}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \\ + \frac{N+2}{2}\,t_0^{N+1}\int_{\mathbb{R}^N} V|u|^2\,dx - \frac{N+p+1}{p+1}\,t_0^{N+p}\int_{\mathbb{R}^N} |u|^{p+1}\,dx = 0,\end{aligned}$$

which implies $G(u_{t_0}) = 0$ and $u_{t_0} \in M$. (a) immediately follows.

(b) Clearly, for any $u \in M$, using the Sobolev embedding theorem, we have

$$\begin{aligned}0 = \frac{N-2}{2}\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{N+2}{2}\int_{\mathbb{R}^N} V|u|^2\,dx \\ - \frac{p+1+N}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx \geq \frac{N-2}{2}\,\|u\|^2 - \frac{C_5(p+1+N)}{p+1}\,\|u\|^{p+1},\end{aligned}$$

which means $0 \notin \partial M$ for $\|u\|$ small enough. This shows that M is complete. Take any $u \in M$, we then deduce that

$$\begin{aligned}(N+2)I(u) &= (N+2)I(u) - G(u)\\ &= 2\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{(p-1)}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx > 0.\end{aligned}$$

This shows that $\inf I|_M > 0$ since $p > 2$.

(c) Reasoning by contradiction, suppose that $G'(u) = 0$. In a weak sense, the equation can be written as

$$(N-2)\triangle^2 u - N\triangle u + (N+2)Vu - (N+p+1)|u|^{p-1}u = 0. \tag{3.2}$$

Then, the following Pohožaev identity holds:

$$\begin{aligned}\frac{(N-2)(N-4)}{2}\int_{\mathbb{R}^N} |\triangle u|^2\,dx + \frac{(N-2)N}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx&\\ + \frac{N(N+2)}{2}\int_{\mathbb{R}^N} V|u|^2\,dx - \frac{N(N+p+1)}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx &= 0.\end{aligned} \tag{3.3}$$

Let

$$\int_{\mathbb{R}^N} |\triangle u|^2\,dx = \alpha, \quad \int_{\mathbb{R}^N} |\nabla u|^2\,dx = \beta, \quad \int_{\mathbb{R}^N} V|u|^2\,dx = \gamma, \quad \int_{\mathbb{R}^N} |u|^{p+1}\,dx = \tau.$$

Then, we have the following identity:

$$\begin{cases} \dfrac{N-2}{2}\alpha + \dfrac{N}{2}\beta + \dfrac{N+2}{2}\gamma - \dfrac{p+1+N}{p+1}\tau = 0,\\ \dfrac{(N-4)(N-2)}{2}\alpha + \dfrac{N(N-2)}{2}\beta + \dfrac{N(N+2)}{2}\gamma - \dfrac{N(p+1+N)}{p+1}\tau = 0,\end{cases}$$

By simple computation, we get $\alpha = -N\beta/(2(N-2))$, which is impossible, since $\alpha \geq 0$, $\beta \geq 0$ and $N > 4$. The proof of (c) is complete.

(d) If $u \in M$ and $(I|_M)'(u) = 0$. Thanks to the Lagrange multiplier rule, there exists $\lambda \in R$ such that $I'(u) + \lambda P'(u) = 0$. We show that $\lambda = 0$. As above, in a weak sense, the equation $I'(u) + \lambda P'(u) = 0$ can be written as

$$\begin{aligned}&[1+\lambda(N-2)]\triangle^2 u - (1+\lambda N)\triangle u\\ &\qquad + [1+\lambda(N+2)]Vu - [1+\lambda(N+p+1)]|u|^{p-1}u = 0.\end{aligned} \tag{3.4}$$

Set

$$\frac{1}{2}\int_{\mathbb{R}^N} |\triangle u|^2\,dx = \alpha_1, \qquad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \beta_1,$$

$$\frac{1}{2}\int_{\mathbb{R}^N} V|u|^2\,dx = \gamma_1, \qquad \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx = \tau_1.$$

Arguing as above, we have

$$\begin{cases} (N-2)\alpha_1 + N\beta_1 + (N+2)\gamma_1 - (p+1+N)\tau_1 = 0, \\ (N-4)[1+\lambda(N-2)]\alpha_1 + (N-2)(1+\lambda N)\beta_1 \\ \qquad\qquad +N[1+\lambda(N+2)]\gamma_1 - N[1+\lambda(p+1+N)]\tau_1 = 0, \\ 2[1+\lambda(N-2)]\alpha_1 + 2(1+\lambda N)\beta_1 \\ \qquad\qquad +2[1+\lambda(N+2)]\gamma_1 - (p+1)[1+\lambda(p+1+N)]\tau_1 = 0. \end{cases}$$

The first equation comes from $u \in M$. The second one is the Pohožaev identity applied to 3.4. The third one holds since $I'(u) + \lambda P'(u) = 0$. By computation, we get

$$\lambda[4(N-2)\alpha_1 + 2N\beta_1 + (p-1)(p+1+N)\tau_1] = 0.$$

If $\lambda \neq 0$, $\alpha_1 = -[(p-1)(p+1+N)\tau_1 + 2N\beta_1]/[4(N-2)]$, which is also impossible since $\alpha_1 > 0$, $\beta_1 > 0$, $\tau_1 > 0$, $p > 2$ and $N > 4$. Thus we obtain that $\lambda = 0$, which completes the proof of (d). □

Lemma 3.2. *If $N > 4$ and $2 < p < 2^* - 1$, then there exists a minimizer u of $\inf_M I$. Moreover, $I'(u) = 0$ in $H^2(\mathbb{R}^N)$.*

Proof. The proof is inspired by [22]. For the reader's convenience, we sketch it here briefly. The main strategy, based on three steps, is the following.

Step 1. Let $\{u_n\} \subset M$ be a sequence such that $I(u_n) \to \inf\limits_M I$. Next we show the boundedness of $\{u_n\}$. Indeed, by using $u_n \in M$ and (3.1), one has

$$\begin{aligned} (p+1+N)I(u_n) &= \frac{p+3}{2}\int_{\mathbb{R}^N} |\triangle u_n|^2\,dx \\ &\quad + \frac{p+1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \frac{p-1}{2}\int_{\mathbb{R}^N} V|u_n|^2\,dx \geq \frac{p-1}{2}\,\|u_n\|^2. \end{aligned}$$

Combining with the fact $I(u_n) \to \inf_M I$, we conclude the boundedness of $\{u_n\}$.

Step 2. Suppose, passing to a subsequence, that $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$. We will prove that $u \in M$ and $u_n \to u$ in $H^2(\mathbb{R}^N)$. Thus $I|_M$ attains its minimum at u. Denote

$$\begin{aligned} a_n &= \int_{\mathbb{R}^N} |\triangle u_n|^2\,dx, & b_n &= \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx, \\ c_n &= \int_{\mathbb{R}^N} V|u_n|^2\,dx, & d_n &= \int_{\mathbb{R}^N} |u_n|^{p+1}\,dx, \\ a &= \int_{\mathbb{R}^N} |\triangle u|^2\,dx, & b &= \int_{\mathbb{R}^N} |\nabla u|^2\,dx, \\ c &= \int_{\mathbb{R}^N} V|u|^2\,dx, & d &= \int_{\mathbb{R}^N} |u|^{p+1}\,dx, \end{aligned}$$

$$\overline{a} = \lim_{n\to\infty} a_n, \qquad \overline{b} = \lim_{n\to\infty} b_n,$$
$$\overline{c} = \lim_{n\to\infty} c_n, \qquad \overline{d} = \lim_{n\to\infty} d_n.$$

Passing to an appropriate subsequence, we can suppose that the above limits exist. Using the compactness of the embedding $H_r^2(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, one has $\overline{d} = d$. Considering the weak convergence, we have that $a \le \overline{a}$, $b \le \overline{b}$ and $c \le \overline{c}$. We will show that the previous inequalities are equalities.

Suppose on the contrary that $a + b + c < \overline{a} + \overline{b} + \overline{c}$. Since $I(u_n) \to \inf I|_M$ and $G(u_n) = 0$, one has

$$\begin{cases} \dfrac{1}{2}\,\overline{a} + \dfrac{1}{2}\,\overline{b} + \dfrac{1}{2}\,\overline{c} - \dfrac{1}{p+1}\,\overline{d} = \inf I|_M, \\ \dfrac{N-2}{2}\,\overline{a} + \dfrac{N}{2}\,\overline{b} + \dfrac{N+2}{2}\,\overline{c} - \dfrac{p+1+N}{p+1}\,\overline{d} = 0. \end{cases} \tag{3.5}$$

(b) of Lemma 3.2 means $\overline{a} + \overline{b} + \overline{c} > \varepsilon$, where $\varepsilon > 0$ is a constant. Consider the second equation of (3.4), we obtain that $d = \overline{d} > 0$, which means that u cannot be identically equal to zero. As a result, $a > 0$, $b > 0$, $c > 0$. Define

$$\overline{g}(t) = \frac{1}{2}\,\overline{a}t^{N-2} + \frac{1}{2}\,\overline{b}t^{N} + \frac{1}{2}\,\overline{c}t^{N+2} - \frac{p+1+N}{p+1}\,\overline{d}t^{N+p+1},$$

$$g(t) = \frac{1}{2}\,at^{N-2} + \frac{1}{2}\,bt^{N} + \frac{1}{2}\,ct^{N+2} - \frac{p+1+N}{p+1}\,dt^{N+p+1}.$$

It follows from Lemma 2.4 that $\overline{g}(t)$ and $g(t)$ have a unique critical point, corresponding to their maximum. By (3.4), we deduce that $\max \overline{g} = \inf I|_M$ as $t = 1$. Since $a + b + c < \overline{a} + \overline{b} + \overline{c}$, then, for all $t > 0$, $g(t) < \overline{g}(t)$. Suppose that t_0 be the point where the maximum of g is achieved. Thus, $g(t_0) \le \max \overline{g} = \inf I|_M$ and $g'(t_0) = 0$. Defining $u_0 = t_0 u(x/t_0)$, one has

$$\begin{aligned} I(u_0) &= \frac{1}{2}\,at_0^{N-2} + \frac{1}{2}\,bt_0^{N} + \frac{1}{2}\,ct_0^{N+2} - \frac{p+1+N}{p+1}\,dt_0^{N+p+1} = g(t_0) < \inf I|_M, \\ G(u_0) &= \frac{N-2}{2}\,at_0^{N-2} + \frac{N}{2}\,bt_0^{N} \\ &\quad + \frac{N+2}{2}\,ct_0^{N+2} - \frac{p+1+N}{p+1}\,dt_0^{N+p+1} = t_0 g'(t_0) = 0. \end{aligned}$$

This means that $u_0 \in M$ and $I(u_0) < \inf I|_M$, a contradiction. Thus $a + b + c = \overline{a} + \overline{b} + \overline{c}$, which implies that $u_n \to u$ and then $u \in M$.

Step 3. We now show that $I'(u) = 0$. Thanks to the Lagrange multiplier rule, there exists $\lambda \in R$ so that $I'(u) + \lambda G'(u) = 0$. Just as the proof of (d) in Lemma 3.1, we can prove that $\lambda = 0$. Thus, $I'(u) = 0$. □

Proof of Theorem 1.1. If $N > 4$ and $2 < p < 2_* - 1$, it follows from Lemma 3.2 that there exists $u \in M$ such that $I(u) = \inf I|_M$ and $I'(u) = 0$.

Then u is a positive critical point of $I|_M$, hence by Lemma 3.1, we see that u is a positive ground state solution of (EQ) with $V(x) = V$. □

4. Nonconstant potential case

In this section, the main goal is to show the proof of Theorem 1.3. By Proposition 2.1, we consider the functional $I_{V,\delta} : E \to \mathrm{R}$ defined by

$$
\begin{aligned}
(4.1)\quad I_{V,\delta}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(x)u^2)\,dx& \\
- \frac{\delta}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx := A(u) - \delta B(u)&
\end{aligned}
$$

for $u \in E$, where

$$
A(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(x)|u|^2)\,dx, \qquad B(u) = \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx,
$$

for $\delta \in [1/2, 1]$. It is clear that this functional is of C^1-class and, for every $u, v \in E$,

$$
(4.2)\quad \langle I'_{V,\delta}(u), v\rangle = \int_{\mathbb{R}^N} (\triangle u \triangle v + \nabla u \nabla v + V(x)uv)\,dx - \delta\int_{\mathbb{R}^N} |u|^{p-1}uv\,dx.
$$

We also need to consider the associated limit problem

$$
(\mathrm{EQ})_\infty \qquad \triangle^2 u - \triangle u + V(\infty)u = \delta|u|^{p-1}u, \quad u \in H^2(\mathbb{R}^N).
$$

It is clear that system $(\mathrm{EQ})_\infty$ is the Euler–Lagrange equations of the functional $I_{\infty,\delta} \colon E \to \mathrm{R}$ defined by

$$
(4.3)\quad I_{\infty,\delta}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(\infty)|u|^2)\,dx - \frac{\delta}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx.
$$

The following lemma ensures that $I_{V,\delta}$ has the mountain pass geometry with the corresponding mountain pass level denoted by $c_{V,\delta}$.

LEMMA 4.1. *Suppose that* (v_1)–(v_2). *If* $p \in (2, 2^* - 1)$ *and* $N > 4$, *then*

(a) *there exists a* $v \in E \setminus \{0\}$ *such that* $I_{V,\delta}(v) < 0$ *for all* $\delta \in [1/2, 1]$.
(b) *for all* $\delta \in [1/2, 1]$

$$
c_{V,\delta} = \inf \max_{\gamma\in\Gamma,\, s\in[0,1]} I_{V,\delta}(\gamma(s)) > \max\{I_{V,\delta}(0), I_{V,\delta}(v)\},
$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ \gamma(1) = v\}$.

PROOF. (a) It follows from (v_2) that

$$
\begin{aligned}
I_{V,\delta}(u) \le I_{\infty,1/2}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(\infty)|u|^2)\,dx& \\
- \frac{1}{2(p+1)}\int_{\mathbb{R}^N} |u|^{p+1}\,dx.&
\end{aligned}
$$

For $u \in E \setminus \{0\}$ fixed, one has

$$I_{\infty,1/2}\bigg(tu\bigg(\frac{x}{t}\bigg)\bigg) = \frac{1}{2}\,t^{N-2}\int_{\mathbb{R}^N}|\triangle u|^2\,dx + \frac{1}{2}\,t^N\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{1}{2}\,t^{N+2}\int_{\mathbb{R}^N}V(\infty)u^2\,dx - \frac{1}{2(p+1)}\,t^{N+p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx.$$

Since $p > 2$, we deduce that $I_{\infty,1/2}(tu(x/t)) \to -\infty$ as $t \to +\infty$. Taking $v = tu(x/t)$, for t large, we have $I_{V,\delta}(v) \leq I_{\infty,1/2}(v) < 0$, which implies (a).

(b) Using the Sobolev embedding theorem, we get

$$I_{V,\delta}(u) \geq \frac{1}{2}\,\|u\|^2 - \frac{C_6}{p+1}\,\|u\|^{p+1}.$$

Then, we deduce that $I_{V,\delta}(u)$ has a strict local minimum in 0 and $c_{V,\delta} > 0$.

Let us introduce the following manifold (see Lemma 3.1 for the details):

$$M_{\infty,\delta} = \{u \in H^2(\mathbb{R}^N) \setminus \{0\} : G_{\infty,\delta}(u) = 0\},$$

where

$$G_{\infty,\delta} = \frac{N-4}{2}\int_{\mathbb{R}^N}|\triangle u|^2\,dx + \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}V(\infty)|u|^2\,dx - \frac{N}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx.$$

Set $m_{\infty,\delta} := \inf\limits_{u\in M_{\infty,\delta}} I_{\infty,\delta}(u)$. According to the above, $M_{\infty,\delta}$ has some similar properties to those of the manifold M, such as containing all the nontrivial critical points of $I_{\infty,\delta}$ and the conclusion which is similar to Lemma 3.1 and the following lemma.

LEMMA 4.2. *If $2 < p < 2^* - 1$, $N > 4$ and $\delta \in [1/2,1]$, $m_{\infty,\delta}$ is obtained at some $u_{\infty,\delta} \in M_{\infty,\delta}$. Moreover, $I'_{\infty,\delta}(u_{\infty,\delta}) = 0$ and*

$$I_{\infty,\delta}(u_{\infty,\delta}) = m_{\infty,\delta} = \inf\{I_{\infty,\delta}(u) : u \neq 0,\ I'_{\infty,\delta}(u) = 0\}.$$

The proof is similar to that of Theorem 1.1 and is omitted here.

LEMMA 4.3. *Suppose that* (v_1)–(v_2) *hold and $2 < p < 2^* - 1$ and $N > 4$. Then $c_{V,\delta} < m_{\infty,\delta}$ for any $\delta \in [1/2,1]$.*

PROOF. Assume that $V(x) \not\equiv V(\infty)$. We get from Lemma 3.1 that

$$I_{\infty,\delta}(u_{\infty,\delta}) = \max_{t>0} I_{\infty,\delta}\bigg(tu\bigg(\frac{x}{t}\bigg)\bigg),$$

where $u_{\infty,\delta}$ is minimizer of $m_{\infty,\delta}$. Thus by choosing $v = tu_{\infty,\delta}(x/t)$ for t large in Lemma 4.1, we have

$$c_{V,\delta} \leq \max_{t>0} I_{V,\delta}\bigg(tu_{\infty,\delta}\bigg(\frac{x}{t}\bigg)\bigg) < \max_{t>0} I_{\infty,\delta}\bigg(tu_{\infty,\delta}\bigg(\frac{x}{t}\bigg)\bigg) = I_{\infty,\delta}(u_{\infty,\delta}) = m_{\infty,\delta}.$$

$\square$

To prove that the functional $I_{V,\delta}$ satisfies $(\mathrm{PS})_{c_{V,\delta}}$ for almost every $\delta \in [1/2,1]$, we have to prove the following global compactness lemma. It is inspired by [14], [19], [20], [30]. It is basic in search for critical points of I_V.

Lemma 4.4. *Suppose that* (v_1)–(v_2) *hold and* $3 \le p < 2^* - 1$, $N > 4$. *For every* $\delta \in [1/2,1]$, *let* $\{u_n\}$ *be a bounded* $(\mathrm{PS})_{c_{V,\delta}}$ *sequence for* $I_{V,\delta}$. *Then there exist a subsequence of* $\{u_n\}$, *still denote* $\{u_n\}$, u_0 *and integer* $\eta \in \mathbb{N} \cup \{0\}$, *sequence* $\{y_n^j\}$, $w^j \subset H^2(\mathbb{R}^N)$ *for* $1 \le j \le \eta$ *such that*

(a) $u_n \rightharpoonup u_0$ *with* $I'_{V,\delta}(u_0) = 0$.
(b) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \ne j, n \to +\infty$.
(c) $w^j \ne 0$ *and* $I'_{\infty,\delta}(w^j) = 0$ *for* $1 \le j \le \eta$.
(d) $\Big\| u_n - u_0 - \sum\limits_{j=1}^{\eta} w^j(\,\cdot\, - y_n^j)\Big\| \to 0$.
(e) $I_{V,\delta} \to I_{V,\delta}(u_0) + \sum\limits_{j=1}^{\eta} I_{\infty,\delta}(w^j)$.

Here we agree that in the case $\eta = 0$ *the above holds without* w^j *and* $\{y_n^j\}$.

Proof. *Step 1.* We obtain from the fact the boundedness of $\{u_n\}$ that, up to subsequence, there exists u_0 such that $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le r < 2^*$ and $u_n \to u_0$ almost everywhere in $\mathbb{R}^N$. Now we prove $I'_{V,\delta}(u_0) = 0$. In fact, it suffices to show that $\langle I'_{V,\delta}(u_0), \varphi\rangle = 0$ for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then, by Hölder's inequality, for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have that

$$
\begin{aligned}
(4.4)\qquad & \bigg| \int_{\mathbb{R}^N} (|u_n|^{s-1}u_n - |u_0|^{s-1}u_0)\varphi\,dx \bigg| \\
&\le \int_{\mathbb{R}^N} |u_n|^{s-1}|u_n - u_0|\varphi\,dx + \int_{\mathbb{R}^N} (|u_n|^{s-1} - |u_0|^{s-1})|u_0\varphi|\,dx \\
&\le \|\varphi\|_\infty \bigg(\int_{\operatorname{supp}\phi} |u_n - u_0|^s \bigg)^{1/s} \|u_n\|_s^{s-1} \\
&\quad + C\|\varphi\|_\infty \|u_0\|_s \bigg(\int_{\operatorname{supp}\phi} |u_n - u_0|^s \bigg)^{(s-1)/s} \to 0,
\end{aligned}
$$

as $n \to \infty$ for $s > 2$. Since $u_n \rightharpoonup u_0$ in E, we get

$$
(4.5)\qquad \langle u_n - u_0, \varphi\rangle \to 0 \quad \text{as } n \to \infty.
$$

Using (4.4)–(4.5), one has

$$
\begin{aligned}
(4.6)\quad & \langle I'_{V,\delta}(u_n), \varphi\rangle - \langle I'_{V,\delta}(u_0), \varphi\rangle \\
&\qquad = \langle u_n - u_0, \varphi\rangle - \delta \int_{\mathbb{R}^N} (|u_n|^{p-1}u_n - |u_0|^{p-1}u_0)\varphi\,dx \to 0.
\end{aligned}
$$

Thus recalling that $I'_{V,\delta}(u_0) = 0$.

Step 2. Next we show that $I_{V,\delta}(u_0) \geq 0$. Define

$$a_1 = \int_{\mathbb{R}^N} |\triangle u_0|^2\,dx, \qquad b_1 = \int_{\mathbb{R}^N} |\nabla u_0|^2\,dx,$$
$$c_1 = \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx, \qquad \overline{c}_1 = \int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_0|^2\,dx,$$
$$d_1 = \int_{\mathbb{R}^N} |u_0|^{p+1}\,dx.$$

Then, by the definition of $I_{V,\delta}(u_0)$, Lemma 2.2, and $\langle I'_{V,\delta}(u_0), u_0\rangle = 0$, we get

$$\begin{cases} \dfrac{1}{2}\,a_1 + \dfrac{1}{2}\,b_1 + \dfrac{1}{2}\,c_1 - \dfrac{\delta}{p+1}\,c_1 = I_{V,\delta}(u_0), \\ \dfrac{N-4}{2}\,a_1 + \dfrac{N-2}{2}\,b_1 + \dfrac{N}{2}\,c_1 + \dfrac{1}{2}\,\overline{c}_1 - \dfrac{\delta N}{p+1}\,d_1 = 0, \\ a_1 + b_1 + c_1 - \delta d_1 = 0. \end{cases} \tag{4.7}$$

From these relations, we deduce that

$$N I_{V,\delta}(u_0) = b_1 + \frac{1}{2}(4c_1 + \overline{c}_1) + \frac{p(N-2)-(N+2)}{p+1} d_1. \tag{4.8}$$

Since $3 \leq p < 2^* - 1$ and $N > 4$, we have

$$p(N-2) - (N+2) \geq 3(N-2) - (N+2) = 2(N-4) > 0. \tag{4.9}$$

It follows from (4.8), (4.9) and (v_2) that $I_{V,\delta}(u_0) \geq 0$.

Step 3. Set $v_n^1 = u_n - u_0$, then we get $v_n^1 \rightharpoonup 0$ in E. Let us define

$$\mu = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |v_n^1|^2\,dx.$$

Vanishing. If $\mu = 0$, then it follows from Lemma 2.3 that $v_n^1 \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$. By similar computation as (4.6), we have

$$\|v_n^1\|^2 = \langle I'_{V,\delta}(u_n), v_n^1\rangle - \langle I'_{V,\delta}(u_0), v_n^1\rangle \to 0,$$

which means $\|v_n^1\| \to 0$ as $n \to \infty$.

Non-vanishing. If $\mu > 0$, we can find sequence $\{y_n^1\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(0)} |\widetilde{v}_n^1|^2\,dx = \int_{B_1(y_n)} |v_n^1|^2\,dx > \frac{\mu}{2} > 0,$$

where $\widetilde{v}_n^1 = v_n^1(\,\cdot\, + y_n^1)$. Note that $\|\widetilde{v}_n^1\| = \|v_n^1(\,\cdot\, + y_n^1)\|$, we see that $\{\widetilde{v}_n^1\}$ is bounded. Going if necessary to a subsequence, we have for a $w^1 \in E$ such that $\widetilde{v}_n^1 \rightharpoonup w^1$ in E, $\widetilde{v}_n^1 \to w^1$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ and $\widetilde{v}_n^1 \to w^1$ almost everywhere in $\mathbb{R}^N$. Since $\int_{B_1(0)} |\widetilde{v}_n^1|^2\,dx > \mu/2$, we see that $w^1 \neq 0$. Moreover, $v_n^1 \rightharpoonup 0$ in E implies that $\{y_n^1\}$ must be unbounded. Consequently, we may assume that $|y_n^1| \to +\infty$.

Now we will prove that $I'_{\infty,\delta}(w^1) = 0$. Similar to the proof of Step 1, for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, it suffices to show that $\langle I'_{\infty,\delta}(\widetilde{v}_n^1), \varphi \rangle \to 0$. By ($\mathrm{v}_2$) and $|y_n^1| \to +\infty$, for n large enough and any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \big(V\big(x + y_n^1\big) - V(\infty)\big)\widetilde{v}_n^1 \varphi \, dx \to 0. \tag{4.10}$$

Since $v_n^1 \rightharpoonup 0$ in E, one has that $\langle I'_{V,\delta}(v_n^1), \varphi(\,\cdot\, - y_n^1) \rangle \to 0$. That is to say that

$$\begin{aligned}\int_{\mathbb{R}^N} \triangle \widetilde{v}_n^1 \triangle \varphi \, dx &+ \int_{\mathbb{R}^N} \nabla \widetilde{v}_n^1 \nabla \varphi \, dx \\ &+ \int_{\mathbb{R}^N} V(x + y_n^1)\widetilde{v}_n^1 \varphi \, dx - \delta \int_{\mathbb{R}^N} |\widetilde{v}_n^1|^{p-1}\widetilde{v}_n^1 \varphi \, dx \to 0\end{aligned} \tag{4.11}$$

as $n \to \infty$. Thus, combining (4.10) with (4.11), one has $\langle I'_{\infty,\delta}(\widetilde{v}_n^1), \varphi \rangle \to 0$. Therefore, $I'_{\infty,\delta}(w^1) = 0$.

Now we show that

$$I_{V,\delta}(u_n) - I_{V,\delta}(u_0) - I_{\infty,\delta}(u_n - u_0) \to 0. \tag{4.12}$$

Indeed, by the Brezis–Lieb lemma, we have

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1), \quad \|v_n^1\|_{p+1}^{p+1} = \|u_n\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1} + o(1). \tag{4.13}$$

In view of (v_2) and the Sobolev inequality, we have

$$\int_{\mathbb{R}^N} (V(x) - V(\infty))|u_n - u_0|^2 \, dx \to 0. \tag{4.14}$$

And we deduce from (4.1) and (4.3) that

$$\begin{aligned} I_{V,\delta}(u_n) \; &- I_{V,\delta}(u_0) - I_{\infty,\delta}(u_n - u_0) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (|\triangle u_n|^2 - |\triangle u_0|^2 - |\triangle(u_n - u_0)|^2) \, dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_0|^2 - |\nabla(u_n - u_0)|^2) \, dx \\ &\quad + \frac{1}{2} \bigg[\int_{\mathbb{R}^N} V(x)(|u_n|^2 - |u_0|^2) \, dx - \int_{\mathbb{R}^N} V(\infty)|u_n - u_0|^2) \, dx \bigg] \\ &\quad - \frac{\delta}{p+1} \int_{\mathbb{R}^N} (|u_n|^{p+1} - |u_0|^{p+1} - |u_n - u_0|^{p+1}) \, dx. \end{aligned} \tag{4.15}$$

It follows from (4.13)–(4.15) that (4.12) holds.

Step 4. Set $v_n^2 = v_n^1 - w^1(\,\cdot\, - y_n)$, then $v_n^2 \rightharpoonup 0$ in E. We get from Brezis–Lieb Lemma again that

$$\begin{aligned} \|\triangle v_n^2\|_2^2 &= \|\triangle u_n\|_2^2 - \|\triangle u_0\|_2^2 - \|\triangle w^1(\,\cdot\, - y_n)\|_2^2 + o(1), \\ \|\nabla v_n^2\|_2^2 &= \|\nabla u_n\|_2^2 - \|\nabla u_0\|_2^2 - \|\nabla w^1(\,\cdot\, - y_n)\|_2^2 + o(1), \\ \|v_n^2\|_{p+1}^{p+1} &= \|u_n\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1} - \|w^1(\,\cdot\, - y_n)\|_{p+1}^{p+1} + o(1), \end{aligned} \tag{4.16}$$

$$\int_{\mathbb{R}^N} V(x)|v_n^2|^2\,dx = \int_{\mathbb{R}^N} V(x)|u_n|^2\,dx - \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx - \int_{\mathbb{R}^N} V(x)|w^1(\,\cdot\, - y_n)|^2\,dx + o(1). \tag{4.17}$$

Using (4.16)–(4.17), we can similarly deduce that

$$\begin{cases} I_{V,\delta}(v_n^2) = I_{V,\delta}(u_n) - I_{V,\delta}(u_0) - I_{\infty,\delta}(w^1) + o(1), \\ I_{\infty,\delta}(v_n^2) = I_{V,\delta}(v_n^1) - I_{\infty,\delta}(w^1) + o(1), \\ \langle I'_{V,\delta}(v_n^2), v_n^2\rangle = \langle I'_{\infty,\delta}(u_n), u_n\rangle - \langle I'_{V,\delta}(u_0), u_0\rangle \\ \qquad\qquad - \langle I'_{\infty,\delta}(w^1), w^1\rangle + o(1) = o(1). \end{cases} \tag{4.18}$$

Using (4.12) and (4.18), one has

$$\begin{aligned} I_{V,\delta}(u_n) &= I_{V,\delta}(u_0) + I_{\infty,\delta}(v_n^1) + o(1) \\ &= I_{V,\delta}(u_0) + I_{\infty,\delta}(v_n^2) + I_{\infty,\delta}(w^1) + o(1). \end{aligned} \tag{4.19}$$

It follows from (b) of Lemma 3.1 that $I_{\infty,\delta}(w^1) \geq 0$. Then, we get from (4.8) that

$$I_{V,\delta}(v_n^2) = c_{V,\delta} - I_{V,\delta}(u_0) - I_{\infty,\delta}(w^1) + o(1) \leq c_{V,\delta}. \tag{4.20}$$

Repeating the same type of arguments explored in Step 3, set

$$\mu_1 = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |v_n^2|^2\,dx.$$

If vanishing occurs, then $\|v_n^2\| \to 0$ in E. Thus Lemma 4.4 holds with $j = 1$. If v_n^2 is non-vanishing, then there exists a sequence $\{y_n^2\}$ and $w^2 \in E$ such that $\widetilde{v}_n^2 = v_n^2(\,\cdot\, + y_n^2) \rightharpoonup w^2$ in E and $I'_{\infty,\delta}(w^2) = 0$. Furthermore, $v_n^2 \rightharpoonup 0$ in E means that $|y_n^2| \to +\infty$ and $|y_n^1 - y_n^2| \to +\infty$. By iterating this techniques we obtain $v_n^j = v_n^{j-1} - w^{j-1}$ with $j \geq 1$ such that

$$v_n^j \to w^j, \qquad I'_{\infty,\delta}(w^j) = 0,$$

and sequences $y_n^j \subset \mathbb{R}^N$ such that $|y_n^j| \to +\infty$ and $|y_n^i - y_n^j| \to +\infty$ if $i \neq j$ as $n \to \infty$, and using the properties of the weak convergence, we have

$$\begin{cases} \|u_n\|^2 - \|u_0\|^2 - \displaystyle\sum_{k=1}^{j-1} \|w^k(\,\cdot\, - y_n^k)\|^2 \\ \qquad = \bigg\| u_n - u_0 - \displaystyle\sum_{k=1}^{j-1} w^k(\,\cdot\, - y_n^k)\bigg\|^2 + o(1), \\ I_{V,\delta}(u_n) \to I_{V,\delta}(u_0) + \displaystyle\sum_{k=1}^{j-1} I_{\infty,\delta}(w^{k-1}) + I_{\infty,\delta}(v_n^j). \end{cases} \tag{4.21}$$

Since $\{u_n\}$is bounded in E, (4.21) implies that the iteration stops at some finite index $\eta + 1$. Therefore $v_n^{\eta+1} \to 0$ in E. And we can verify that conclusions (d) and (e) hold by (4.21). $\square$

LEMMA 4.5. *Assume that* (v_1)–(v_2) *hold,* $3 \le q < 2^*-1$ *and* $N > 4$. *Let* $\{u_n\}$ *be a bounded* $(\mathrm{PS})_{c_{V,\delta}}$ *sequence of* $I_{V,\delta}$. *Then there exists a nontrivial* $u_{V,\delta} \in E$ *such that* $I'_{V,\delta}(u_{V,\delta}) = 0$ *and* $I_{V,\delta}(u_{V,\delta}) = c_{V,\delta}$ *for almost all* $\delta \in [1/2, 1]$.

PROOF. For $\delta \in [1/2, 1]$, let $u_{\infty,\delta}$ be the minimizer of $m_{\infty,\delta}$. Then by Lemma 4.3, we have

$$c_{V,\delta} < m_{\infty,\delta}. \tag{4.22}$$

It follows from Lemma 4.4 that there exist $u_{V,\delta}$ and integer $\eta \in \mathbb{N} \cup \{0\}$, sequence $\{y_n^j\}$, $w^j \subset H^2(\mathbb{R}^N)$ for $1 \le j \le \eta$ such that

$$I'_{V,\delta}(u_{V,\delta}) = 0, \quad u_n \rightharpoonup u_{V,\delta}, \quad I_{V,\delta}(u_n) \to I_{V,\delta}(u_{V,\delta}) + \sum_{j=1}^{\eta} I_{\infty,\delta}(w^j), \tag{4.23}$$

where w^j is the critical point of $I_{\infty,\delta}$. Define

$$\begin{aligned} a_2 &= \int_{\mathbb{R}^N} |\triangle u_{V,\delta}|^2 \, dx, & b_2 &= \int_{\mathbb{R}^N} |\nabla u_{V,\delta}|^2 \, dx, \\ c_2 &= \int_{\mathbb{R}^N} V(x) |u_{V,\delta}|^2 \, dx, & \overline{c}_2 &= \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u_{V,\delta}|^2 \, dx, \\ d_2 &= \lambda \int_{\mathbb{R}^N} |u_{V,\delta}|^{p+1} \, dx. \end{aligned} \tag{4.24}$$

Then, using the definition of $I_{V,\delta}(u_{V,\delta})$, Lemma 2.2 and $\langle I'_{V,\delta}(u_{V,\delta}), u_{V,\delta} \rangle = 0$, one has

$$\begin{cases} \dfrac{1}{2} a_2 + \dfrac{1}{2} b_2 + \dfrac{1}{2} c_2 - \dfrac{\delta}{p+1} d_2 = I_{V,\delta}(u_{V,\delta}), \\ \dfrac{N-4}{2} a_2 + \dfrac{N-2}{2} b_2 + \dfrac{N}{2} c_2 + \dfrac{1}{2} \overline{c}_2 - \dfrac{\delta N}{p+1} d_2 = 0, \\ a_2 + b_2 + c_2 - \delta d_2 = 0. \end{cases}$$

Similar to the arguments of (4.8), we also have

$$N I_{V,\delta}(u_{V,\delta}) = b_2 + \frac{1}{2} (4c_2 + \overline{c}_2) + \frac{p(N-2) - (N+2)}{p+1} d_2 \ge 0$$

since $3 \le p < 2^* - 1$, $N > 4$ and (v_1). If $\eta \ne 0$, then by (4.23)

$$c_{V,\delta} = I_{V,\delta}(u_{V,\delta}) + \sum_{j=1}^{\eta} I_{\infty,\delta}(w^j) \ge m_{\infty,\delta},$$

which contradicts to (4.22). So $\eta = 0$, which implies $u_n \to u_{V,\delta}$ in E and $I_{V,\delta}(u_{V,\delta}) = c_{V,\delta}$. $\square$

PROOF OF THEOREM 1.3. The main strategy, based on two steps, is the following.

Step 1. It follows from Proposition 2.1, Lemma 4.1 that, for almost every $\delta \in [1/2,]$ there exists a bounded $(\mathrm{PS})_{c_{V,\delta}}$ sequence for $I_{V,\delta}$. Then Lemma 4.5 implies that there exists a nontrivial critical point $u_{V,\delta} \in E$ for $I_{V,\delta}$ and $I_{V,\delta}(u_{V,\delta}) = c_{V,\delta}$. Choose $\delta_n \to 1$ such that I_{V,δ_n} has a critical point u_{V,δ_n}, still denoted by $\{u_n\}$. Now we show that $\{u_n\}$ is bounded in E. Similar to (4.24), we denote that

$$\begin{aligned} a_n &= \int_{\mathbb{R}^N} |\triangle u_n|^2\,dx, & b_n &= \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx, \\ c_n &= \int_{\mathbb{R}^N} V(x)|u_n|^2\,dx, & \overline{c}_n &= \int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_n|^2\,dx, \\ d_n &= \lambda \int_{\mathbb{R}^N} |u_n|^{p+1} dx. \end{aligned}$$

Then

$$\begin{cases} \dfrac{1}{2}\,a_n + \dfrac{1}{2}\,b_n + \dfrac{1}{2}\,c_n - \dfrac{\delta_n}{p+1}\,d_n = c_{V,\delta_n}, \\ \dfrac{N-4}{2}\,a_n + \dfrac{N-2}{2}\,b_n + \dfrac{N}{2}\,c_n + \dfrac{1}{2}\,\overline{c}_n - \dfrac{\delta_n N}{p+1}\,d_n = 0, \\ a_n + b_n + c_n - \delta_n d_n = 0. \end{cases}$$

From these relations, one has

$$(p-1)(a_n + b_n + c_n) = (p+1)c_{V,\delta} \le c_{V,1/2}, \tag{4.25}$$

which implies that $a_n + b_n + c_n$ is bounded. That is, $\{u_n\}$ is bounded in E. Therefore, using the fact that the map $\delta \to c_{V,\delta}$ is left-continuous (see Proposition 2.1), we have

$$\begin{aligned} \lim_{n\to\infty} I_V(u_n) = \lim_{n\to\infty} \bigg\{ I_{V,\delta_n}(u_n) + (\delta_n - 1)\bigg[\frac{1}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1}\,dx\bigg]\bigg\} \\ = \lim_{n\to\infty} c_{V,\delta_n} = c_{V,1} \end{aligned} \tag{4.26}$$

and

$$\lim_{n\to\infty} \langle I'_V(u_n), \psi\rangle = \lim_{n\to\infty} \bigg\{ \langle I'_{V,\delta_n}(u_n), \psi\rangle + (\delta_n - 1)\bigg[\int_{\mathbb{R}^N} |u_n|^p\,dx\bigg]\bigg\} = 0. \tag{4.27}$$

(4.26) and (4.27) show that $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_{V,1}}$ sequence for $I_V := I_{V,1}$. Then by Lemma 4.5, there exists a nontrivial critical point $u_0 \in E$ for I_V and $I_V(u_0) = c_{V,1}$.

Now we prove the existence of a ground state solution for (EQ). Set

$$m_V := \inf\{I_V(u) : u \ne 0,\ I'_V(u) = 0\}.$$

As in the proof of Step 2 of Lemma 4.4, we can see that every critical point of I_V has nonnegative energy. Thus $0 \le m_V \le I_V(u_0) < c_{V,1} < +\infty$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I_V satisfying $I_V(u_n) \to m_V$. Since $I_V(u_n)$ is bounded, using the similar arguments as (4.25), we can conclude that $\{u_n\}$ is bounded $(\mathrm{PS})_{m_V}$ sequence of I_V. Similar arguments in Lemma 4.5, there exists a nontrivial $u^* \in E$ such that $I_V(u^*) = m_V$. □

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Liping Xu (corresponding author)
Department of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471003, P.R. CHINA
E-mail address: x.liping@126.com

Haibo Chen
School of Mathematics and Statistics
Central South University
Changsha 410075, P.R. CHINA
E-mail address: math_chb@csu.edu.cn