Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 1, 2018, 31–47
DOI: 10.12775/TMNA.2017.035

ON TWO SYMMETRIES IN THE THEORY OF m-HESSIAN OPERATORS

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Dedicated to the memory of Marek Burnat

Abstract. The modern theory of fully nonlinear operators had been inspired by the skew symmetry of minors in cooperation with the symmetry of symmetric functions. We present some consequences of this interaction for m-Hessian operators. One of them is setting of the isoperimetric variational problem for Hessian integrals. The m-admissible minimizer is found that allows a new simple proof of the well-known Poincaré-type inequalities for Hessian integrals. Also a new set of inequalities, generated by a special finite set of functions, is presented.

1. Introduction

The modern theory of fully nonlinear second-order partial differential equations counts more than 35 years and has been initiated in the papers [8], [19], where the a priori estimates of Hölder constants for the second derivatives of solutions have been established. It reduced the problem of classical solvability of the Dirichlet problem for fully nonlinear second-order partial differential equations to finding the a priori estimate of solutions in C^2. For an attempt to give a general description of obtaining this estimate for fully nonlinear operators we refer to [3], [4], [19].

2010 *Mathematics Subject Classification.* Primary: 58J70, 47J20; Secondary: 15B48, 47H05.

Key words and phrases. Partial differential fully nonlinear operators; m-Hessian operators; skew symmetry; symmetric functions; Hessian integrals; isoperimetric variational problem; Poincaré-type inequalities.

The paper was supported by the RFBR grants 15-31-20600, 15-01-07650.

There are other trends in this theory. One of them is to extend some qualitative results known in the theory of linear elliptic operators to fully nonlinear operators. The first examples of such pattern are the embedding-type theorems for Hessian integrals introduced in the papers [5], [29], [27]. A discussion on some other problems inherited from the linear case may be found, for instance, in the recent papers [28], [7] and many others.

On the other hand, there are developments, which have no analogs in the linear theory, and these are of interest in our paper. It singles out the fully nonlinear operators of very special structure. A classical representative of this kind is the Monge–Ampère operator

$$\det u_{xx}, \quad u \in C^2(\Omega), \quad \Omega \subset \mathbb{R}^n,$$

where u_{xx} is the Hessian matrix of u. Up to 1970, investigation of the Monge–Ampère equation had been performed in the framework of differential geometry (see [21] and references therein). Since 1975, the Dirichlet problem for Monge–Ampère equations has become a model to modify methods developed in the theory of linear second-order partial differential equations to fully nonlinear equations. In particular, it became the basis for the study of m-Hessian operators:

$$T_m[u] = T_m(u_{xx}), \quad 0 \le m \le n. \tag{1.1}$$

Here $T_0(S) \equiv 1$, $T_m(S)$ is the m-trace of the symmetric matrix S, that is the sum of all the principal minors of order m. The set of operators (1.1) includes the Laplace and Monge–Ampère operators, with $m = 1$, $m = n$, respectively.

The m-Hessian operator is m-homogeneous and has two kinds of symmetries. The first is the orthogonal invariance of m-traces. Namely, if B is an $n \times n$ orthogonal matrix, then

$$T_m(S) = T_m(BSB^T), \quad BB^T = \mathrm{id}. \tag{1.2}$$

Such symmetry admits a substitute of the m-traces of symmetric matrix by the elementary symmetric functions of order m of its eigenvalues $\lambda(S)$:

$$T_m(S) = S_m(\lambda(S)) = \sum_{i_1<\ldots<i_m} \lambda_{i_1} \ldots \lambda_{i_m}.$$

It follows from the papers [3], [25] that such symmetry is sufficient for classical solvability of the Dirichlet problem for m-Hessian equations. May be this is the reason that up to now the majority of scholars prefer to write m-Hessian operators (1.1) in terms of eigenvalues of the Hesse matrix $D^2u = u_{xx}$:

$$T_m[u] = S_m(\lambda[D^2u]). \tag{1.3}$$

The orthogonal invariance is a well-known type of symmetry of m-Hessian operators but in this paper we focus on the second type of symmetry, which we call a skew symmetry. In mid 70s this unnamed symmetry was discovered and investigated in quite different areas of mathematics. It brought out new nonlinear differential operators and mathematical models.

In Section 2 of this paper we give a brief outline of this story and show that the skew symmetric operators are divergence free, if homogeneous, generate exterior n-forms, etc. In fact, all this is a straightforward consequence of the skew symmetry of minors and that is why we discuss skew symmetric functions and operators. In this paper we give a survey of some well-known facts for the set of m-Hessian operators as a consequences of this type of symmetry.

The approach of Section 2 suggests that we may interpret Hessian integrals

$$I_m[u] := \int_\Omega -uT_m[u]\,dx, \quad m = 1, \dots, n,$$

as a collection of new type of volumes related to a bounded domain $\Omega \subset \mathbb{R}^n$ and functional sets

$$\{u \in C^2(\Omega) : T_m[u] > 0\}.$$

In order to compare these volumes we set up and solve a variational isoperimetric problem in Section 3. Somewhat unexpectedly this setting has led to the new Poincaré-type inequalities. These inequalities were first discovered by N.S. Trudinger and Xu-Jia Wang in [27]. A different, straightforward approach to Hessian Poincaré-type inequalities was given in [28], based on convexity methods developed originally in [29].

We deduce these inequalities by a different method but the most essential link is the same as in [27]. Namely, it is the nontrivial solvability of the Dirichlet problem

$$T_m[w] - T_l[w] = 0, \quad w|_{\partial\Omega} = 0, \quad 0 \le l < m \le n. \tag{1.4}$$

Equation (1.4) may be rewritten as $T_{m,l}[u] = 1$ and in this form qualified as the simplest equation with Hessian quotient operator

$$T_{m,l} := \frac{T_m[u]}{T_l[u]}, \quad 1 \le l < m \le n, \tag{1.5}$$

introduced in the papers of N.S. Trudinger [24], [25]. Notice that a quotient operator $T_{m,l}[u]$ is not skew symmetric. A sufficient condition close to necessary conditions for classical solvability of the Dirichlet problem for the equation $T_{m,l}[u] = f > 0$ was found in the paper [25]. The following theorem is a particular case of Theorem 1.1 from loc.cit.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega \in C^{4+\alpha}$. Assume that $\partial\Omega$ is $(m-1)$-convex. Then problem (1.4) has a unique in $C^2(\Omega)$ nontrivial solution $w \in C^{4+\alpha}(\overline{\Omega})$ for odd $q = ml$ and two solutions, $w, -w$, otherwise.*

The notion of p-convexity of the hypersurface via its p-curvature $\mathbf{k}_p[\partial\Omega]$ may be found in [13], [16]. With its help the assumption from Theorem 1.1 is equivalent to the inequality $\mathbf{k}_{m-1}[\partial\Omega] > 0$, $\mathbf{k}_{m-1}[\partial\Omega]$ is the $(m-1)$-curvature of $\partial\Omega$.

A brief outline of the theory of Hessian quotients $T_{m,l}$ is given in Section 4.

In Section 5 we consider a direct approach to deduction of the Poincaré-type inequalities, that is, to finding an m-admissible minimizer to the functional

$$J_{m,l}[u] := \frac{I_m^{1/(m+1)}[u]}{I_l^{1/(l+1)}[u]}, \quad u|_{\partial\Omega} = 0, \quad 0 \le l < m \le n. \tag{1.6}$$

The answer is known, see Section 3. Namely, the unique nontrivial solution of problem (1.4) with $w = w_{m,l} \le 0$ provides minimum to the functional (1.6) on the set of m-admissible functions. Hence, $\delta^2 J_{m,l}[w_{m,l}] \ge 0$ on this set. The latter leads to a collection of regulated by functions $\{w_{m,l}\}$ new inequalities. The following theorem is a typical result of this type.

Theorem 1.2. *Let $\partial\Omega \in C^{4+\alpha}$, $u \in \overset{\circ}{W}{}^2_1(\Omega)$. Assume that the Gauss curvature of $\partial\Omega$ is positive. Then*

$$\frac{n-1}{\displaystyle\int_\Omega |w_x|^2\,dx}\bigg(\int_\Omega u\Delta w\,dx\bigg)^2 + \int_\Omega |u_x|^2\,dx \le \int_\Omega u_i u_j \frac{\partial}{w_{ij}}(\det w_{xx})\,dx, \tag{1.7}$$

where $w \le 0$ is the nontrivial solution to problem (1.4) *with $l = 1$, $m = n$.*

2. On skew symmetry of fully nonlinear differential operators

In order to indicate the idea of the formalism introduced in the mid-seventies (see, for instance, [22], [23], [9], [1], [20], [2]), we present a slightly updated version of Theorem 2.1 from [10].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $v = (v^1, \ldots, v^n)^T \in C^1(\overline{\Omega})$:*

$$v_i := \frac{\partial v}{\partial x^i}, \quad v_x := (v_i^k)_1^n.$$

The following statements are equivalent:

(a) *the Lagrangian $F[v] = F(v_x)$ belongs to the kernel of variational derivative, i.e. $\int_\Omega F(v_x)\,dx$ does not depend on $v(x)$, $x \in \Omega$;*

(b) *the identities*

$$\frac{\partial}{\partial x^i}\frac{\partial F[v]}{\partial v_i^k} \equiv 0, \quad k = 1, \ldots, n, \tag{2.1}$$

are valid;

(c) *the operator $F[v] = F(v_x)$ is a linear combination of minors of* $\det v_x$ *of arbitrary order.*

The skew symmetry of minors is of common knowledge and it has turned out that (a), (b) are consequences of this property via (c).

Definition 2.2. We say an operator $F[v] = F(v_x)$, $v = (v^1, \dots, v^n)^T \in C^1(\overline{\Omega})$, is skew symmetric if it is a linear combination of minors of $\det v_x$ of arbitrary order.

Notice that it does not make sense to speak about skew symmetry when only minors of the first order are taken in (c). In this case Theorem 2.1 is trivial. Nevertheless, the divergence free linear differential operators might be qualified as generated by skew symmetric ones.

This amazing property had been a starting point to some important developments in quite different areas of mathematics and not surprisingly the choice of v as well as notations were different therein. For instance, the authors of [23], [1] worked with vector-fields $v \in \mathbb{R}^n$. In the paper [22] the vector-functions $v = u_x/\sqrt{1+u_x^2}$, $u \in C^2$, are under consideration and geometric curvature operators were investigated from this point of view.

In the present paper the case $v = u_x$, i.e. Hessian operators, generated by the Hessian matrix u_{xx}, is of main interest. The following proposition has been known for a long time. In order to underline its connection with skew symmetry, we formulate it in our terminology.

Corollary 2.3. *Let $v = u_x$, $u \in C^2$. Assume that the operator $F[u] = F(u_{xx})$ is m-homogeneous and skew symmetric. Then*

$$F[u] \equiv \frac{1}{m}\frac{\partial}{\partial x^i}\left(u_j \frac{\partial F[u]}{\partial u_{ij}}\right) \equiv \frac{1}{m}\frac{\partial^2}{\partial x^i \partial x^j}\left(u \frac{\partial F[u]}{\partial u_{ij}}\right). \tag{2.2}$$

The simplest example of m-homogeneous and skew symmetric operator is m-Hessian operator (1.1):

$$T_m[u] = T_m(u_{xx}).$$

Recall that by the symbol $T_m(u_{xx})$ we denote the m-trace of the matrix u_{xx}, that is the sum of all m-order principal minors, $T_0 \equiv 1$.

The skew symmetry of minors may be considered as an equivalent of the skew symmetry of exterior n-forms. Such approach to m-homogeneous fully nonlinear operators was described, for instance, in the paper [12]. Namely, denote by $\omega_{m,n-m}[v]$ the exterior form

$$\omega_{m,n-m}[v] = \sum_{\substack{(i_1<\dots<i_m)\\(i_{m+1}<\dots<i_n)}} \sigma(\mathbf{i}) dv^{i_1} \wedge \dots \wedge dv^{i_m} \wedge dx^{i_{m+1}} \wedge \dots \wedge dx^{i_n}, \tag{2.3}$$

where $\sigma(\mathbf{i})$ equals to 1 either -1 depending on evenness of permutation $(i_1, \dots i_m, i_{m+1}, \dots, i_n)$. Denote also $\omega_n(x) = \omega_{0,n}[v]$. The following proposition is the result of straightforward computation via (2.3), (2.1).

Theorem 2.4. *Let $v \in C^1$. Then*

$$\omega_{m,n-m}[v] = T_m(v_x)\omega_n(x), \quad 0 \le m \le n. \tag{2.4}$$

It looks reasonable to interpret m-homogeneous skew symmetric operators as operator-densities of some measures in Ω, which leads to the restriction $T_m(v_x) > 0$. Let, for instance, in (2.4)

$$\begin{aligned} v &= u_x && \Rightarrow \quad \omega_{m,n-m}[v] = T_m(u_{xx})\omega_n(x), \\ v &= \frac{u_x}{\sqrt{1+u_x^2}} && \Rightarrow \quad \omega_{m,n-m}[v] = \mathbf{k}_m[\Gamma(u)]\omega_n(x), \end{aligned} \tag{2.5}$$

where $\mathbf{k}_m[\Gamma(u)]$ is the m-curvature of the graph of u (see [13], [16]). So, if one plans to deal with geometric measures in the sense (2.5), it is necessary to require $\mathbf{k}_m[\Gamma(u)] > 0$.

If $T_m[u](x) > 0$, the m-Hessian operator $T_m[u] = T_m(u_{xx})$, $x \in \overline{\Omega}$, may be interpreted as an m-Hessian operator-density of some measure in Ω. Possibly, this was the reason to introduce the notion of "Hessian measures" in [26] under similar circumstances.

In order to describe some properties of $\omega_{m,n-m}[u_x]$, we fix orientation by the requirement $\int_\Omega \omega_n(x) > 0$, Ω is a bounded domain in $\mathbb{R}^n$. This agreement and the above argumentation single out a functional set $\{u \in C^2(\overline{\Omega}) : T_m[u] > 0\}$, $1 \le m \le n$. The following theorem (see, for instance, [16]) indicates some complications with these sets.

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^k$, $k \ge 2$. Assume there is a point $x_0 \in \partial\Omega$ such that $\mathbf{k}_{m-1}[\partial\Omega](x_0) = 0$. Then*

$$\{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = \mathrm{const},\ T_m[u] > 0\} = \emptyset, \tag{2.6}$$

for all $1 < m \le n$.

Notice that $m = 1$ is excluded from Theorem 2.5 because $\mathbf{k}_0[\partial\Omega] = 1$ by definition. Relation (2.6) shows that in contrast to the linear elliptic equations the theory of m-Hessian operators, $m > 1$, is nonlocal.

On the other hand, Theorem 3 from the paper [3], page 264, contains some positive information. In our notations a slightly modified version of this theorem reads as

Theorem 2.6. *Let $f \in C^{2+\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{4+\alpha}$, $0 < \alpha < 1$. Assume that $f > 0$ in $\overline{\Omega}$, $\mathbf{k}_{m-1}[\partial\Omega] > 0$. Then the Dirichlet problem*

$$T_m(u_{xx}) = f, \quad u|_{\partial\Omega} = \mathrm{const}, \quad 1 \le m \le n, \tag{2.7}$$

admits a solution $u \in C^{4+\alpha}(\overline{\Omega})$. Moreover, if in (2.7) $m = 2k-1$, u is a unique solution in $C^2(\Omega)$. In the case $m = 2k$, there are two solutions $u = \pm u_0 + \mathrm{const}$

in $C^2(\Omega)$ *where* u_0 *satisfies the problem*

$$T_m(u_{xx}) = f, \quad u|_{\partial\Omega} = 0.$$

Further development is restricted to the following functional sets, supported by Theorem 2.6:

$$\overset{\circ}{\mathbb{K}}_m(\overline{\Omega}) = \{u \in \overset{\circ}{C}{}^2(\overline{\Omega}) : T_m[u] > 0,\, u \leq 0\}, \quad 1 \leq m \leq n, \tag{2.8}$$

which are sub-cones of the well-known cones of m-admissible in $\overline{\Omega}$ functions. They admit many equivalent definitions (see for instance [17]) and are denoted by different symbols (compare [11], [3], [26]). The constructive definition of the cone of m-admissible functions was given in the paper [11] and in updated notations reads as

$$\mathbb{K}_m(\overline{\Omega}) = \{u \in C^2(\overline{\Omega}) : T_p[u] > 0,\, p = 1, \ldots, m\}, \quad 1 \leq m \leq n. \tag{2.9}$$

We show that

$$\mathbb{K}_m(\bar{\Omega}) \cap \{u|_{\partial\Omega} = 0\} = \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}).$$

If $u \in \mathbb{K}_m(\overline{\Omega})$, then u_{xx} cannot be a negative definite matrix in any point of Ω. So u has no maximums in Ω and the requirement $u|_{\partial\Omega} = 0$ provides $u \leq 0$ in $\overline{\Omega}$. In order to prove the reverse implication we consider a matrix analog of cone (2.9). Denote by $\mathrm{Sym}(n)$ the space of symmetric $n \times n$-matrices:

$$K_m = \{S \in \mathrm{Sym}(n) : T_p(S) > 0,\, p = 1, \ldots, m\}, \quad 1 \leq m \leq n. \tag{2.10}$$

Let S_0 be a positive definite matrix. It is well known that K_m is a connected in $\mathrm{Sym}(n)$ component of the set $\{S : T_m(S) > 0\}$, containing S_0 (see for instance [15]–[18]). A function $u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ attains minimum (may be not strong) in Ω. Hence, the connected set $\{u_{xx} : x \in \overline{\Omega}\}$ contains a positive definite matrix and the requirement $T_m(u_{xx}) > 0$, $x \in \overline{\Omega}$, implies $u_{xx} \in K_m$, $x \in \overline{\Omega}$, i.e. $u \in \mathbb{K}_m(\overline{\Omega})$.

The matching of definitions (2.8) and (2.9) demonstrates once again the nonlocal nature of m-admissible functions.

3. On variational problems I

It is natural to associate with forms (2.3) the following integrals:

$$\int_\Omega h(x)\omega_{p,n-p}[v], \quad \Omega \subset \mathbb{R}^n, \quad v = (v^1, \ldots, v^n), \quad p = 1, \ldots, n, \tag{3.1}$$

and speak about some volumes generated by v if $h(x) > 0$, $x \in \Omega$. If $v = u_x$, $h = -u$, integrals (3.1) may be written in the following form (see (1.3)):

$$H_m[u] := -\int_\Omega u S_m[D^2 u]\, dx.$$

The functional $H_n[u]$ was introduced in the paper [5], while the paper [29] covers all $0 < m \leq n$ and functionals $H_m[u]$, $m = 1, \ldots, n$. Therein these functionals were named Hessian integrals. Later on the ideas from this paper were developed further by many authors. For instance, in [6] Hessian integrals were applied to study some analogs of problems from the theory of semi-linear elliptic equations. Some properties of Hessian integrals discovered in the paper [27] are of particular interest in the context of our further proceeding.

We consider Hessian integrals from a different point of view and to begin with write them in our notations:

$$I_p[u] := \int_\Omega (-u)\omega_{p,n-p}[u_x] = \int_\Omega (-u)T_p[u]\,dx, \quad u \in \overset{\circ}{\mathbb{K}}_p(\overline{\Omega}), \tag{3.2}$$

$p = 0,\dots,n$. Our goal is to compare these functionals for different p and we set up the following isoperimetric problem: find $\underline{u}$, which minimizes $I_m[u]$ in $\overset{\circ}{\mathbb{K}}_m[\Omega]$ under condition $I_l[u] = 1$, $0 \le l < m \le n$. In other words, we are looking for $\underline{u}$ such that

$$I_m[\underline{u}] \le I_m[u], \quad \underline{u}, u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}) \cap \{I_l[u] = 1\}, \quad 0 \le l < m \le n. \tag{3.3}$$

The correctness of setting (3.3) confirms

Lemma 3.1. *Let $u \in C^2(\Omega) \cap \overset{\circ}{C}{}^1(\overline{\Omega})$. Assume $I_p[u] = 1$. Then the first variation of the functional* (3.2) *is nonzero on u.*

Proof. Indeed, let $\widetilde{u} = u + th$, where h is an arbitrary function from $C^2(\Omega) \cap \overset{\circ}{C}{}^1(\overline{\Omega})$, $t \in \mathbb{R}$. Then

$$\frac{d}{dt}\, I_p[\widetilde{u}] = -\int_\Omega (hT_p[\widetilde{u}] + \widetilde{u}T_p^{ij}[\widetilde{u}]h_{ij})\,dx, \quad T_p^{ij}[\widetilde{u}] = \frac{\partial T_p[\widetilde{u}]}{\partial \widetilde{u}_{ij}}, \quad 1 \le i,j \le n.$$

It follows from integration by parts and (2.2) that

$$\frac{d}{dt}\, I_p[\widetilde{u}] = -(p+1)\int_\Omega hT_p[\widetilde{u}]\,dx. \tag{3.4}$$

Assume that

$$\delta I_p[u] = \frac{d}{dt}\, I_p[\widetilde{u}]|_{t=0} = 0.$$

Then relation (3.4) is equivalent to $T_p[u] \equiv 0$. But it contradicts the assumption $I_p[u] = 1$, what validates Lemma 3.1. □

Notice that the correctness of problem (3.3) is a consequence of identity (2.2), i.e. of the skew symmetry of m-Hessian operators. Next, we discuss the link between the isoperimetric problem (3.3) and Hessian quotients (1.5).

Theorem 3.2. *Let $0 \le l < m \le n$. Assume there is $w \in \overset{\circ}{\mathbb{K}}_m[\overline{\Omega}]$ such that*

$$T_{m,l}[w] := \frac{T_m[w]}{T_l[w]} = 1. \tag{3.5}$$

Then there exists $\underline{u}$ satisfying (3.3) *and*

$$I_m[u] \ge I_m[\underline{u}] = I_m^{(l-m)/(l+1)}[w], \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}) \cap \{I_l[u] = 1\}. \tag{3.6}$$

Proof. Problem (3.3) is a classical isoperimetric variational problem. Due to Lemma 3.1 there exists a Lagrange multiplier λ such that a minimizer to the functional

$$\int_\Omega -u(T_m[u] - \lambda T_l[u])\,dx, \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}),$$

solves problem (3.3). Hence, we are looking for solutions to the following Euler–Lagrange equation:

$$(m+1)T_m[u] - (l+1)\lambda T_l[u] = 0, \tag{3.7}$$

what follows from (3.4). Since only functions $u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ are of interest, a multiplier λ has to be positive. Denote

$$\mu^{m-l} = \frac{l+1}{m+1}\,\lambda.$$

Then equation (3.7) turns into $T_{m,l}[u] = \mu^{m-l}$. The function $\underline{u} = \mu w$, where w is a solution to (3.5), satisfies condition in (3.3), and hence solves problem (3.3). Moreover, we have the sharp estimate:

$$I_m[w] = I_l[w] = \frac{1}{\mu^{l+1}}\,I_l[\underline{u}] = \frac{1}{\mu^{l+1}} \quad \Rightarrow \quad \mu = I^{-1/(l+1)}[w],$$

$$I_m[u] \geq I_m[\underline{u}] = \mu^{m+1} I_m[w] = I_m^{(l-m)/(l+1)}[w], \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}) \cap \{I_m[u] = 1\}. \ \square$$

An auxiliary Dirichlet problem (3.5) appeared in the paper [27] as a crucial tool to derive Poincaré-type inequalities for functionals $I_m[u]$, $1 \leq m \leq n$, interpreted in a weak sense. For $u \in C^2(\overline{\Omega})$ these inequalities spring up as a simple consequence of (3.6) and we write out their equivalents in

Corollary 3.3. *Let $0 \leq l \leq m \leq n$ and w satisfy equation* (3.5)*. Then*

$$\left(\frac{I_m[u]}{I_m[w]}\right)^{1/(m+1)} \geq \left(\frac{I_l[u]}{I_l[w]}\right)^{1/(l+1)}, \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}). \tag{3.8}$$

Proof. Indeed, defined by the line $u = I_l^{1/(l+1)}[u]\widetilde{u}$, the function $\widetilde{u}$ belongs to $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega}) \cap \{I_l[u] = 1\}$. It follows from (3.6) that

$$\begin{aligned} I_m^{1/(m+1)}[u] &= I_l^{1/(l+1)}[u] I_m^{1/(m+1)}[\widetilde{u}] \\ &\geq I_l^{1/(l+1)}[u] I_m^{(l-m)/((l+1)(m+1))}[w] = I_l^{1/(l+1)}[u] I_m^{1/(m+1)-1/(l+1)}[w]. \ \square \end{aligned} \tag{3.9}$$

Notice that inequality (3.8) is a symmetrized form of restricted to $u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ inequality (1.13) from [27]. Also a solution $w_\mu \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ to equation $T_{m,l}[w] = \mu^2$ with an arbitrary $\mu \in \mathbb{R}^+$ may be taken in capacity of w in relation (3.8).

Properties of solutions to equation (3.5) from $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ are of our special interest and the first one we present is a consequence of sharp inequalities (3.6).

Theorem 3.4. *Let $0 \leq l \leq p < m$. Assume that, for every p, there is a solution $w_{m,p} \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ to equations $T_{m,p}[w_{m,p}] = 1$. Then*

$$I_m^{m-l}[w_{m,l}] \geq I_m^{(l+1)(m-p)/(p+1)}[w_{m,p}] I_l^{(m+1)(p-l)/(p+1)}[w_{p,l}]. \tag{3.10}$$

Proof. We use inequality (3.8) in the form (3.9):

$$I_m^{1/(m+1)}[u] \geq c_{m,l} I_l^{1/(l+1)}[u], \quad c_{m,l} = I_m^{1/(m+1)-1/(l+1)}[w_{m,l}], \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}).$$

A constant $c_{m,l}$ is sharp, because the above inequality turns into equality, when $u = w_{m,l}$. Using inequality (3.9) twice, we derive

$$I_m^{1/(m+1)}[u] \geq c_{m,p} c_{p,l} I_l^{1/(l+1)}[u], \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}),$$

where a constant $c_{m,p}c_{p,l}$ is not sharp. Hence, $c_{m,l} \geq c_{m,p}c_{p,l}$, what coincides with relation (3.10). □

4. Some properties of Hessian quotients

To make Theorem 3.2 credible it is necessary to confirm solvability of problem (3.5) and we present some extraction from general theory. The existence of admissible solutions to the Dirichlet problem for the Hessian quotient equations was proved in the paper [25, Theorem 1.1, p. 153], and in the author's notations it reads as

Theorem 4.1. *Let $0 \leq l < m \leq n$ and Ω be a bounded uniformly $(m-1)$-convex domain in $\mathbb{R}^n$, with $\partial\Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\partial\Omega)$ and let ψ be a positive function in $C^{1,1}(\overline{\Omega})$. Then the Dirichlet problem,*

$$F(D^2u) = S_{m,l}(\lambda[D^2u]) = \psi \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega, \tag{4.1}$$

is uniquely solvable for admissible $u \in C^{3,\alpha}(\overline{\Omega})$ for any $0 < \alpha < 1$.

It is more than 20 years since this amazing theorem has been proved and now we suggest to slightly update its formulation. Namely,

(i) the basis of Theorem 4.1 is a construction of a priori estimates of solutions at the boundary and the requirement "uniformly $(m-1)$-convex domain" is equivalent to the inequality $\mathbf{k}_{m-1}[\partial\Omega] > 0$, what means that the hyper-surface $\partial\Omega$ is $(m-1)$-convex. The definition of $\mathbf{k}_{m-1}$-curvature of the hyper-surface $\partial\Omega$ and reasons for such substitute may be found in [13], [16];

(ii) in our argument we do not allude to the eigenvalues $\lambda[D^2u]$ and write equation in (4.1) as $T_{m,l}[u] = \psi$ (see (1.3), (1.5)), what allows to differentiate our equations, when necessary, without preliminary passes;

(iii) the assertion of Theorem 4.1 is equivalent to "there exists a unique in $\mathbb{K}_m(\overline{\Omega})$ solution u to problem (4.1) and $u \in C^{3+\alpha}(\overline{\Omega})$ for any $0 < \alpha < 1$".

Notice that if $\varphi = 0$, the unique solution from Theorem 4.1 belongs to $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ (see description of the cones (2.8), (2.9)), what means that it is unique in

$$\overset{\circ}{C}{}^2_-(\overline{\Omega}) := \{u \in \overset{\circ}{C}{}^2(\overline{\Omega}) : u \leq 0\}. \tag{4.2}$$

More precisely, the following consequence of Theorem 2.5 and properties of cones (2.8)–(2.10) are valid.

Lemma 4.2. *Let $0 \leq l < m \leq n$, $\partial\Omega \in C^2$. There are two possibilities*:

(a) *if there exists $x_0 \in \partial\Omega$ such that $\mathbf{k}_{m-1}[\partial\Omega](x_0) = 0$, then*

$$\{u \in \overset{\circ}{C}{}^2(\overline{\Omega}) : T_{m,l}[u] > 0\} = \emptyset;$$

(b) *if x_0 from* (a) *does not exist, then*

$$\{u \in \overset{\circ}{C}{}^2_-(\overline{\Omega}) : T_{m,l}[u] > 0\} = \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}).$$

It follows from Theorem 4.1, Lemma 4.2 that the cone $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ is a natural set of solvability of the problem

$$T_{m,l}[u] = \psi > 0, \quad u|_{\partial\Omega} = 0 \tag{4.3}$$

and the requirement of $(m-1)$-convexity of $\partial\Omega$ is necessary. Notice that the half-space (4.2) was introduced to avoid pecularity of even values of the number $m+l$. Similarly to situation in Theorem 2.6, in this case the inequality $T_{m,l}[u] > 0$ leads to two cones in $\overset{\circ}{C}{}^2(\overline{\Omega})$.

A correct setting of the Dirichlet problem (4.1) assumes that the operator F is elliptic on the set of admissible functions, what was proved in the paper [24] by combinatorical methods. We offer somewhat different approach and consider ellipticity of F as a consequence of positive monotonicity of operators $T_{m,l}[u]$ in $\mathbb{K}_m(\overline{\Omega})$.

To begin with we consider a set of functions $\{T_p = T_p(S)\}_1^n$ in the matrix cone (2.10) and denote

$$T_p^{ij}(S) := \frac{\partial T_p}{\partial s_{ij}}(S), \quad 1 \leq i, j \leq n. \tag{4.4}$$

Notice that

$$T_{m-1;i}(S) := \frac{\partial T_m}{\partial s_{ii}}(S)$$

is the $(m-1)$-trace of the matrix S with deleted i-th row and column. It is known that

$$T_{m-1;i}(S) > 0, \quad S \in K_m, \ m = 1, \dots, n. \tag{4.5}$$

In this course we associate with the quotient operator $T_{m,l}[u]$ a functional quotient

$$T_{m,l}(S) := \frac{T_m(S)}{T_l(S)}, \quad 0 \leq l < m \leq n, \ S \in \mathrm{Sym}(n), \tag{4.6}$$

and prove its monotonicity in the matrix cone K_m.

Denote by $\mathrm{Sym}^+(n) \subset \mathrm{Sym}(n)$ the set of positive definite matrices.

THEOREM 4.3. *Let* $S^0 \in \overline{\mathrm{Sym}^+}(n)$, $0 \le l < m \le n$. *Assume that* $S^0 \ne \mathbf{0}$. *Then*

$$T_{m,l}(S+S^0) > T_{m,l}(S), \quad S \in K_m. \tag{4.7}$$

PROOF. The proof consists of three steps.

Step1. We fix a matrix $S \in K_m$, an index $1 \le i \le n$ and associate with them an auxiliary matrix:

$$S(t;i) = (s_{kl} + t\delta_{ki}\delta_{li})_1^n, \quad t \in \mathbb{R}.$$

When $l = 0$, $T_{m,0} = T_m$ and due to (4.5) we have

$$T_m(S(t;i)) = T_m(S) + tT_{m-1;i}(S) > T_m(S), \quad t > 0.$$

Let

$$\underline{t} := -\frac{T_m(S)}{T_{m-1;i}(S)}.$$

Then $T_m(S(\underline{t};i)) = 0$. Moreover, $S(t,i) \in K_m$ for all $t > \underline{t}$ because the cone K_m is a connected component of the set $\{S : T_m(S) > 0\}$.

For the case $l > 0$ we introduce an auxiliary function:

$$y(t) = T_{m,l}(S(t;i)), \quad t \in \mathbb{R}. \tag{4.8}$$

Due to the Maclaurin inequality (see for instance [11], [15])

$$\left(\frac{T_l(S)}{C_n^l}\right)^{1/l} \ge \left(\frac{T_m(S)}{C_n^m}\right)^{1/m}, \quad S \in K_m, \tag{4.9}$$

we have the estimate

$$y(t) \le c(m,n)(T_m(S(t;i)))^{1-l/m}, \quad t > \underline{t}.$$

Hence, $y(t) \to 0$ when $t \to \underline{t}$.

Step 2. The differentiating the function (4.8) we obtain

$$y'(t) = \frac{T_{l-1;i}(S)}{T_l(S(t;i))}\left(\frac{T_{m-1;i}(S)}{T_{l-1;i}(S)} - y(t)\right), \tag{4.10}$$

$$y''(t) = -2\,\frac{T_{l-1;i}(S)}{T_l(S(t;i))}\,y'(t). \tag{4.11}$$

Integrating the ODE in (4.11) we derive

$$y'(t) = y'(t_0)\,\frac{T_l^2(S(t_0;i))}{T_l^2(S(t;i))}.$$

Consider the initial value $y'(t_0)$. It follows from Step 1 and (4.5) that there exists t_0 such that $\underline{t} < t_0 < 0$ and $y'(t_0) > 0$. Hence $y'(t) > 0$ for $t \geq t_0$ and we have arrived at the inequality

$$T_{m,l}(S) < T_{m,l}(S(t;i)) < \frac{T_{m-1;i}(S)}{T_{l-1;i}(S)} = \lim_{t\to+\infty} T_{m,l}(S(t;i)), \quad t > 0,\ i = 1,\dots,n.$$

Step 3. Consider first a diagonal matrix $S_d^0 \in \overline{\mathrm{Sym}^+(n)}$, $S_d^0 \neq \mathbf{0}$. Inequality (4.7) with $S^0 = S_d^0$ follows from Step 2. Since p-traces are orthogonal invariant (see (1.2)), inequality (4.7) is also true for an arbitrary nonzero matrix $S_0 \in \overline{\mathrm{Sym}^+(n)}$. □

Inequality

$$(T^{ij}_{m,l}(S)\xi,\xi) > 0, \quad S \in K_m,\ \xi \in \mathbb{R}^n,\ |\xi| = 1, \tag{4.12}$$

is a straightforward consequence of monotonicity (4.7). An operator version of (4.12) reads as

$$(T^{ij}_m - T_{m,l}T^{ij}_l)[u]\xi_i\xi_j > 0, \quad T^{ij}_p[u] = \frac{\partial T_p(u_{xx})}{\partial u_{ij}}, \quad u \in \mathbb{K}_m(\overline{\Omega}), \tag{4.13}$$

which means that operator quotients $T_{m,l}[u]$ are elliptic onto $\mathbb{K}_m(\overline{\Omega})$. So, equation (3.5) is uniquely solvable in $\mathbb{K}^0_m(\overline{\Omega})$ due to Theorem 4.1 and the following consequence is of the principal interest in our paper.

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^{4+\alpha}$, $0 < \alpha < 1$. Assume $\mathbf{k}_{m-1}[\partial\Omega] > 0$. Then there exists a unique in $C^2_{-0}(\overline{\Omega})$ solution w to the problem*

$$T_{m,l}[w] = 1, \quad w|_{\partial\Omega} = 0, \quad 0 \leq l < m \leq n. \tag{4.14}$$

Moreover, $w \in \mathbb{K}_m(\overline{\Omega}) \cap C^{4+\alpha}(\overline{\Omega})$ and it satisfies the inequality

$$(T^{ij}_m - T^{ij}_l)[w]\xi_i\xi_j > 0, \quad |\xi| = 1. \tag{4.15}$$

Notice that the existence part of Theorem 4.4 is identical with Theorem 1.1, while inequality (4.15) coincides with ellipticity condition (4.13) with $u = w$.

Remark 4.5. Quotients operators $T_{m,l}[u]$ are not divergence free, when $l \geq 1$. It means that skew symmetry does not matter for solvability of the Dirichlet problem for Hessian equations. However, equation (4.14) may be written as $(T_m - T_l)[w] = 0$ in $\mathbb{K}_m(\overline{\Omega})$. Due to identities (2.2) the latter is equivalent to

$$\frac{\partial}{\partial x^i} A^{ij}[w]w_j = 0, \qquad A^{ij}[w] = \left(\frac{1}{m}T^{ij}_m - \frac{1}{l}T^{ij}_l\right)[w].$$

For the fixed $1 < m \leq n$ we consider now the set of solutions $\{w_{m,l},\ l = 0\dots,m-1\}$ from Theorem 4.4. It is natural to expect some connections between these functions. At the moment we know the following result.

Lemma 4.6. *Under conditions of Theorem* 4.4 *the inequalities*

$$T_p[w_{m,l}] > 1, \quad w_{m,l} < w_{m,0},\ x \in \Omega, \tag{4.16}$$

hold true for all $1 \leq l, p \leq m-1$.

Proof. To prove the left-hand side of (4.16) we apply a strong version of the Maclaurin inequality (4.9):

$$T_m^{1/m}[w] < T_l^{1/l}[w], \quad 1 \leq l \leq m-1,\ w \in \mathbb{K}_m(\overline{\Omega}). \tag{4.17}$$

Let $w = w_{m,l}$. By definition $T_{m,l}[w_{m,l}] = 1$ and due to (4.17) we have

$$1 = \frac{T_m[w_{m,l}]}{T_l[w_{m,l}]} < T_m^{(m-l)/m}[w_{m,l}] < T_p^{(m-l)/p}[w_{m,l}]. \tag{4.18}$$

The second part of (4.16) is a consequence of the well-known comparison theorem for m-Hessian operators. Indeed, it follows from (4.18) that $T_m[w_{m,l}] > 1$. On the other hand, $T_m[w_{m,0}] = 1$ by definition. Via the comparison principle the inequality for m-Hessian operators guarantees the reverse inequality for functions from $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$, i.e. the second inequality in (4.16). □

5. On variational problems II

Theorems 3.2 and 4.4 yield

Theorem 5.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $\partial\Omega \in C^{4+\alpha}$, $0 \leq l < m \leq n$. *Assume* $\mathbf{k}_{m-1}[\partial\Omega] > 0$. *Then there is a sharp constant* $\mathbf{c} = \mathbf{c}(l, m, \mathbf{k}_{m-1}[\partial\Omega]) > 0$ *such that*

$$J_{m,l}[u] := \frac{\left(\displaystyle\int_\Omega -uT_m[u]\,dx\right)^{1/(m+1)}}{\left(\displaystyle\int_\Omega -uT_l[u]\,dx\right)^{1/(l+1)}} \geq \mathbf{c}, \quad u \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega}). \tag{5.1}$$

Indeed, due to the assumption $\mathbf{k}_{m-1}[\partial\Omega] > 0$ there exists a unique in $C^2(\overline{\Omega})$ solution $w = w_{l,m} \in \overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ to problem (4.14). Therefore inequality (5.1) with $\mathbf{c} = J_{m,l}[w_{l,m}]$ is a replica of (3.8).

Notice that inequalities (5.1) are equivalent to the Poincaré-type inequalities from the paper [27]. If the principal goal of our paper had been to give a straightforward proof of those, it would be reasonable to set up a classical variational problem of minimization of the functional $J_{m,l}[u]$ over the cone $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$. In order to produce some new analogs of the classic Poincaré inequality we outline this approach.

Theorem 5.2. *Assume conditions of Theorem* 5.1 *are satisfied and let* u *be from the Sobolev space* $\overset{\circ}{W}{}^2_1(\Omega)$, w *be a solution to problem* (4.14). *Then*

$$\frac{m-l}{I_m[w]}\left(\int_\Omega uT_m[w]\,dx\right)^2 + \int_\Omega T_l^{ij}[w]u_iu_j\,dx \leq \int_\Omega T_m^{ij}[w]u_iu_j\,dx. \tag{5.2}$$

Proof. Let $\widetilde{w} = w + th$, $t \in \mathbb{R}$, $h \in \overset{\circ}{C}{}^2(\overline{\Omega})$. Similarly to (3.4) we derive

$$\frac{d^2}{dt^2} I_p[w] \equiv (p+1) \int_\Omega T_p^{ij}[\widetilde{w}] h_i h_j \, dx, \quad p = 1, \dots, n. \tag{5.3}$$

It follows from (5.1) that w minimizes $J_{m,l}[u]$ over $\overset{\circ}{\mathbb{K}}_m(\overline{\Omega})$ and hence $\delta J_{m,l}[w] = 0$, $\delta^2 J_{m,l}[w] \geq 0$. Keeping in mind that $T_m[w] = T_l[w]$, $I_m[w] = I_l[w]$, we compute via (3.4), (5.3) the second variation of the functional $J_{m,l}[\widetilde{w}]$:

$$\delta^2 J_{m,l}[w] = \frac{J_{m,l}[w]}{I_m[w]} \left(\frac{l-m}{I_m[w]} \left(\int_\Omega h T_m[w] \, dx \right)^2 + \int_\Omega (T_m^{ij} - T_l^{ij})[w] h_i h_j \, dx \right). \tag{5.4}$$

Since the case $t = 0$ is of interest, we may without loss of generality assume that $\widetilde{w} \in \overset{\circ}{\mathbb{K}}_m(\Omega)$ for an arbitrary $h \in C^2(\overline{\Omega}) \cap \overset{\circ}{C}{}^1_m(\overline{\Omega})$. Therefore, relation (5.4) and a choice of w provide $\delta^2 J_{m,l}[w] \geq 0$, hence inequality (5.2) is valid for an arbitrary function $u = h \in C^2(\overline{\Omega})$. The case of $u \in \overset{\circ}{W}{}^2_1(\Omega)$ may be derived by approximation. □

Letting $l = 1$, $m = n$ in Theorem 5.2 one sees exactly Theorem 1.2. The case $l = 0$ in Theorem 5.2 is of special interest and we extract it as

Corollary 5.3. *Let $u \in \overset{\circ}{W}{}^2_1(\Omega)$ be an arbitrary function, $w_m \in C^2(\overline{\Omega})$ a solution to the problem $T_m[w_m] = 1$, $w_m|_{\partial\Omega} = 0$, $w_m \leq 0$. Then the inequalities*

$$m \left(\int_\Omega u \, dx \right)^2 \leq \int_\Omega -w_m \, dx \int_\Omega T_m^{ij}[w_m] u_i u_j \, dx, \quad m = 1, \dots, n, \tag{5.5}$$

are true.

Notice that Corollary 5.3 implicitly contains the requirement of $(m-1)$-convexity of $\partial\Omega$.

Inequality (5.5) with $m = 1$ and under requirement $\Delta u > 0$ in a weak sense was attributed to Poincaré in the paper [27]. Theorem 5.2 along with Corollary 5.3 is valid for an arbitrary function u from $\overset{\circ}{W}{}^2_1(\Omega)$ and speaking formally inequality (5.5) with $m = 1$ is more general than its analog from [27].

All inequalities (5.2) are sharp and the set (5.5) might be considered as a set of depending on the p-convexity of $\partial\Omega$ analogs to the classical Poincaré inequality.

Remark 5.4. There are two questions concerning our inequalities:

(a) Assume that $\mathbf{k}_{n-1}[\partial\Omega] > 0$ in Corollary 5.3. Then we have a set of functions $\{w_m\}_1^n$ and relevant sharp inequalities (5.5). Is it possible to compare them for different values of m?

(b) Let $m > 1$ be fixed and assumptions of Theorem 5.2 be satisfied. Then we have a collection of functions $\{w_{l,m}\}_0^{m-1}$. Are they comparable?

Eventually we rewrite general inequality (5.2) in the invariant under dilation form. Denote

$$\langle u, v\rangle_p = \int_\Omega T_p^{ij}[w] u_i v_j \, dx, \quad p = 1, \dots, n,$$

and let w be a solution to the problem $T_{m,l}[w] = \mu$, $w|_{\partial\Omega} = 0$, $u \in \overset{\circ}{W}{}_1^2(\Omega)$. Then the inequality

$$(m-l)\,\frac{\langle u, w\rangle_l}{\langle w, w\rangle_l}\,\frac{\langle u, w\rangle_m}{\langle w, w\rangle_m} \le m\,\frac{\langle u, u\rangle_m}{\langle w, w\rangle_m} - l\,\frac{\langle u, u\rangle_l}{\langle w, w\rangle_l} \tag{5.6}$$

is equivalent to (5.2), whatever $\mu \in \mathbb{R}^+$ is. It follows from (5.6) that the constant $\mathbf{c}$ in (5.1) is invariant under dilation.

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Manuscript received December 3, 2016

accepted May 30, 2017

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