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PERIODIC ORBITS FOR MULTIVALUED MAPS WITH CONTINUOUS MARGINS OF INTERVALS

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Abstract. Let I be a bounded connected subset of $\mathbb{R}$ containing more than one point, and $\mathcal{L}(I)$ be the family of all nonempty connected subsets of I. Each map from I to $\mathcal{L}(I)$ is called a multivalued map. A multivalued map $F\colon I \to \mathcal{L}(I)$ is called a multivalued map with continuous margins if both the left endpoint and the right endpoint functions of F are continuous. We show that the well-known Sharkovskiĭ theorem for interval maps also holds for every multivalued map with continuous margins $F\colon I \to \mathcal{L}(I)$, that is, if F has an n-periodic orbit and $n \succ m$ (in the Sharkovskiĭ ordering), then F also has an m-periodic orbit.

1. Introduction

Let X be a set and $\mathbb{N} = \{1, 2, \ldots\}$. An infinite sequence $(x_1, x_2, \ldots)$ of elements in X is said to be *periodic* if there is $n \in \mathbb{N}$ such that

$$x_{i+n} = x_i \quad \text{for all } i \in \mathbb{N}. \tag{1.1}$$

In this case, we also write $(x_1, \ldots, x_n)^\circ$ for $(x_1, x_2, \ldots)$, where we put the small circle $\circ$ at the top-right corner of the finite sequence $(x_1, \ldots, x_n)$, which means that we repeat this finite sequence infinitely many times. The least n such that (1.1) holds is called the *period* of $(x_1, x_2, \ldots)$. Note that if we cannot clearly

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mention the period of the infinite sequence $(x_1, \ldots, x_n)^\circ$, then it may be a proper factor of n. A periodic sequence of period n is also called an *n-periodic sequence*.

Denote by $2^X - \{\emptyset\}$ the family of all nonempty subsets of X. Each map from X to $2^X - \{\emptyset\}$ is called a *multivalued map* on X. An infinite sequence $(x_1, x_2, \ldots)$ of elements in X is called an *orbit* of $F\colon X \to 2^X - \{\emptyset\}$ if $x_{i+1} \in F(x_i)$ for all $i \in \mathbb{N}$. The sequence $(x_1, x_2, \ldots)$ is called a *periodic orbit* of F if it is both a periodic sequence and an orbit of F. If $\mathcal{O} = (x_1, x_2, \ldots) = (x_1, \ldots, x_n)^\circ$ is an n-periodic orbit of F, then, for any $i \in \mathbb{N}$, the finite sequence $(x_i, x_{i+1}, \ldots, x_{i+n-1})$ with length n is called a *periodic segment* of the orbit $\mathcal{O}$. If $F\colon X \to 2^X - \{\emptyset\}$ is a multivalued map and F contains only one element for each $x \in X$, then F is a single-valued map from X to X. Note that if $f\colon X \to X$ is a single-valued map, then any period segment of a periodic orbit of f contains no repeating element, and if $F\colon X \to 2^X - \{\emptyset\}$ is a multivalued map, then a period segment of some periodic orbit of F may contain repeating elements. This is a difference between single-valued maps and multivalued maps. Since there may appear repeating elements in a period segment when we study periodic orbits of multivalued maps, it will meet some additional trouble.

Let I be a bounded connected subset of $\mathbb{R}$ containing more than one point, that is, I is a closed interval, or an open interval, or a half-open interval. Denote by $\overline{I}$ the closure of I in $\mathbb{R}$ and by $\mathcal{L}(I)$ the family of all nonempty connected subsets of I. Each map from I to $\mathcal{L}(I)$ is called a *connected-multivalued map* on I. Obviously, for any connected-multivalued map $F\colon I \to \mathcal{L}(I)$, there exists a unique pair of functions $\alpha\colon I \to \overline{I}$ and $\beta\colon I \to \overline{I}$, called the *left endpoint function* and the *right endpoint function* of F, respectively, satisfying the following two conditions:

(i) $\alpha(x) \leq \beta(x)$ for any $x \in I$;
(ii) $(\alpha(x), \beta(x)) \subset F(x) \subset [\alpha(x), \beta(x)]$ for any $x \in I$.

If $\alpha(x) = \beta(x)$, then $F(x) = [\alpha(x), \beta(x)] = \{\alpha(x)\}$.

A connected-multivalued map $F\colon I \to \mathcal{L}(I)$ is said to be a *multivalued map with continuous margins* if both the left endpoint and the right endpoint functions of F are continuous.

In 1964, Sharkovskiĭ found the following order relation in $\mathbb{N}$:

$$3 \succ 5 \succ 7 \succ \ldots \succ 3\cdot 2 \succ 5\cdot 2 \succ 7\cdot 2 \succ \ldots \succ 3\cdot 2^2 \succ 5\cdot 2^2 \succ 7\cdot 2^2 \succ \ldots$$
$$\ldots \succ 3\cdot 2^k \succ 5\cdot 2^k \succ 7\cdot 2^k \succ \ldots \succ 2^4 \succ 2^3 \succ 2^2 \succ 2 \succ 1,$$

and proved the following theorem.

Theorem 1.1 (Sharkovskiĭ's theorem, see [17]). *Let J be a connected subset of $\mathbb{R}$ and $f\colon J \to J$ be a single-valued continuous map. For any $m, n \in \mathbb{N}$ with $n \succ m$, if f has an n-periodic orbit, then f has an m-periodic orbit.*

Note that the above Sharkovskiĭ's order is well-ordered. If $n \succ m$ in this order, then we also write $m \prec n$.

In [1], Alseda and Llibre showed that Theorem 1.1 holds for triangular maps on a rectangle. In [12], Minc and Transue showed that Sharkovskiĭ's theorem also holds for continuous maps on hereditarily decomposable chainable continua. In [5], Andres et al. also obtained a full analogy of Sharkovskiĭ's theorem for lower-semicontinuous maps (i.e. for every closed subset $V \subset \mathbb{R}$, the set $\{x \in \mathbb{R} : F(x) \subset V\}$ is closed) with nonempty, connected and compact values.

Recently, there has been a lot of work on the dynamics of multivalued maps (see [11], [13]–[16]). In [3], Andres et al. studied the periodic orbits of a class of multivalued maps and obtained the following theorem.

THEOREM 1.2. *Let $\mathcal{C}(\mathbb{R})$ be the family of all nonempty compact connected subsets of $\mathbb{R}$ and $F\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be upper-semicontinuous (i.e. for every open $V \subset \mathbb{R}$, the set $\{x \in \mathbb{R} : F(x) \subset V\}$ is open). If F has an n-periodic orbit for some odd integer n, but F has no l-periodic orbit for any $l \succ n$, then for any $n \succ m$, F has an m-periodic orbit, except $m = 4$.*

Further, Andres and Pastor [9] (also see [10]) obtained the following theorem.

THEOREM 1.3. *Let $F\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be upper-semicontinuous. For any $m, n \in \mathbb{N}$ with $n \succ m$, if F has an n-periodic orbit, then F has an m-periodic orbit with at most two exceptions.*

For some other papers in the area, see also [2], [4], [6]–[8] and the references therein. In this paper, we study connected-multivalued maps on the bounded connected set I. Our main result is the following theorem.

THEOREM 1.4. *Let I be a bounded connected subset of $\mathbb{R}$ and $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins. For any $m, n \in \mathbb{N}$ with $n \succ m$, if F has an n-periodic orbit, then F has an m-periodic orbit.*

REMARK 1.5. In [2]–[10], the set of every value of upper-semicontinuous and lower-semicontinuous maps is nonempty, it is a connected and compact set. But for multivalued maps with continuous margins of intervals studied in this paper, the set of every value need not be compact.

REMARK 1.6. In [3], the authors constructed upper-semicontinuous maps $F\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ and $G\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ such that F has a 3-periodic orbit but has no 2-periodic orbit and G has a 5-periodic orbit but has no 4-periodic orbit. While for multivalued maps with continuous margins of intervals studied in this paper, Sharkovskiĭ's theorem holds, without exception.

Example 1.7. Define a connected-multivalued map $F\colon [0,1] \to \mathcal{L}([0,1])$ by

$$F(x) = \begin{cases} [0,0] & \text{if } x = 0, \\ [0, \sqrt{2}x) & \text{if } x \in (0, \sqrt{2}/2), \\ [0,1] & \text{if } x = \sqrt{2}/2, \\ [0, (2+\sqrt{2})(1-x)) & \text{if } x \in (\sqrt{2}/2, 0), \\ [0,0] & \text{if } x = 1, \end{cases}$$

$x \in [0,1]$. Then, according to our definition, F is a multivalued map with continuous margins. But according to the definitions in [2]–[4], [6]–[10], F is not upper-semicontinuous since the set $\{x \in [0,1] : F(x) \subset [0,y)\} = [0, y/\sqrt{2}] \cup [1 - y/(2+\sqrt{2}), 1]$ is closed for any $y \in (0,1]$. What means that continuity of margins does not necessarily implies upper-semicontinity of the multivalued map with continuous margins under consideration.

2. Periodic orbits for multivalued maps

Let X be a set. Let F and G be maps from X to $2^X - \{\emptyset\}$. Define the composite map $G \circ F\colon X \to 2^X - \{\emptyset\}$ by

$$G \circ F(x) = \bigcup \{G(y) : y \in F(x)\}, \tag{2.1}$$

$x \in X$. Denote by F^0 the identity map on X, $F^1 = F$, and $F^{n+1} = F \circ F^n$ for each $n \in \mathbb{N}$. For $n \geq 0$, F^n is called the *n-th iterate* of F.

Remark 2.1. We see from the definition that for any $n \geq 2$ and any $x \in X$,

$$F^n(x) = \{y \in X : \text{there exists } \{x_i\}_{i=0}^n \subset X \\ \text{such that } x_0 = x,\ x_n = y,\ x_i \in F(x_{i-1}) \text{ for } 1 \leq i \leq n\}.$$

Let $S = (x_1, x_2, \ldots)$ be an infinite sequence. For any $k, i \in \mathbb{N}$, the sequence

$$(x_i, x_{k+i}, x_{2k+i}, x_{3k+i}, \ldots)$$

is called the i-th *k-subsequence* of S. Obviously, if the sequence S is an orbit of $F\colon X \to 2^X - \{\emptyset\}$, then any k-subsequence of S is an orbit of F^k.

The following lemma is well-known, but we still give a simplified proof.

Lemma 2.2. *Suppose that $(x_1, x_2, \ldots)$ is an infinite sequence. Let $n, m \in \mathbb{N}$ and $k = \gcd(n,m)$ be the greatest common factor of n and m. If $x_{i+n} = x_i$ and $x_{i+m} = x_i$ for all $i \in \mathbb{N}$, then $x_{i+k} = x_i$ for all $i \in \mathbb{N}$.*

Proof. As $k = \gcd(n,m)$, there exist $p, q \in \mathbb{N}$ such that $pn - qm = k$. Then we have $x_i = x_{i+pn} = x_{i+pn-qm} = x_{i+k}$ for any $i \in \mathbb{N}$. □

Corollary 2.3. *If $(x_1, x_2, \ldots)$ is a periodic sequence, which can be written as $(x_1, \ldots, x_n)^\circ$, then the period of this sequence is a factor of n.*

DEFINITION 2.4. Two positive integers k and n are said to *have the same prime factor* if for any prime number p, p is a factor of k if and only if p is a factor of n.

The following lemma is trivial.

LEMMA 2.5. *Suppose that integers k and n have the same prime factor. Then*:

(a) $k = 1$ *if and only if* $n = 1$.

(b) *If* $k > 1$, *then there exist prime numbers* $p_1, \ldots, p_m$ *with* $m \geq 1$ *and positive integers* $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m$ *such that*

$$k = \prod_{i=1}^{m} p_i^{\lambda_i} \quad \textit{and} \quad n = \prod_{i=1}^{m} p_i^{\mu_i}.$$

LEMMA 2.6. *Let $k, n \in \mathbb{N}$. Then there exists a unique sequence* (k_1, k_2, n_1, n_2) *of positive integers such that*

(a) $k = k_1 k_2$ *and* $n = n_1 n_2$,

(b) k_1 *and* n_1 *have the same prime factor,*

(c) $\gcd(k_2, n) = 1$ *and* $\gcd(n_2, k) = 1$.

PROOF. Let $k_2 = \max\{\lambda : \lambda \text{ is a factor of } k \text{ and } \gcd(\lambda, n) = 1\}$ and $n_2 = \max\{\mu : \mu \text{ is a factor of } n \text{ and } \gcd(\mu, k) = 1\}$. Put $k_1 = k/k_2$ and $n_1 = n/n_2$. Then the sequence (k_1, k_2, n_1, n_2) satisfies three conditions in Lemma 2.6. Moreover, it is easy to show that the sequence (k_1, k_2, n_1, n_2) satisfying these three conditions is unique, so the process can be omitted. $\square$

The main result in this section is the following lemma.

LEMMA 2.7. *Suppose that* $S = (x_1, x_2, \ldots) = (x_1, \ldots, x_{nk})^\circ$ *is a periodic sequence with* $n > 1$ *and* $k > 1$, *and* $S_k = (x_1, x_{k+1}, x_{2k+1}, \ldots, x_{(n-1)k+1})^\circ$ *is a k-subsequence of S. Let* (k_1, k_2, n_1, n_2) *be the same as in Lemma* 2.6. *If the period of S_k is n, then there is a factor λ of k_2 such that the period of S is $k_1 \lambda n$.*

PROOF. Let m be the period of the sequence S. According to Corollary 2.3, m is a factor of kn. Then $\gcd(m, kn) = m$. Write $n_3 = \gcd(m, n_2)$. Then $n_1 n_3$ is a factor of $n = n_1 n_2$. As $\gcd(n_2, kn_1) = 1$, we have $m = \gcd(m, kn) = \gcd(m, kn_1 n_2) = \gcd(m, kn_1) \cdot \gcd(m, n_2) = \gcd(m, kn_1) \cdot n_3$. Hence m is a factor of $kn_1 n_3$, which implies

$$x_{i + kn_1 n_3} = x_i \quad \text{for all } i \in \mathbb{N}. \tag{2.2}$$

On the other hand, if $n_1 n_3 < n$, then $x_{j + kn_1 n_3} = x_j$ does not hold for some $j \in \{1, k+1, 2k+1, \ldots, (n-1)k+1\}$ since the period of the sequence S_k is n. This will contradict to (2.2). Thus we must have $n_1 n_3 = n$, which means that $n_2 = n_3 = \gcd(m, n_2)$. Hence we obtain:

CLAIM 1. $m = n_2 r$ for some $r \in \mathbb{N}$.

Let $k_3 = \gcd(m, k_1n_1)$. As $\gcd(k_1n_1, k_2n_2) = 1$, we have

$$\begin{aligned} m &= \gcd(m, kn) = \gcd(m, k_1n_1 \cdot k_2n_2) \\ &= \gcd(m, k_1n_1) \cdot \gcd(m, k_2n_2) = k_3 \cdot \gcd(m, k_2n_2). \end{aligned}$$

Let k_4 $=\mathrm{lcm}(k_1, k_3)$ be the least common multiple of k_1 and k_3. Then m is a factor of $k_4k_2n_2$, and hence

$$x_i = x_{i+k_4k_2n_2} \quad \text{for all } i \in \mathbb{N}. \tag{2.3}$$

On the other hand, if k_3 is a proper factor of k_1n_1, then $k_1 > 1$, $n_1 > 1$, and from the condition (b) of Lemma 2.6, we see that k_4 is also a proper factor of k_1n_1, which implies that $k_4k_2n_2$ is a proper factor of $nk = k_1n_1k_2n_2$. Thus there is a proper factor n_4 of n such that $k_4k_2n_2 = kn_4$. However, $x_{j+kn_4} = x_j$ does not hold for some $j \in \{1, k+1, 2k+1, \ldots, (n-1)k+1\}$ since the period of the sequence S_k is n. This will contradict to (2.3). Thus we must have $k_3 = \gcd(m, k_1n_1) = k_1n_1$ and hence we obtain

Claim 2. $m = k_1n_1r$ for some $r \in \mathbb{N}$.

As $\gcd(k_1n_1, n_2) = 1$, by Claims 1 and 2, we see that $m = k_1n_1n_2r = k_1nr$ for some $r \in \mathbb{N}$. Hence there exists a factor λ of k_2 such that $m = k_1\lambda n$ since m is a factor of $kn = k_1k_2n_1n_2$. □

Conversely, we have

Lemma 2.8. *Let k, n and (k_1, k_2, n_1, n_2) be the same as in Lemma 2.6. Then, for any factor λ of k_2, there exists a $k_1\lambda n$-periodic sequence $S = (x_1, x_2, \ldots) = (x_1, \ldots, x_{kn})^\circ$ such that the period of the k-subsequence $S_k = (x_1, x_{k+1}, x_{2k+1}, \ldots, x_{(n-1)k+1}, \ldots)$ is n.*

Proof. Let $m = k_1\lambda n$. Then m is a factor of kn. Take an m-periodic sequence $S = (x_1, x_2, \ldots) = (x_1, \ldots, x_m)^\circ$ such that $x_1, \ldots, x_m$ are pairwise different elements. Noting that $x_{i+m} = x_i$ for all $i \in \mathbb{N}$, we can also write $S = (x_1, \ldots, x_{kn})^\circ$. For $0 \le i < j \le n-1$, we have $(j-i)k_2/n \notin \mathbb{N}$ since $\gcd(k_2, n) = 1$, which implies that $(j-i)k/(k_1\lambda n) = (j-i)k_2/(\lambda n) \notin \mathbb{N}$, and hence $jk + 1 \not\equiv ik + 1 \pmod{k_1\lambda n}$. Thus $x_1, x_{k+1}, x_{2k+1}, \ldots, x_{(n-1)k+1}$ are pairwise different elements, and hence the period of S_k is n. □

Lemma 2.9. *Suppose that $S = (x_1, x_2, \ldots) = (x_1, \ldots, x_{kn})^\circ$ is a kn-periodic sequence with $k \ge 2$ and $n \ge 2$. Let $S_i = (x_i, x_{k+i}, x_{2k+i}, \ldots, x_{(n-1)k+i})^\circ$, for each $i \in \mathbb{N}$, be the i-th k-subsequence of S. Then:*

(a) *There exists $i \in \{1, \ldots, k\}$ such that the period of S_i is a factor of n greater than 1.*
(b) *If there exist a prime number p and $\lambda \in \mathbb{N}$ such that $n = p^\lambda$, then there exists $i \in \{1, \ldots, k\}$ such that the period of S_i is n.*

PROOF. Since the length of the finite sequence $(x_i, x_{k+i}, x_{2k+i}, \dots, x_{(n-1)k+i})$ is n, by Corollary 2.3, the period of S_i must be a factor of n.

(a) is obvious, since, otherwise, if for each $i \in \{1, \dots, k\}$, the period of S_i is 1, then the period of S will be a factor of k, which contradicts the condition of the lemma that period of S is kn.

(b) is also obvious, since, otherwise, if for each $i \in \{1, \dots, k\}$, the period of S_i is a proper factor of $n = p^\lambda$, then the period of S will be a proper factor of kn, which also contradicts the condition of the lemma. $\square$

REMARK 2.10. In Lemma 2.9, if n is not an integral power of some prime number, then it is possible that the period of any k-subsequence of S is a proper factor of n. For example, let $k = 2$, $n = 6$, and let x_1, x_2, y_1, y_2, y_3 be pairwise different elements. Then the period of any 2-subsequence of the 12-periodic sequence $S = (x_1, y_1, x_2, y_2, x_1, y_3, x_2, y_1, x_1, y_2, x_2, y_3)^\circ$ is a proper factor of 6.

From Lemma 2.7 we get

COROLLARY 2.11. *Suppose that X is a set and $F\colon X \to 2^X - \{\emptyset\}$ is a multivalued map. Let k, n and (k_1, k_2, n_1, n_2) be the same as in Lemma 2.6. If F^k has an n-periodic orbit, then F itself has a periodic orbit, of which the period is a factor of kn and is an integral multiple of $k_1 n$.*

PROOF. Let $O_k = (x_1, x_{k+1}, x_{2k+1}, \dots, x_{(n-1)k+1})^\circ$ be an n-periodic orbit of F^k. By Remark 2.1, O_k can be extended to be a periodic orbit

$$O = (x_1, \dots, x_k, x_{k+1}, \dots, x_{2k}, x_{2k+1}, \dots, x_{(n-1)k+1}, \dots, x_{nk})^\circ$$

of F. By Lemma 2.7, the period of O is $k_1 \lambda n$ for some factor λ of k_2. $\square$

From Lemma 2.9 we get the following corollary at once.

COROLLARY 2.12. *Let $F\colon X \to 2^X - \{\emptyset\}$ be a multivalued map. Suppose that F has a kn-periodic orbit $O = (x_1, x_2, \dots) = (x_1, \dots, x_{kn})^\circ$ with $k \geq 2$ and $n \geq 2$. Then:*

(a) *The k-th iterate F^k has a periodic orbit, whose period is a factor of n greater than 1.*
(b) *If there exist a prime number p and $\lambda \in \mathbb{N}$ such that $n = p^\lambda$, then F^k has an n-periodic orbit.*

3. Multivalued maps with continuous margins of intervals

Let I be a bounded connected subset of $\mathbb{R}$. Recall that each map $F\colon I \to \mathcal{L}(I)$ is called a *connected-multivalued map* on I, and F is a *multivalued map with continuous margins* if both the left endpoint $\alpha\colon I \to \overline{I}$ and the right endpoint functions $\beta\colon I \to \overline{I}$ of F are continuous.

Lemma 3.1. *Let $F\colon I \to \mathcal{L}(I)$ and $G\colon I \to \mathcal{L}(I)$ be multivalued maps with continuous margins. Then the composite function $G \circ F$ also is a multivalued map with continuous margins from I to $\mathcal{L}(I)$.*

Proof. Let α_1, β_1 and α_2, β_2 be the left endpoint and right endpoint functions of F and G, respectively. Define $\alpha_3\colon I \to \overline{I}$ and $\beta_3\colon I \to \overline{I}$ by

$$\alpha_3(x) = \inf\{\alpha_2(y) : y \in F(x)\} \quad \text{and} \quad \beta_3(x) = \sup\{\beta_2(y) : y \in F(x)\},$$

$x \in I$. For any $u, v \in \mathbb{R}$, denote by $\langle u, v\rangle$ the smallest connected subset containing u and v in $\mathbb{R}$. Then we have

$$(\alpha_3(x), \beta_3(x)) \subset G \circ F(x) \subset [\alpha_3(x), \beta_3(x)]$$

since $F(x)$ is connected and α_2 is continuous. It is easy to see that for any $x, w \in I$, $|\alpha_3(w) - \alpha_3(x)| \leq \max\{S_1, S_2\}$, where

$$\begin{aligned} S_1 &= \sup\{\alpha_2(u) - \alpha_2(v) : \{u, v\} \subset \langle \alpha_1(x), \alpha_1(w)\rangle \cap I\}, \\ S_2 &= \sup\{\alpha_2(u) - \alpha_2(v) : \{u, v\} \subset \langle \beta_1(x), \beta_1(w)\rangle \cap I\}. \end{aligned}$$

Noting that α_1, β_1 and α_2 are continuous, we derive that $\alpha_3(w) \to \alpha_3(x)$ as $w \to x$. Thus α_3 is continuous. In a similar fashion, we can show that β_3 is also continuous. Hence $G \circ F$ is a multivalued map with continuous margins from I to $\mathcal{L}(I)$. □

Definition 3.2. Let $F\colon X \to 2^X - \{\emptyset\}$ be a multivalued map, and $f\colon X \to X$ be a single-valued map. We say that F *contains* f or f *is contained by* F if $f(x) \in F(x)$ for any $x \in X$. If f is contained by F, then we write $f \dot{\in} F$.

The following is one of the key lemmas in this paper.

Lemma 3.3. *Let $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins and $n \in \mathbb{N}$. Then for any pairwise different points $x_1, \ldots, x_n$ in I and any given $y_i \in F(x_i)$, $1 \leq i \leq n$, there exists a continuous map $f\colon I \to I$ such that $f \dot{\in} F$ and $f(x_i) = y_i$ for every $1 \leq i \leq n$.*

Proof. Let α and β be the left endpoint and right endpoint functions of F, respectively. For any $i \in \{1, \ldots, n\}$, obviously, there is a real number $t_i \in [0, 1]$ such that $y_i = t_i\alpha(x_i) + (1 - t_i)\beta(x_i)$. Take a continuous function $t\colon I \to [0, 1]$ such that

$$t(x_i) = t_i \quad \text{for } i \in \{1, \ldots, n\}, \tag{3.1}$$

$$t(x) \in (0, 1) \quad \text{for any } x \in I - \{x_1, \ldots, x_n\}. \tag{3.2}$$

Define $f\colon I \to I$ by

$$f(x) = t(x) \cdot \alpha(x) + (1 - t(x)) \cdot \beta(x) \quad \text{for any } x \in I. \tag{3.3}$$

Then f is continuous. By (3.1) and (3.3), we have $f(x_i) = y_i$ for $i \in \{1, \ldots, n\}$. By (3.2) and (3.3), we get

$$f(x) \in (\alpha(x), \beta(x)) \subset F(x) \quad \text{for any } x \in I - \{x_1, \ldots, x_n\}.$$

Thus $f \dot{\in} F$. □

From Lemma 3.3 we obtain the following corollary at once.

COROLLARY 3.4. *Let $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins, and $O = (x_1, x_2, \ldots) = (x_1, \ldots, x_n)^\circ$ be an n-periodic orbit of F, where $n \in \mathbb{N}$. If $x_1, \ldots, x_n$ are pairwise different, then F contains a continuous map $f\colon I \to I$ such that $O = (x_1, \ldots, x_n)^\circ$ is also an n-periodic orbit of f, and hence, for any $m \in \mathbb{N}$ with $n \succ m$, f and F have an m-periodic orbit.*

If $(x_1, x_2, \ldots) = (x_1, x_2)^\circ$ is a 2-periodic sequence, then we must have $x_1 \neq x_2$. Therefore, from Corollary 3.4 we get

COROLLARY 3.5. *If a multivalued map with continuous margins $F\colon I \to \mathcal{L}(I)$ has a 2-periodic orbit, then F has a 1-periodic orbit.*

COROLLARY 3.6. *Let $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins. If F has a 3-periodic orbit $(x_1, x_2, \ldots) = (x_1, x_2, x_3)^\circ$, then F has an m-periodic orbit for any $m \in \mathbb{N}$.*

PROOF. By Corollary 3.4, we can consider only the case that $x_i = x_j$ for some $1 \leq i < j \leq 3$, that is, there exists $k \in \{1, 2, 3\}$ such that $x_k = x_{k+1} \neq x_{k+2}$. From this we see that F has a 1-periodic orbit $(x_k)^\circ$, a 2-periodic orbit $(x_k, x_{k+2})^\circ$, and an m-periodic orbit $(x_k, x_{k+2}, y_1, \ldots, y_{m-2})^\circ$ for any $m \geq 3$, where $y_1 = \ldots = y_{m-2} = x_k$. □

COROLLARY 3.7. *Let $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins. If F has a 4-periodic orbit $(x_1, x_2, \ldots) = (x_1, x_2, x_3, x_4)^\circ$, then F has a 2-periodic orbit.*

PROOF. By Corollary 3.4, we can consider only the case that $x_i = x_j$ for some $1 \leq i < j \leq 5$ with $i \leq 4$ and $j \leq i+2$. If $j = i+1$, then F has a 3-periodic orbit $(x_{j+1}, x_{j+2}, x_{j+3})^\circ$, and hence has a 2-periodic orbit. If $j = i+2$, then at least one of the two orbits $(x_i, x_{i+1})^\circ$ and $(x_j, x_{j+1})^\circ$ is a 2-periodic orbit. □

LEMMA 3.8. *Let $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins. If F has a 2^λ-periodic orbit, then F has a $2^{\lambda-1}$-periodic orbit.*

PROOF. It follows from Corollaries 3.5 and 3.7 that Lemma 3.8 holds for the case that $\lambda \in \{1, 2\}$. In what follows we can assume that $\lambda \geq 3$. By (b) of Corollary 2.12, we see that $F^{2^{\lambda-2}}$ has a 4-periodic orbit. This combining with Corollary 3.7 implies that $F^{2^{\lambda-2}}$ has a 2-periodic orbit. Using Corollary 2.11

in the case that $k = k_1 = 2^{\lambda-2}$ and $n = 2$, we see that F has a $2^{\lambda-1}$-periodic orbit. □

Now we give the main result of this paper and its proof.

Theorem 3.9. *Let I be a bounded connected subset of $\mathbb{R}$ and $F\colon I \to \mathcal{L}(I)$ be a multivalued map with continuous margins. For any $m, n \in \mathbb{N}$ with $n \succ m$, if F has an n-periodic orbit, then F has an m-periodic orbit.*

Proof. If $x_1, \ldots, x_n$ are pairwise different points, then by Corollary 3.4, we see that Theorem 3.9 holds. We can add the following hypothesis:

(H_1) There exist $1 \le i < j \le i + n - 2$ such that $x_i = x_j \ne x_{j+1}$ and $j - i$ is the least, that is, if there exist $1 \le i' < j' \le i' + n - 2$ such that $x_{i'} = x_{j'} \ne x_{j'+1}$, then $j' - i' \ge j - i$. Further, we may assume that $x_{j+1} > x_j$.

By Lemmas 3.6 and 3.8, we can add the following hypothesis:

(H_2) For any $\lambda \in \mathbb{N}$, $3 \succ n \succ 2^\lambda$, and it has been proved that, for any $n_0 \in \mathbb{N}$ with $3 \succeq n_0 \succ n$ and for any multivalued map with continuous margins $G\colon I \to \mathcal{L}(I)$, if G has an n_0-periodic orbit, then for any $m \in \mathbb{N}$ with $n_0 \succ m$, G has an m-periodic orbit.

There are three cases to be considered.

Case 1. $n > 3$ is odd and $j - i \ge 2$.

In this case, by (H_1), $O_1 \equiv (x_i, \ldots, x_{j-1})^\circ$ and $O_2 \equiv (x_j, \ldots, x_{i+n-1})^\circ$ are also periodic orbits of F, whose periods are greater than 1 and are factors of $j-i$ and $i+n-j$, respectively. Hence, since one of the integers $j - i$ and $i + n - j$ is odd, F has an n_0-periodic orbit for some odd n_0 with $3 \succeq n_0 \succ n$. Therefore, by (H_2), for any $m \in \mathbb{N}$ with $n \succ m$, F has an m-periodic orbit.

Case 2. $n > 3$ is odd and $j - i = 1$.

There are two subcases.

Subcase 2.1. There is $k \in \{3, \ldots, n-1\}$ such that $x_{i+k} = x_i$. In this subcase, $O_1 \equiv (x_i, \ldots, x_{i+k-1})^\circ$ and $O_2 \equiv (x_{i+1}, \ldots, x_{i+k-1})^\circ$ are periodic orbits of F, whose periods are greater than 1 and are factors of k and $k - 1$, respectively. Since one of the integers k and $k - 1$ is odd, similar to Case 1, for any $m \in \mathbb{N}$ with $n \succ m$, F has an m-periodic orbit.

Subcase 2.2. $x_{i+\lambda} \ne x_i$ for any $\lambda \in \{2, \ldots, n-1\}$. In this subcase, there is $k \in \{2, \ldots, n-1\}$ such that $x_{i+k+1} \le x_i$ and $x_{i+\lambda} > x_i$ for $\lambda \in \{2, \ldots, k\}$. Let $Z_0 = \{\lambda : \lambda \in \{2, \ldots, k\}$ and $x_{i+\lambda} \ge x_{i+k}\}$. Then $k \in Z_0$. Let $q = \min Z_0$. If $q > 2$, then $x_i < x_{i+q-1} < x_{i+k} \le x_{i+q}$. By Lemma 3.3, F contains a continuous map $f\colon I \to I$ such that $f(x_i) = x_i$, $f(x_{i+q-1}) = x_{i+q} \ge x_{i+k}$ and $f(x_{i+k}) = x_{i+k-1} \le x_1$. Thus f is turbulent since $f([x_i, x_{i+q-1}]) \cap f([x_{i+q-1}, x_k]) \supset [x_i, x_k]$.

It is well-known that a turbulent interval map, f (and hence F), has an m-periodic orbit for any $m \in \mathbb{N}$.

If $q = 2$, then $x_{i+k} \in (x_i, x_{i+2}] \subset F(x_i)$. By Lemma 3.3, F contains a continuous map $f\colon I \to I$ such that $f(x_i) = x_{i+2} \geq x_{i+k} > x_i$ and $f(x_{i+k}) = x_{i+k+1} \leq x_i$, which implies that there is a point $y \in (x_i, x_{i+k}]$ such that $f(y) = x_i$, and hence F has a 3-periodic orbit $(x_i, x_i, y)^\circ$. By Lemma 3.6, F has an m-periodic orbit for any $m \in \mathbb{N}$.

Case 3. $n = 2^\lambda(2\mu + 1)$ for some $\lambda, \mu \in \mathbb{N}$.

In this case, from (a) of Corollary 2.12 we see that F^{2^λ} has a periodic orbit which period is a factor of $2\mu+1$ greater than 1. By Lemma 3.8, we may assume that $n \succ m \succ 2^\lambda$.

If $n \succ m \succ 3 \cdot 2^{\lambda+1}$, then there is $\mu_0 \in \mathbb{N}$ such that $m = 2^\lambda(2\mu + 2\mu_0 + 1)$. By hypothesis (H_2), F^{2^λ} has a $(2\mu + 2\mu_0 + 1)$-periodic orbit. By Corollary 2.11, there is a factor k_2 of 2^λ such that F has a $k_2(2\mu + 2\mu_0 + 1)$-periodic orbit O_m. If $k_2 = 2^\lambda$, O_m itself is an m-periodic orbit of F. If k_2 is a proper factor of 2^λ, then $3 \succeq k_2(2\mu + 2\mu_0 + 1) \succ n$, and from (H_2) we see that F has an m-periodic orbit.

If $3 \cdot 2^{\lambda+1} \succeq m \succ 2^\lambda$, then there is $m_0 \in \mathbb{N}$ such that $m = 2^\lambda \cdot 2m_0$. By hypothesis (H_2), F^{2^λ} has a $2m_0$-periodic orbit. Using Corollary 2.11 to the case that $k = k_1 = 2^\lambda$, we see that F has an m-periodic orbit. $\square$

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