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EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF GROUND STATE SOLUTION FOR QUASILINEAR SCHRÖDINGER–POISSON SYSTEMS IN $\mathbb{R}^3$

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Abstract. In this paper, we are concerned with existence and asymptotic behavior of ground state in the whole space $\mathbb{R}^3$ for quasilinear Schrödinger–Poisson systems

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u), & x \in \mathbb{R}^3, \\ -\mathrm{div}[(1+\varepsilon^4|\nabla\phi|^2)\nabla\phi] = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

when the nonlinearity coefficient $\varepsilon > 0$ goes to zero, where $f(t)$ is asymptotically linear with respect to t at infinity. Under appropriate assumptions on K, a and f, we establish existence of a ground state solution $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ of the above system. Furthermore, for all ε sufficiently small, we show that $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ converges to $(u_0, \phi_{0,K}(u_0))$ which is the solution of the corresponding system for $\varepsilon = 0$.

1. Introduction and main results

Consider the following quasilinear Schrödinger–Poisson systems

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u), & x \in \mathbb{R}^3, \\ -\mathrm{div}[(1+\varepsilon^4|\nabla\phi|^2)\nabla\phi] = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $K \in L^2(\mathbb{R}^3)$, $K \geq (\not\equiv)\, 0$, a is a positive bounded function, and $f \in C(\mathbb{R}, \mathbb{R}^+)$. When $\varepsilon = 0$, this Schrödinger–Poisson system arises in an interesting physical model which describes the interaction of a charged particle with electromagnetic field (see [3] and the references therein). When $\varepsilon \neq 0$, system (1.1) firstly arises this form like

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta u + (V + \phi(x))u, & x \in \mathbb{R}^3, \\ -\mathrm{div}[\varepsilon(\nabla\phi)\nabla\phi] = |u|^2 - n^*, & x \in \mathbb{R}^3, \\ u(x,0) = u(x), & x \in \mathbb{R}^3, \end{cases}$$

which corresponds to a quantum mechanical model where the quantum effects are important, as in the case of microstructures (see for example Markowich, Ringhofer and Schmeiser [22]). The charge density $n(x,t)$ derives from the Schrödinger wave function $u(x,t)$ by $n(x,t) = |u(x,t)|^2$, while n^* and V represent respectively a dopant-density and a real effective potential which are time-independent. More details dealing with the phenomenon may be found in [17], [18] and references therein. After that, in [1], [16], that the field dependent dielectric constant in Poisson equation has the form

$$\varepsilon(\nabla\phi) = \varepsilon^0 + \varepsilon^1|\nabla\varphi|^2, \quad \varepsilon^0, \varepsilon^1 > 0.$$

Existence and uniqueness of global strong solutions and existence results of solutions of the form $u(x,t) = e^{i\omega t}u(x)(\omega,\ u(x) \in \mathbb{R})$ are obtained under suitable conditions, respectively. Moreover, in [4], authors obtained that the existence of standing waves (actually ground states) solutions for the Schrödinger–Poisson system with $\varepsilon^0 = 1$ and $\varepsilon^1 = \varepsilon^4$ of

$$\begin{cases} -\dfrac{1}{2}\Delta u + (V + \phi(x))u = 0, & x \in \mathbb{R}^3, \\ -\mathrm{div}[(1 + \varepsilon^4|\nabla\phi|^2)\nabla\phi] = |u|^2 - n^*, & x \in \mathbb{R}^3 \end{cases} \tag{1.2}$$

and with their asymptotic behavior when the nonlinearity coefficient in the Poisson equation ε goes to zero with suitable potential V.

From the mathematical view, if the Schrödinger equation with only one nonlinear nonlocal term $\phi(x)u$ in system (1.2) is replaced by the other different version of Schrödinger equations which have other nonlinear terms besides the nonlinear nonlocal term, we want to know that whether ground state solutions exist and if exists whether they converge to ones of the corresponding system for $\varepsilon = 0$. In this paper, we shall answer these questions about system (1.1).

When $\phi \equiv 0$, system (1.1) becomes into a single equation

$$-\Delta u + u = a(x)f(u). \tag{1.3}$$

Problem (1.3) has been studied extensively in the last decade, see [9]–[27] and so on. In these mentioned papers, the condition: $f(t)/t$ is nondecreasing in $t > 0$ is usually assumed to prove that a (PS) sequence is bounded.

When $\phi(x) \not\equiv 0$ and $\varepsilon \equiv 0$, Cerami and Vaira in [6] studied system (1.1) with $f(t) = |t|^{p-1}u(p \in (3,5))$ and obtained the existence of positive ground state solutions by minimizing the corresponding energy functional restricted to the Nehari manifold when K and a satisfy different assumptions, respectively. Sun, Chen and Nieto in [28] also studied system (1.1) with general f which is asymptotically linear at infinity and obtain the existence of a positive ground state solution under suitable assumptions about K and a by Mountain Pass Theorem. Furthermore, there are abundant results with respect to Schrödinger–Poisson system, see [10]–[31] and so on.

When $\phi(x) \not\equiv 0$ and $\varepsilon \not\equiv 0$, there are some results with respect to Schrödinger–Possion systems depending on a parameter ε, see [12], [11], [25], [26], [15] and the references therein. In [12], [11], [25], [26], the perturbation parameter ε appears in the first Schrödinger equation of system, the domain is a flat domain or a Riemannian manifold in $\mathbb{R}^3$ and concentration of solutions were mainly studied. In [15], the perturbation parameter ε appears in the exponent of the nonlinearity ($f(t) = t^{6-\varepsilon}$) and multiplicity positive solutions are obtained. But there are few results for system (1.1) with $\phi(x) \not\equiv 0$ where the perturbation parameter $\varepsilon \not\equiv 0$ appears in the second equation. So, we want to fill this gap. To our best knowledge, this is the first paper which consider this type of problem.

Before stating our main results, we give some notations. For any $1 \leq q \leq +\infty$, we denote by $\|\cdot\|_q$ the usual norm of the Lebesgue space $L^q(\mathbb{R}^3)$. Define the function space

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the standard product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\,dx, \qquad \|u\| := \bigg(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\,dx\bigg)^{1/2}.$$

Define the function space

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the usual norm $\|u\|_{D^{1,2}} := \|\nabla u\|_2$. Define

$$D^{1,4}(\mathbb{R}^3) := \{u \in C_0(\mathbb{R}^3) : \nabla u \in L^4(\mathbb{R}^3)\}$$

with the usual norm $\|u\|_{D^{1,4}} := \|\nabla u\|_4$. And the space $D^{1,2} \cap D^{1,4}(\mathbb{R}^3)$ is equipped with the natural norm $\|u\|_{D^{1,2}\cap D^{1,4}} = \|\nabla u\|_2 + \|\nabla u\|_4$. Recall that Sobolev's inequalities with the best constant S and S^* are

$$\|v\|_6^2 \leq S\|\nabla v\|_2^2, \qquad \|v\|_6^2 \leq S^*\|v\|^2.$$

Here are the main results of this paper.

Theorem 1.1. *Suppose that the following conditions hold*:

(f1) $f \in C(\mathbb{R}, \mathbb{R}^+)$, $f(0) = 0$, *and* $f(t) \equiv 0$ *for* $t < 0$.

(f2) $\lim\limits_{t \to 0} f(t)/t = 0$.

(f3) $\lim\limits_{t \to +\infty} f(t)/t = l < +\infty$.

(A1) $a(x)$ *is a positive continuous function and there exists* $R_0 > 0$ *such that*

$$\sup\{f(t)/t : t > 0\} < \inf\{1/a(x) : |x| \geq R_0\}.$$

(A2) *There exists a constant* $\beta \in (0,1)$ *such that*

$$(1-\beta)l > \mu^* := \inf\bigg\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\, dx : u \in H^1(\mathbb{R}^3, \mathbb{R}^+), \\ \int_{\mathbb{R}^3} a(x)F(u)\, dx \geq \frac{l}{2},\ \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K} u^2\, dx < \frac{4}{3}\beta l \bigg\},$$

where $F(t) = \int_0^t f(s)\, ds$.

(K1) $K \in L^2(\mathbb{R}^3)$, $K \geq (\not\equiv) 0$ *for all* $x \in \mathbb{R}^3$.

Then system (1.1) *possesses a ground state solution* $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ *in* $H^1(\mathbb{R}^3)$ *for all* $\varepsilon > 0$.

Remark 1.2. When $\varepsilon \equiv 0$, system (1.1) has been studied in [28] and possesses a ground state solution in $H^1(\mathbb{R}^3)$ under the same conditions of Theorem 1.1. In this case, solvability of such Schrödinger–Poisson systems begins the unique positive solution of the linear Poisson equation in $D^{1,2}(\mathbb{R}^3)$ denoted by $\phi_{0,K}(u)$ which is the Newtonian potential of $K(x)u^2$ and has the explicit formula

$$\phi_{0,K}(u(x)) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}\, dy.$$

Clearly, this solution has some good properties. But when $\varepsilon > 0$, we will solve a quasilinear Poisson equation

$$-\mathrm{div}[(1 + \varepsilon^4 |\nabla\phi|^2)\nabla\phi] = K(x)|u|^2$$

which has a unique weak nonnegative solution $\phi_{\varepsilon,K}(u)$ in the space $D^{1,2} \cap D^{1,4}(\mathbb{R}^3)$ in the following Lemma 2.2. Moreover, Theorem 1.1 generalizes Theorem 1.1 in [28] where the author only studied the special situation, that is, $\varepsilon = 0$. Functions K, a, f satisfying Theorem 1.1 can be constructed by the same method as Remark 1.1 in [4].

Furthermore, we want to know the asymptotic behavior u_ε when $\varepsilon \to 0$. We have the following result.

Theorem 1.3. *If* $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ *denotes the ground state solution of system* (1.1) *obtained by Theorem* 1.1, *then* u_ε *is bounded in* $H^1(\mathbb{R}^3)$ *and any limit point* *of* $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ *when* $\varepsilon \to 0$ *is a solution* $(u_0, \phi_{0,K}(u_0))$ *of system* (1.1) *with* $\varepsilon \equiv 0$.

In order to obtain our results, we have to overcome various difficulties. First, the competing effect of the quasilinear non-local term in the functional of system (1.1) gives rise to some difficulties. Second, since the embedding of $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in [2,6]$, is not compact, condition (A1) is crucial to obtain the boundedness of Cerami sequence. Furthermore, in order to recover the compactness, we establish a compactness result $\int_{|x|\geq R}(|\nabla u_n|^2 + |u_n|^2)\,dx \leq \varepsilon'$ similar to the one in [28] but different from the one in [6]. In fact, this difficulty can be avoided, when autonomous problems are considered, restricting the corresponding functional to the subspace of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions, or, when one is looking for semi-classical states, by using perturbation methods or a reduction to a finite dimension by the projections method. Third, it is not difficult to find that every (PS) sequence is bounded when $3 < p < 5$ in [6] because a variant of global Ambrosetti–Rabinowitz condition is satisfied when $3 < p < 5$ (see [10]). However, for the asymptotically linear case, we have to find another method to verify the boundedness of (PS) sequence.

This paper is organized as follows. In Section 2, some important preliminaries are listed out. In Sections 3 and 4, we manage to give proofs of Theorems 1.1 and 1.3. In the following discussion, we denote various positive constants as C or C_i $(i = 0, 1, \ldots)$ for convenience.

2. Preliminaries

System (1.1) has a variational structure. Its corresponding functional

$$J_\varepsilon \colon H^1(\mathbb{R}^3) \times (D^{1,2} \cap D^{1,4}(\mathbb{R}^3)) \to \mathbb{R}$$

defined by

$$\begin{aligned} J_\varepsilon(u,\phi) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi u^2\,dx \\ - \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx - \frac{\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi|^4\,dx - \int_{\mathbb{R}^3} a(x)F(u)\,dx. \end{aligned}$$

Evidently, from conditions of Theorem 1.1, the action functional $J_\varepsilon \in C^1(H^1(\mathbb{R}^3) \times(D^{1,2} \cap D^{1,4}(\mathbb{R}^3)), \mathbb{R})$ and the partial derivatives in (u,ϕ) are given, for $\zeta \in H^1(\mathbb{R}^3)$ and $\eta \in D^{1,2} \cap D^{1,4}(\mathbb{R}^3)$, we have

$$\begin{aligned} \left\langle \frac{\partial J_\varepsilon}{\partial u}(u,\phi), \zeta \right\rangle &= \int_{\mathbb{R}^3} (\nabla u \cdot \nabla\zeta + u\zeta + K(x)\phi u\zeta - a(x)f(u)\zeta)\,dx, \\ \left\langle \frac{\partial J_\varepsilon}{\partial \phi}(u,\phi), \eta \right\rangle &= -\frac{1}{2}\int_{\mathbb{R}^3} (\nabla\phi\cdot\nabla\eta + \varepsilon^4|\nabla\phi|^2\nabla\phi\cdot\nabla\eta - K(x)u^2\eta)\,dx. \end{aligned}$$

Thus, we have the following result:

Proposition 2.1. *The pair (u,ϕ) is a weak solution of the system* (1.1) *if and only if it is a critical point of J_ε in $H^1(\mathbb{R}^3)\times(D^{1,2}\cap D^{1,4}(\mathbb{R}^3))$.*

Lemma 2.2. *Assume that* (K1) *holds. For any $u\in H^1(\mathbb{R}^3)$ and all $\varepsilon>0$, there is a unique nonnegative weak solution $\phi_{\varepsilon,K}(u)\in D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$ for*

$$-\mathrm{div}[(1+\varepsilon^4|\nabla\phi|^2)\nabla\phi]=K(x)u^2,\quad x\in\mathbb{R}^3. \tag{2.1}$$

Furthermore, for any $\psi\in D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3}(1+\varepsilon^4|\nabla\phi_{\varepsilon,K}(u)|^2)\nabla\phi_{\varepsilon,K}\cdot\nabla\psi\,dx=\int_{\mathbb{R}^3}K(x)u^2\psi\,dx.$$

Proof. Equation (2.1) is the special case of one of Lemma 3.1 in [4], so we write its proof for completeness.

For any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$ and $K\in L^2(\mathbb{R}^3)$, by the Hölder inequality and Sobolev inequality, we have

$$\int_{\mathbb{R}^3}(K(x)u^2)^{6/5}\,dx\le\|K(x)\|_2^{6/5}\|u\|_6^{12/5}\le(S^*)^{6/5}\|K(x)\|_2^{6/5}\|u\|^{12/5}.$$

Therefore, $K(x)u^2\in L^{6/5}$. The corresponding functional of (2.1) is

$$\widetilde{J}_\varepsilon(\phi)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx+\frac{\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi|^4\,dx-\int_{\mathbb{R}^3}K(x)u^2\phi\,dx$$

for $\phi\in D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$. Therefore, by the Hölder inequality and Sobolev inequality, we get

$$\begin{aligned}\widetilde{J}_\varepsilon(\phi)&\ge\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx+\frac{\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi|^4\,dx-\|K(x)u^2\|_{6/5}\|\phi\|_6\\&\ge\frac{1}{2}\|\nabla\phi\|_2^2+\frac{\varepsilon^4}{4}\|\nabla\phi\|_4^4-S^{1/2}\|K(x)u^2\|_{6/5}\|\nabla\phi\|_2\to+\infty\end{aligned}$$

as $\|\phi\|_{D^{1,2}\cap D^{1,4}}\to+\infty$. That is, the functional $\widetilde{J}_\varepsilon(\phi)$ is coercive. So, $\widetilde{J}_\varepsilon$ has a bounded minimizing sequence $\{\phi_n\}$ such that

$$\widetilde{J}_\varepsilon(\phi_n)\to\inf_{D^{1,2}\cap D^{1,4}(\mathbb{R}^3)}\widetilde{J}_\varepsilon(\phi)$$

whenever $n\to\infty$. Let

$$G_\varepsilon(\phi_n)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi_n|^2\,dx+\frac{\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi_n|^4\,dx\quad\text{and}\quad L(\phi_n)=\int_{\mathbb{R}^3}K(x)u^2\phi_n\,dx.$$

Clearly, G_ε is a strictly convex functional and L is a linear functional. So, $\widetilde{J}_\varepsilon(\phi_n)=G_\varepsilon(\phi_n)-L(\phi_n)$ is a strictly convex functional. Furthermore, $\widetilde{J}_\varepsilon$ is C^1. So, by Mazur's theorem (see, e.g. Theorem V.1.2 in [32]), $\widetilde{J}_\varepsilon$ is weakly lower semi-continuous on $D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$. It follows from the least action principle (see, e.g. Theorem 1.1 in [23]) that $\widetilde{J}_\varepsilon$ has a minimum on $D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$.

We claim that the minimum point of $\widetilde{J}_\varepsilon$ is unique. Otherwise, suppose that $\phi_{\varepsilon,K,1}(u)$ and $\phi_{\varepsilon,K,2}(u)$ are both minimum points of $\widetilde{J}_\varepsilon$. That is,

$$\widetilde{J}(\phi_{\varepsilon,K,1}(u)) = \widetilde{J}(\phi_{\varepsilon,K,2}(u)) = \inf_{D^{1,2}\cap D^{1,4}(\mathbb{R}^3)} \widetilde{J}_\varepsilon(\phi).$$

Because $\widetilde{J}_\varepsilon$ is strictly convex, for each $\alpha \in (0,1)$, we obtain

$$\begin{aligned}
&\widetilde{J}_\varepsilon(\alpha\phi_{\varepsilon,K,1}(u) + (1-\alpha)\phi_{\varepsilon,K,2}(u)) \\
&\qquad < \alpha\widetilde{J}_\varepsilon(\phi_{\varepsilon,K,1}(u)) + (1-\alpha)\widetilde{J}_\varepsilon(\phi_{\varepsilon,K,2}(u)) = \inf_{D^{1,2}\cap D^{1,4}(\mathbb{R}^3)} \widetilde{J}_\varepsilon(\phi).
\end{aligned}$$

This is a contradiction. So, $\widetilde{J}_\varepsilon$ has a unique minimum, then equation (2.1) has a unique weak solution $\phi_{\varepsilon,K}(u)$.

Next, we shall prove that the solution $\phi_{\varepsilon,K}(u)$ of equation (2.1) is nonnegative. Denote by $\phi^{\pm}_{\varepsilon,K}(u) := \max\{\pm\phi_{\varepsilon,K}(u), 0\}$ the positive (negative) part of $\phi_{\varepsilon,K}(u)$. Since $K(x)u^2 \geq 0$ and $\phi_{\varepsilon,K}(u)$ is a solution of equation (2.1), we deduce

$$-\mathrm{div}[(1+\varepsilon^4|\nabla\phi_{\varepsilon,K}(u)|^2)\nabla\phi_{\varepsilon,K}(u)] \geq 0, \quad x \in \mathbb{R}^3.$$

Multiplying this equation by $\phi^-_{\varepsilon,K}(u)$ with $\phi^-_{\varepsilon,K}(u) = \max\{-\phi_{\varepsilon,K}(u), 0\}$ and integrating on $\mathbb{R}^3$ by parts, we obtain

$$-\int_{\mathbb{R}^3}(|\nabla\phi^-_{\varepsilon,K}(u)|^2 + \varepsilon^4|\nabla\phi^-_{\varepsilon,K}(u)|^4) \geq 0.$$

This yields that $\|\phi^-_{\varepsilon,K}(u)\|_{D^{1,2}\cap D^{1,4}(\mathbb{R}^3)} = 0$, so, $\phi^-_{\varepsilon,K}(u) = 0$. Therefore, we obtain $\phi_{\varepsilon,K}(u) = \phi^+_{\varepsilon,K}(u) - \phi^-_{\varepsilon,K}(u) = \phi^+_{\varepsilon,K}(u) \geq 0$. Thus, $\phi_{\varepsilon,K}(u)$ is a nonnegative weak solution of (2.1).

From the above discussion, $\widetilde{J}_\varepsilon$ achieves its minimum at a unique nonnegative $\phi_{\varepsilon,K}(u) \in D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$ and therefore

$$\langle \widetilde{J}_\varepsilon{}'(\phi_{\varepsilon,K}(u)), \psi\rangle = 0 \quad \text{for all } \psi \in D^{1,2}\cap D^{1,4}(\mathbb{R}^3). \qquad \square$$

By Lemma 2.2, there exists a unique function $0 \leq \phi_{\varepsilon,K}(u) \in D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$ such that

$$-\mathrm{div}[(1+\varepsilon^4|\nabla\phi_{\varepsilon,K}(u)|^2)\nabla\phi_{\varepsilon,K}(u)] = K(x)u^2. \tag{2.2}$$

Substitute the solution $\phi_{\varepsilon,K}(u)$ in the first (Schrödinger) equation of the system (1.1), then get the corresponding functional $E_\varepsilon\colon H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by $E_\varepsilon(u) = J_\varepsilon(u, \phi_{\varepsilon,K}(u))$. After multiplying (2.2) by $\phi_{\varepsilon,K}(u)$ and integration by parts, we obtain

$$\int_{\mathbb{R}^3}(1+\varepsilon^4|\nabla\phi_{\varepsilon,K}(u)|^2)|\nabla\phi_{\varepsilon,K}(u)|^2\,dx = \int_{\mathbb{R}^3} K(x)u^2\phi_{\varepsilon,K}(u)\,dx. \tag{2.3}$$

Therefore, the reduced functional takes the form

$$E_\varepsilon(u) = \frac{1}{2}\|u\|^2 + I_\varepsilon(u) - \int_{\mathbb{R}^3} a(x)F(u)\,dx,$$

where

$$I_\varepsilon(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_{\varepsilon,K}(u)|^2\,dx + \frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla\phi_{\varepsilon,K}(u)|^4\,dx.$$

Clearly, $I_\varepsilon(u) \geq 0$ and $\int_{\mathbb{R}^3} K(x)u^2\phi_{\varepsilon,K}(u)\,dx \geq 0$. Now, we give the following definition:

Definition 2.3. $(u,\phi_{\varepsilon,K}(u))$ with $u \in H^1(\mathbb{R}^3)$ is a ground state solution of system (1.1), we mean that $(u,\phi_{\varepsilon,K}(u))$ is a solution of system (1.1) which has the least energy among all solutions of system (1.1), that is, $E'_\varepsilon(u)=0$ and $E_\varepsilon(u) = \inf\{E_\varepsilon(v) : v \in H^1(\mathbb{R}^3)\setminus\{0\}$ and $E'_\varepsilon(v)=0\}$.

Moreover, we have

Lemma 2.4. *For any $\varepsilon > 0$ the functional $u \mapsto I_\varepsilon(u)$ is C^1 on $H^1(\mathbb{R}^3)$ and its Fréchet-derivative satisfies*

$$\langle I'_\varepsilon(u),\psi\rangle = \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u)u\psi\,dx, \quad \textit{for all } u,\psi \in H^1(\mathbb{R}^3).$$

By suitably modifying the proof of Proposition 4.1 in [4], this lemma can be proved. Here we omit its proof.

By (2.3), the Hölder's inequality and Sobolev's inequalities, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla\phi_{\varepsilon,K}(u)|^2\,dx + \varepsilon^4\int_{\mathbb{R}^3} |\nabla\phi_{\varepsilon,K}(u)|^4\,dx &= \int_{\mathbb{R}^3} K(x)u^2\phi_{\varepsilon,K}(u)\,dx\\ &\leq \|\phi_{\varepsilon,K}(u)\|_6\|K\|_2\|u\|_6^2 \leq S^{1/2}S^*\|K\|_2\|\nabla\phi_{\varepsilon,K}(u)\|_2\|u\|^2.\end{aligned}$$

This yields

$$\|\nabla\phi_{\varepsilon,K}(u)\|_2 \leq S^{1/2}S^*\|K\|_2\|u\|^2 := C_0\|u\|^2 \tag{2.4}$$

and

$$\begin{aligned}\|\nabla\phi_{\varepsilon,K}(u)\|_2^2 &+ \varepsilon^4\|\nabla\phi_{\varepsilon,K}(u)\|_4^4\\ &\leq S^{1/2}S^*\|K\|_2\|\nabla\phi_{\varepsilon,K}(u)\|_2\|u\|^2 \leq S(S^*)^2\|K\|_2^2\|u\|^4.\end{aligned} \tag{2.5}$$

From (2.5), we obtain

$$\varepsilon^4\|\nabla\phi_{\varepsilon,K}(u)\|_4^4 \leq S(S^*)^2\|K\|_2^2\|u\|^4 := C_1\|u\|^4. \tag{2.6}$$

Combining (2.4), (2.6) and (f1), E_ε is well defined. Furthermore, together with Lemma 2.4, E_ε is a C^1 functional with derivative given, for all $v \in H^1(\mathbb{R}^3)$, by

$$\langle E'_\varepsilon(u),v\rangle = \int_{\mathbb{R}^3} (\nabla u\cdot\nabla v + uv + K(x)\phi_{\varepsilon,K}(u)uv - a(x)f(u)v)\,dx. \tag{2.7}$$

Now, we can apply Theorem 2.3 of [7] to the functional E_ε and obtain

Proposition 2.5. *The following statements are equivalent*:

(a) $(u,\phi) \in H^1(\mathbb{R}^3)\times(D^{1,2}\cap D^{1,4}(\mathbb{R}^3))$ *is a critical point of J_ε (i.e. (u,ϕ) is a solution of the system* (1.1));

(b) *u is a critical point of E_ε and $\phi = \phi_{\varepsilon,K}(u)$.*

Furthermore, in order to obtain our results, we also need the following lemma.

LEMMA 2.6 ([4, Lemma 3.2]). *For all $\varepsilon > 0$ and $f_\varepsilon, f \in L^{6/5}(\mathbb{R}^3)$, let $\phi_\varepsilon(f_\varepsilon) \in D^{1,2} \cap D^{1,4}(\mathbb{R}^3)$ be a unique solution of $-\operatorname{div}[(1+\varepsilon^4|\nabla\phi|^2)\nabla\phi] = f_\varepsilon$ in $\mathbb{R}^3$ and $\phi_0(f) \in D^{1,2}(\mathbb{R}^3)$ be a unique solution of $-\Delta\phi = f$ in $\mathbb{R}^3$. Then:*

(a) *if $f_\varepsilon \rightharpoonup f$ weakly in $L^{6/5}(\mathbb{R}^3)$ then $\phi_\varepsilon(f_\varepsilon) \rightharpoonup \phi_0(f)$ in $D^{1,2}(\mathbb{R}^3)$ as $\varepsilon \to 0$;*

(b) *if $f_\varepsilon \to f$ strongly in $L^{6/5}(\mathbb{R}^3)$, then:*

$$\begin{aligned} \phi_\varepsilon(f_\varepsilon) &\to \phi_0(f) && \textit{strongly in } D^{1,2}(\mathbb{R}^3), \\ \varepsilon\phi_\varepsilon(f_\varepsilon) &\to 0 && \textit{strongly in } D^{1,4}(\mathbb{R}^3) \end{aligned}$$

as $\varepsilon \to 0$.

3. Proof of Theorem 1.1

Now we prove that system (1.1) has a mountain pass type solution. For this purpose, we use a variant version of Mountain Pass Theorem [14], which allows us to find a so-called Cerami type (PS) sequence (Cerami sequence, in short). The properties of this kind of (PS) sequence are very helpful in showing its boundedness in the asymptotically linear case. The following lemmas show that E_ε has the so-called mountain pass geometry.

LEMMA 3.1. *Suppose that* (f1)–(f3), (A1) *and* (K1) *hold. Then there exist $\rho > 0$ and $\alpha > 0$ such that $E_\varepsilon(u)|_{\|u\|=\rho} \geq \alpha > 0$.*

PROOF. For any $\widetilde{\varepsilon} > 0$, it follows from (f1)–(f3) that there exists $C_{\widetilde{\varepsilon}} > 0$ such that

$$|f(t)| \leq \widetilde{\varepsilon}|t| + C_{\widetilde{\varepsilon}}|t|^5 \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Therefore, we have

$$|F(t)| \leq \frac{1}{2}\widetilde{\varepsilon}|t|^2 + \frac{C_{\widetilde{\varepsilon}}}{6}|t|^6 \quad \text{for all } t \in \mathbb{R}. \tag{3.2}$$

Furthermore, by (f1)–(f3) and (A1), there exists $C_2 > 0$ such that

$$a(x) \leq C_2 \quad \text{for all } x \in \mathbb{R}^3. \tag{3.3}$$

According to (3.2), (3.3) and the Sobolev inequality, we duduce

$$\left|\int_{\mathbb{R}^3} a(x)F(u)\,dx\right| \leq \frac{\widetilde{\varepsilon}C_2}{2}\int_{\mathbb{R}^3}|u|^2\,dx + \frac{C_2C_{\widetilde{\varepsilon}}}{6}\int_{\mathbb{R}^3}|u|^6\,dx \leq \frac{\widetilde{\varepsilon}C_2}{2}\|u\|^2 + C_3\|u\|^6$$

for some $C_3 > 0$. Together with $I_\varepsilon(u) \geq 0$, one has

$$\begin{aligned} E_\varepsilon(u) &= \frac{1}{2}\|u\|^2 + I_\varepsilon(u) - \int_{\mathbb{R}^3} a(x)F(u)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\widetilde{\varepsilon}C_2}{2}\|u\|^2 - C_3\|u\|^6 \geq \|u\|\left(\frac{1-\widetilde{\varepsilon}C_2}{2}\|u\| - C_3\|u\|^5\right). \end{aligned} \tag{3.4}$$

Taking $\widetilde{\varepsilon} \in (0, 1/2C_2)$. From (3.4), letting $\|u\| = \rho > 0$ small enough, there exists $\alpha > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$. □

Lemma 3.2. *Suppose that* (f1)–(f3), (A1)–(A2) *and* (K1) *hold. Then there exists $v \in H^1(\mathbb{R}^3)$ with $\|v\| > \rho$, ρ is given by Lemma* 3.1, *such that $E_\varepsilon(v) < 0$ for all $\varepsilon > 0$.*

Proof. By (A2), in view of the definition of μ^* and $(1-\beta)l > \mu^*$, there is a nonnegative function $v \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} a(x)F(v)\,dx \geq \frac{l}{2}, \qquad \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(v)v^2\,dx < \frac{4}{3}\beta l,$$

and $\mu^* \leq \|v\|^2 < (1-\beta)l$. From (2.3) and the definition of I_ε, we obtain

$$I_\varepsilon(v) \leq \frac{3}{8}\int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(v)v^2\,dx < \frac{1}{2}\beta l.$$

Therefore, we have

$$\begin{aligned} E_\varepsilon(v) &= \frac{1}{2}\|v\|^2 + I_\varepsilon(v) - \int_{\mathbb{R}^3} a(x)F(v)\,dx \\ &\leq \frac{1}{2}\|v\|^2 + \frac{\beta l}{2} - \frac{l}{2} = \frac{1}{2}(\|v\|^2 - (1-\beta)l) < 0. \end{aligned}$$

Choosing $\rho > 0$ small enough in Lemma 3.1 such that $\|v\| > \rho$, then this lemma is proved. □

From Lemmas 3.1 and 3.2 and Mountain Pass Lemma in [14], there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$\|E'_\varepsilon(u_n)\|_{H^{-1}}(1 + \|u_n\|) \to 0 \quad \text{and} \quad E_\varepsilon(u_n) \to c \quad \text{as } n \to \infty, \tag{3.5}$$

where H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$. In the following, we shall prove that sequence $\{u_n\}$ has a convergence subsequence.

Lemma 3.3. *Suppose that* (f1)–(f3), (A1) *and* (K1) *hold. Then $\{u_n\}$ defined in* (3.5) *is bounded in $H^1(\mathbb{R}^3)$.*

Proof. By contradiction, let $\|u_n\| := \alpha_n \to \infty$. Define $w_n = u_n\|u_n\|^{-1} = \alpha_n^{-1}u_n$. Clearly, $\|w_n\| = 1$ and $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and there is a $w \in H^1(\mathbb{R}^3)$ such that, up to a sequence(still denoted by $\{w_n\}$),

$$\begin{cases} w_n \rightharpoonup w & \text{weakly in } H^1(\mathbb{R}^3), \\ w_n \to w & \text{a.e. in } \mathbb{R}^3, \\ w_n \to w & \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3), \end{cases}$$

as $n \to \infty$.

Firstly, we claim that w is nontrivial, that is $w \not\equiv 0$. Otherwise, if $w \equiv 0$, the Sobolev embedding implies that $w_n \to 0$ strongly in $L^2_{\mathrm{loc}}(B_{R_0})$, R_0 is given by (A1). Define

$$\widetilde{f}(t) = \begin{cases} \dfrac{f(t)}{t} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Together with (f1)–(f3) with $l < +\infty$, there exists $C_4 > 0$ such that

$$\widetilde{f}(t) \le C_4 \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Then, for all $n \in N$, we have

$$0 \le \int_{|x|<R_0} a(x)\widetilde{f}(u_n)w_n^2 dx \le C_4|a|_\infty \int_{|x|<R_0} w_n^2\,dx \to 0.$$

This yields

$$\lim_{n\to\infty} \int_{|x|<R_0} a(x)\widetilde{f}(u_n)w_n^2\,dx = 0. \tag{3.7}$$

Furthermore, by (A1), there exists a constant $\theta \in (0,1)$ such that

$$\sup\{f(t)/t : t > 0\} \le \theta \inf\{1/a(x) : |x| \ge R_0\}. \tag{3.8}$$

Then, for all $n \in N$, we have

$$\begin{aligned} \int_{|x|\ge R_0} a(x)\widetilde{f}(u_n)w_n^2\,dx &\le \theta \int_{|x|\ge R_0} w_n^2\,dx \\ &< \theta \int_{|x|\ge R_0} (|\nabla w_n|^2 + w_n^2)\,dx < \theta\|w\|^2 = \theta < 1. \end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} a(x)\widetilde{f}(u_n)w_n^2\,dx < 1. \tag{3.10}$$

By (3.5), we get

$$0 \le |\langle E'_\varepsilon(u_n), u_n\rangle| \le \|E'_\varepsilon(u_n)\|_{H^{-1}}\|u_n\| \le \|E'_\varepsilon(u_n)\|_{H^{-1}}(1 + \|u_n\|) \to 0$$

as $n \to \infty$. Together with $\alpha_n \to \infty$ as $n \to \infty$, it follows that $\alpha_n^{-2}\langle E'_\varepsilon(u_n), u_n\rangle = o(1)$. So, by (2.7) and (2.3), we have

$$\begin{aligned} o(1) &= \|w_n\|^2 + \alpha_n^{-2} \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n^2\,dx - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}w_n^2\,dx \\ &= \|w_n\|^2 + \alpha_n^{-2} \int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\varepsilon,K}(u_n)|^2)|\nabla\phi_{\varepsilon,K}(u_n)|^2\,dx \\ &\quad - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}w_n^2\,dx \\ &\ge 1 - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}w_n^2\,dx, \end{aligned}$$

where, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \to \infty$. Therefore, we deduce that

$$\int_{\mathbb{R}^3} a(x) \frac{f(u_n)}{u_n} w_n^2 \, dx + o(1) \geq 1,$$

which contradicts (3.10). So, $w \not\equiv 0$. By (3.5) and the definition of E_ε, we get

$$\begin{aligned} o(1) + \alpha_n^{-2} c &= \alpha_n^{-2} E_\varepsilon(u_n) \\ &= \frac{1}{2}\|w_n\|^2 + \alpha_n^{-2} I_\varepsilon(u_n) - \int_{\mathbb{R}^3} a(x) \frac{F(u_n)}{u_n^2} w_n^2 \, dx \\ &= \frac{1}{2} + \alpha_n^{-2} I_\varepsilon(u_n) - \int_{\mathbb{R}^3} a(x) \frac{F(u_n)}{u_n^2} w_n^2 \, dx. \end{aligned} \tag{3.11}$$

Define

$$\widetilde{F}(t) = \begin{cases} \dfrac{F(t)}{t^2} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

By (f1)–(f3) with $l < +\infty$ and (3.11), there exists $C_5 > 1/2$ such that $\widetilde{F}(t) \leq C_5/C_2$ for all $t \in \mathbb{R}$. So, together with (3.3), we obtain

$$\int_{\mathbb{R}^3} a(x) \widetilde{F}(u_n) w_n^2 \, dx \leq C_5 \int_{\mathbb{R}^3} w_n^2 \, dx \leq C_5 \|w_n\|^2 = C_5. \tag{3.12}$$

Combining (3.11) and (3.12), we deduce

$$\alpha_n^{-2} I_\varepsilon(u_n) \leq C_5 - \frac{1}{2} + o(1). \tag{3.13}$$

From Lemma 2.2, we know that $\phi_{\varepsilon,K}(u_n)$ is the unique solution of the equation

$$-\mathrm{div}[(1 + \varepsilon^4 |\nabla \phi|^2) \nabla \phi] = K(x) u_n^2.$$

That is, the following equality holds:

$$-\mathrm{div}[(1 + \varepsilon^4 |\nabla \phi_{\varepsilon,K}(u_n)|^2) \nabla \phi_{\varepsilon,K}(u_n)] = K(x) u_n^2. \tag{3.14}$$

Multiplying (3.14) by $\phi_{\varepsilon,K}(u_n)$ and integrating by parts, we find that

$$\widetilde{J}_\varepsilon(\phi_{\varepsilon,K}(u_n)) = -\frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon,K}(u_n)|^2 \, dx - \frac{3\varepsilon^4}{4} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon,K}(u_n)|^4 \, dx = -2 I_\varepsilon(u_n).$$

Together with (3.13), we get

$$-\frac{1}{2} \alpha_n^{-2} \widetilde{J}_\varepsilon(\phi_{\varepsilon,K}(u_n)) = \alpha_n^{-2} I_\varepsilon(u_n) \leq C_5 - \frac{1}{2} + o(1). \tag{3.15}$$

From Lemma 2.2, $\phi_{\varepsilon,K}(w_n)$ satisfies

$$\begin{aligned} &\int_{\mathbb{R}^3} K(x) \phi_{\varepsilon,K}(w_n) w_n^2 \, dx \\ &\qquad = \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon,K}(w_n)|^2 \, dx + \varepsilon^4 \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon,K}(w_n)|^4 \, dx. \end{aligned} \tag{3.16}$$

Since $\phi_{\varepsilon,K}(u_n)$ is the minimizer of $\widetilde{J}_\varepsilon$ on $D^{1,2}\cap D^{1,4}(\mathbb{R}^3)$ and (3.16), we may write

$$
\begin{aligned}
\widetilde{J}_\varepsilon(\phi_{\varepsilon,K}(u_n)) &\le \widetilde{J}_\varepsilon(\alpha_n^{2/3}\phi_{\varepsilon,K}(w_n))\\
&= \frac{\alpha_n^{4/3}}{2}\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx + \frac{\alpha_n^{8/3}\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx\\
&\quad - \alpha_n^{2/3}\int_{\mathbb{R}^3}K(x)u_n^2\phi_{\varepsilon,K}(w_n)\,dx\\
&= \frac{\alpha_n^{4/3}}{2}\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx + \frac{\alpha_n^{8/3}\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx\\
&\quad - \alpha_n^{8/3}\int_{\mathbb{R}^3}K(x)w_n^2\phi_{\varepsilon,K}(w_n)\,dx\\
&= \left(\frac{\alpha_n^{4/3}}{2} - \alpha_n^{8/3}\right)\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx - \frac{3\alpha_n^{8/3}\varepsilon^4}{4}\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx\\
&\le -\frac{3\alpha_n^{8/3}}{4}\left(\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx + \varepsilon^4\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx\right),
\end{aligned}
$$

because $\alpha_n^{4/3}/2-\alpha_n^{8/3}\le -3\alpha_n^{8/3}/4$ for n large enough since $\alpha_n\to+\infty$ as $n\to\infty$. Together with (3.15), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx + \varepsilon^4\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx\\
&\qquad\qquad \le -\frac{4}{3}\alpha_n^{-8/3}\widetilde{J}_\varepsilon(\phi_{\varepsilon,K}(u_n)) \le \frac{8\alpha_n^{-2/3}}{3}\left(C_5 - \frac{1}{2} + o(1)\right).
\end{aligned}
$$

So, we have

$$
\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^2\,dx + \varepsilon^4\int_{\mathbb{R}^3}|\nabla\phi_{\varepsilon,K}(w_n)|^4\,dx \to 0 \quad \text{as } n\to\infty. \tag{3.17}
$$

By (3.16) and (3.17), we have

$$
\int_{\mathbb{R}^3}K(x)\phi_{\varepsilon,K}(w_n)w_n^2\,dx \to 0 \quad \text{as } n\to\infty. \tag{3.18}
$$

We can easily verify that

$$
\int_{\mathbb{R}^3}K(x)\phi_{\varepsilon,K}(w_n)w_n^2\,dx \to \int_{\mathbb{R}^3}K(x)\phi_{\varepsilon,K}(w)w^2\,dx \quad \text{as } n\to\infty. \tag{3.19}
$$

Indeed, in view of the Sobolev embedding theorems and (3) of Lemma 2.1 in [6], $w_n\rightharpoonup w$ weakly in $H^1(\mathbb{R}^3)$, we obtain

$$
\begin{array}{lrcll}
\text{(a)} & w_n & \rightharpoonup & w & \text{weakly in } L^6(\mathbb{R}^3),\\
\text{(b)} & w_n^2 & \to & w^2 & \text{strongly in } L^3_{\mathrm{loc}}(\mathbb{R}^3),\\
\text{(c)} & \phi_{\varepsilon,K}(w_n) & \rightharpoonup & \phi_{\varepsilon,K}(w) & \text{weakly in } D^{1,2}(\mathbb{R}^3),\\
\text{(d)} & \phi_{\varepsilon,K}(w_n) & \to & \phi_{\varepsilon,K}(w) & \text{strongly in } L^6_{\mathrm{loc}}(\mathbb{R}^3).
\end{array} \tag{3.20}
$$

For any choice of $\overline{\varepsilon} > 0$ and $\rho > 0$, the relation

$$\|w_n - w\|_{6,B_\rho(0)} < \overline{\varepsilon} \tag{3.21}$$

holds for large n. Using (c) of (3.20), for large n, we have

$$\left| \int_{\mathbb{R}^3} K(x)(\phi_{\varepsilon,K}(w_n) - \phi_{\varepsilon,K}(w))w^2\,dx \right| = o(1). \tag{3.22}$$

Because w_n is bounded in $H^1(\mathbb{R}^3)$ and the continuity of the Sobolev embedding of $D^{1,2}(R^3)$ in $L^6(\mathbb{R}^3)$, then $\phi_{\varepsilon,K}(w_n)$ is bounded in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$. Moreover, $K \in L^2(\mathbb{R}^3)$ implies that Kw_n^2 and Kw^2 belong to $L^{6/5}(\mathbb{R}^3)$ and that to any $\overline{\varepsilon} > 0$ there exists $\overline{\rho} = \overline{\rho}(\overline{\varepsilon})$ such that

$$\|K\|_{2,R^3\setminus B_\rho(0)} < \overline{\varepsilon}, \quad \text{for all } \rho \geq \overline{\rho} \tag{3.23}$$

By (3.22), (3.21) and (3.23), we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(w_n)w_n^2\,dx - \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(w)w^2\,dx \right| \\
&\quad = \left| \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(w_n)(w_n^2 - w^2)\,dx - \int_{\mathbb{R}^3} K(x)(\phi_{\varepsilon,K}(w) - \phi_{\varepsilon,K}(w_n))w^2\,dx \right| \\
&\quad \leq \int_{\mathbb{R}^3} |K(x)\phi_{\varepsilon,K}(w_n)(w_n^2 - w^2)|\,dx + \int_{\mathbb{R}^3} |K(x)(\phi_{\varepsilon,K}(w_n) - \phi_{\varepsilon,K}(w))w^2|\,dx \\
&\quad \leq \|\phi_{\varepsilon,K}(w_n)\|_6 \bigg(\int_{\mathbb{R}^3} |K(x)(w_n^2 - w^2)|^{6/5}\,dx \bigg)^{5/6} + \overline{\varepsilon} \\
&\quad \leq C_6 \bigg(\int_{R^3\setminus B_\rho(0)} |K(x)(w_n^2 - w^2)|^{6/5}\,dx + \int_{B_\rho(0)} |K(x)(w_n^2 - w^2)|^{6/5}\,dx \bigg)^{5/6} + \overline{\varepsilon} \\
&\quad \leq C_6 \big(\|K\|_{2,\mathbb{R}^3\setminus B_\rho(0)}^{6/5} \|w_n^2 - w^2\|_3^{6/5} + \|K\|_2^{6/5} \|w_n^2 - w^2\|_{3,B_\rho(0)}^{6/5} \big)^{5/6} + \overline{\varepsilon} \leq C_7\overline{\varepsilon}.
\end{aligned}$$

This proves (3.19). So, by (3.18) and (3.19), we obtain

$$\int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(w)w^2\,dx = 0,$$

which implies that $w \equiv 0$. That is a contradiction. Therefore, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. □

Lemma 3.4. *Suppose that* (f1)–(f3), (A1) *and* (K1) *hold. Then for any* $\varepsilon' > 0$, *there exist* $R(\varepsilon') > R_0$ *and* $n(\varepsilon') > 0$ *such that* $\{u_n\}$ *defined in* (3.5) *satisfies*

$$\int_{|x|\geq R} (|\nabla u_n|^2 + |u_n|^2)\,dx \leq \varepsilon'$$

for $n > n(\varepsilon')$ *and* $R \geq R(\varepsilon')$.

Proof. Let $\xi_R\colon \mathbb{R}^3 \to [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0 & \text{if } 0 \leq |x| \leq R/2, \\ 1 & \text{if } |x| \geq R. \end{cases} \tag{3.24}$$

Moreover, there exists a constant C_8 independent of R such that

$$|\nabla\xi_R(x)| \le C_8/R \quad \text{for all } x \in \mathbb{R}^3. \tag{3.25}$$

Then, for all $n \in N$ and $R \ge R_0$, by (3.24), (3.25) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla(u_n\xi_R)|^2\,dx \\
&\quad\le \int_{\mathbb{R}^3} |\nabla u_n|^2|\xi_R|^2\,dx + \int_{\mathbb{R}^3} |u_n|^2|\nabla\xi_R|^2\,dx + 2\int_{\mathbb{R}^3} |u_n||\xi_R||\nabla u_n||\nabla\xi_R|\,dx \\
&\quad\le \int_{R/2<|x|<R} |\nabla u_n|^2\,dx + \int_{|x|\ge R} |\nabla u_n|^2\,dx + \frac{C_8^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2\,dx \\
&\qquad + 2\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2|\xi_R^2|\,dx\bigg)^{1/2}\bigg(\int_{\mathbb{R}^3} |u_n|^2|\nabla\xi_R|^2\,dx\bigg)^{1/2} \\
&\quad\le \int_{R/2<|x|<R} |\nabla u_n|^2\,dx + \int_{|x|\ge R} |\nabla u_n|^2\,dx + \frac{C_8^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2\,dx \\
&\qquad + 2\bigg(\int_{R/2<|x|<R} |\nabla u_n|^2\,dx + \int_{|x|\ge R} |\nabla u_n|^2\,dx\bigg)^{1/2}\bigg(\frac{C_8^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2\,dx\bigg)^{1/2} \\
&\quad\le \bigg(2 + \frac{C_8^2}{R^2} + \frac{2\sqrt{2}C_8}{R}\bigg)\|u_n\|^2 \le \bigg(2 + \frac{C_8^2}{R_0^2} + \frac{2\sqrt{2}C_8}{R_0}\bigg)\|u_n\|^2.
\end{aligned}$$

This implies that

$$\|u_n\xi_R\| \le C_9\|u_n\| \tag{3.26}$$

for all $n \in N$ and $R \ge R_0$, where $C_9 = (3 + C_8^2/R_0^2 + 2\sqrt{2}C_8/R_0)^{1/2}$. From Lemma 3.3, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Together with (3.5), we obtain that $E'_\varepsilon(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$. Moreover, by (3.26), for $\varepsilon' > 0$, there exists $n(\varepsilon') > 0$ such that

$$\langle E'_\varepsilon(u_n), \xi_R u_n\rangle \le C_9\|E'_\varepsilon(u_n)\|_{H^{-1}(\mathbb{R}^3)}\|u_n\| \le \varepsilon'/4$$

for $n > n(\varepsilon')$ and $R > R_0$. Note that

$$\begin{aligned}
\langle E'_\varepsilon(u_n), \xi_R u_n\rangle &= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)\xi_R\,dx + \int_{\mathbb{R}^3} u_n\nabla u_n\cdot\nabla\xi_R\,dx \\
&\quad + \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n^2\xi_R dx - \int_{\mathbb{R}^3} a(x)f(u_n)u_n\xi_R\,dx \le \frac{\varepsilon'}{4}.
\end{aligned}$$

Together with Lemma 2.2, (K1) and the definition of ξ_R, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} [(|\nabla u_n|^2 + |u_n|^2)\xi_R + u_n\nabla u_n\cdot\nabla\xi_R]\,dx \\
&\quad\le \int_{\mathbb{R}^3} a(x)f(u_n)u_n\xi_R\,dx - \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n^2\xi_R\,dx + \frac{\varepsilon'}{4} \\
&\quad\le \int_{\mathbb{R}^3} a(x)f(u_n)u_n\xi_R\,dx + \frac{\varepsilon'}{4}.
\end{aligned} \tag{3.27}$$

By (3.8) and (A1), we have

$$a(x)f(u_n)u_n \le \theta u_n^2 \quad \text{for } \theta \in (0,1) \text{ and } |x| \ge R_0.$$

This yields

$$\int_{\mathbb{R}^3} a(x)f(u_n)u_n\xi_R\,dx \le \theta \int_{\mathbb{R}^3} u_n^2\xi_R\,dx \tag{3.28}$$

for all $n \in N$ and $|x| \ge R_0$. For any $\varepsilon' > 0$, there exists $R(\varepsilon') \ge R_0$ such that

$$\frac{1}{R^2} \le \frac{4\varepsilon'^2}{C_8^2} \quad \text{for all } R > R(\varepsilon'). \tag{3.29}$$

By the Young inequality, (3.25) and (3.29), for all $n \in N$ and $R > R(\varepsilon')$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n \nabla u_n \cdot \nabla \xi_R|\,dx &= \int_{\mathbb{R}^3} \sqrt{2\varepsilon'}|\nabla u_n| \frac{1}{\sqrt{2\varepsilon'}}|u_n||\nabla \xi_R|\,dx \\ &\le \varepsilon' \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{1}{4\varepsilon'}\int_{|x|\le R} |u_n|^2 \frac{C_8^2}{\mathbb{R}^2}\,dx \\ &\le \varepsilon' \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \varepsilon' \int_{|x|\le R} |u_n|^2\,dx \le \varepsilon' \|u_n\|^2.\end{aligned} \tag{3.30}$$

Combining (3.27), (3.28) and (3.30), there exists $C_6 > 0$ such that

$$\begin{aligned}(1-\theta)&\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)\xi_R\,dx \\ &\le \int_{\mathbb{R}^3} (|\nabla u_n|^2 + (1-\theta)|u_n|^2)\xi_R\,dx \le \frac{\varepsilon'}{4} + \varepsilon'\|u_n\|^2 \le C_{10}\varepsilon'\end{aligned}$$

for all $R > R(\varepsilon')$. Noting that C_{10} is independent of ε'. So, for any $\varepsilon > 0$, we can choose $R(\varepsilon') > R_0$ and $n(\varepsilon') > 0$ such that

$$\int_{|x|\ge R} (|\nabla u_n|^2 + |u_n|^2)\,dx \le \varepsilon'$$

holds. □

Lemma 3.5. *Suppose that* (f1)–(f3)*,* (A1)*,* (A2) *and* (K1) *hold. Then the sequence* $\{u_n\}$ *in* (3.5) *has a convergent subsequence. Moreover,* E_ε *possesses a nonzero critical point* u *in* $H^1(R^3)$ *and* $E_\varepsilon(u) > 0$.

Proof. By Lemma 3.3, the sequence $\{u_n\}$ in (3.5) is bounded in $H^1(\mathbb{R}^3)$. We may assume that, up to a subsequence $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ for some $u \in H^1(\mathbb{R}^3)$. Now, we shall show that $\|u_n\| \to \|u\|$ as $n \to \infty$. By (3.5), we have

$$\begin{aligned}&\langle E'_\varepsilon(u_n), u_n\rangle \\ &\quad = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2 + K(x)\phi_{\varepsilon,K}(u_n)u_n^2 - a(x)f(u_n)u_n)\,dx = o(1),\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}(3.32)\quad &\langle E'_\varepsilon(u_n), u\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + u_n u + K(x)\phi_{\varepsilon,K}(u_n)u_n u - a(x)f(u_n)u)\,dx = o(1).\end{aligned}$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + u_n u)\,dx = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\,dx + o(1). \tag{3.33}$$

Moreover, by the Hölder inequality, (3.8), Lemma 3.4 and $u_n \to u$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, for any $\varepsilon' > 0$ and n large enough, one has

$$\begin{aligned}&\int_{|x|\ge R(\varepsilon')} a(x)f(u_n)u_n\,dx - \int_{|x|\ge R(\varepsilon')} a(x)f(u_n)u\,dx \\ &\quad= \int_{|x|\ge R(\varepsilon')} a(x)f(u_n)(u_n-u)\,dx \le \int_{|x|\ge R(\varepsilon')} |a(x)f(u_n)||u_n-u|\,dx \\ &\quad\le \bigg(\int_{|x|\ge R(\varepsilon')} |a^2(x)f^2(u_n)|\,dx\bigg)^{1/2}\bigg(\int_{|x|\ge R(\varepsilon')} |u_n-u|^2\,dx\bigg)^{1/2} \\ &\quad\le \theta\bigg(\int_{|x|\ge R(\varepsilon')} |u_n^2|\,dx\bigg)^{1/2}\bigg(\int_{|x|\ge R(\varepsilon')} |u_n-u|^2\,dx\bigg)^{1/2} \\ &\quad\le \theta\bigg(\int_{|x|\ge R(\varepsilon')} (|\nabla u_n|^2 + |u_n^2|)\,dx\bigg)^{1/2}\bigg(\int_{|x|\ge R(\varepsilon')} |u_n-u|^2\,dx\bigg)^{1/2} \\ &\quad\le C_{11}\varepsilon'.\end{aligned}$$

This and the compactness of embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^3)$ imply

$$\int_{\mathbb{R}^3} a(x)f(u_n)u_n\,dx = \int_{\mathbb{R}^3} a(x)f(u_n)u\,dx + o(1). \tag{3.34}$$

Furthermore, because $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, we obtain

(a) $u_n \rightharpoonup u$ weakly in $L^6(\mathbb{R}^3)$,
(b) $u_n^2 \to u^2$ stongly in $L^3_{\mathrm{loc}}(\mathbb{R}^3)$,
(c) $\phi_{\varepsilon,K}(u_n) \rightharpoonup \phi_{\varepsilon,K}(u)$ weakly in $D^{1,2}(\mathbb{R}^3)$,
(d) $\phi_{\varepsilon,K}(u_n) \to \phi_{\varepsilon,K}(u)$ stongly in $L^6_{\mathrm{loc}}(\mathbb{R}^3)$.

For any choice of $\varepsilon' > 0$ and $\rho > 0$, the relation

$$\|u_n - u\|_{6,B_\rho(0)} < \varepsilon' \tag{3.35}$$

holds for large n. Because u_n is bounded in $H^1(\mathbb{R}^3)$ and the continuity of the Sobolev embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, then $\phi_{\varepsilon,K}(u_n)$ is bounded in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$. Moreover, $K \in L^2(\mathbb{R}^3)$ implies that Ku_n^2 and Ku^2 belong to $L^{6/5}(\mathbb{R}^3)$ and that to any $\varepsilon' > 0$ there exists $\widetilde{\rho} = \widetilde{\rho}(\varepsilon')$ such that

$$\|K\|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \varepsilon', \quad \text{for all } \rho \ge \widetilde{\rho}. \tag{3.36}$$

By the Hölder inequality, (3.35) and (3.36), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n^2\,dx - \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n u\,dx \\
&\le \int_{\mathbb{R}^3} |K(x)\phi_{\varepsilon,K}(u_n)u_n(u_n-u)|\,dx \\
&\le \|\phi_{\varepsilon,K}(u_n)\|_6 \bigg(\int_{\mathbb{R}^3} |K(x)u_n(u_n-u)|^{6/5}\,dx\bigg)^{5/6} \\
&= \|\phi_{\varepsilon,K}(u_n)\|_6 \bigg(\int_{R^3\setminus B_\rho(0)} |K(x)u_n(u_n-u)|^{6/5}\,dx \\
&\quad + \int_{B_\rho(0)} |K(x)u_n(u_n-u)|^{6/5}\,dx\bigg)^{5/6} \\
&\le C_{12}\bigg(\|K\|^{6/5}_{2,\mathbb{R}^3\setminus B_\rho(0)}\|u_n(u_n-u)\|_3^{6/5} \\
&\quad + \|K\|_2^{6/5}\bigg[\int_{B_\rho(0)} |u_n|^6\,dx\bigg]^{1/5}\bigg[\int_{B_\rho(0)} |u_n-u|^6\,dx\bigg]^{1/5}\bigg)^{5/6} \\
&\le C_{12}\Big(\varepsilon'^{6/5}\|u_n(u_n-u)\|_3^{6/5} + \|K\|_2^{6/5}\|u_n\|^{5/6}_{6,B_\rho(0)}\|u_n-u\|^{5/6}_{6,B_\rho(0)}\Big)^{5/6} \\
&\le C_{12}\Big(\varepsilon'^{6/5}\|u_n(u_n-u)\|_3^{6/5} + \varepsilon'^{6/5}\|K\|_2^{6/5}\|u_n\|^{5/6}_{6,B_\rho(0)}\Big)^{5/6} \le C_{13}\varepsilon'.
\end{aligned}$$

This yields

$$\int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n^2\,dx = \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_n)u_n u\,dx + o(1). \tag{3.37}$$

By (3.31)–(3.34) and (3.37), we have

$$\langle E'_\varepsilon(u_n), u_n-u\rangle = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)\,dx - \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\,dx = o(1).$$

This yields that $\|u_n\| \to \|u\|$ as $n\to\infty$ and u is a nonzero critical point of E_ε in $H^1(\mathbb{R}^3)$ and $E_\varepsilon(u) > 0$ by Mountain Pass Theorem in [14]. □

Proof of Theorem 1.1. Set the Nehari manifold

$$N_\varepsilon = \{u_\varepsilon \in H^1(\mathbb{R}^3)\setminus\{0\} : \langle E'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = 0\}.$$

By Lemma 3.5, we know that N_ε is not empty. For any $u_\varepsilon \in N_\varepsilon$, by Lemma 2.2 and (K1), we have

$$\begin{aligned}
o(1) = \langle E'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = \|u_\varepsilon\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{\varepsilon,K}(u_\varepsilon)u_\varepsilon^2\,dx - \int_{\mathbb{R}^3} a(x)f(u_\varepsilon)u_\varepsilon\,dx \\
\ge \|u_\varepsilon\|^2 - \int_{\mathbb{R}^3} a(x)f(u_\varepsilon)u_\varepsilon\,dx.
\end{aligned}$$

Now, choose $\widetilde{\varepsilon}$ such that $0 < \widetilde{\varepsilon} < \min\{1, C_2^{-1}\}$ where C_2 is as in (3.3). By (3.1), (3.3) and the Sobolev inequality, we deduce

$$\left| \int_{\mathbb{R}^3} a(x) f(u_\varepsilon) u_\varepsilon \, dx \right| \le \widetilde{\varepsilon} C_2 \int_{\mathbb{R}^3} |u_\varepsilon|^2 \, dx + C_2 C_\varepsilon \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx$$
$$\le \widetilde{\varepsilon} C_2 \|u_\varepsilon\|^2 + C_2 C_\varepsilon (S^*)^3 \|u_\varepsilon\|^6.$$

Therefore, for every $u_\varepsilon \in N$, we have

$$o(1) \ge \|u_\varepsilon\|^2 - \widetilde{\varepsilon} C_2 \|u_\varepsilon\|^2 - C_2 C_\varepsilon (S^*)^3 \|u_\varepsilon\|^6. \tag{3.38}$$

We recall that $u_\varepsilon \neq 0$ whenever $u_\varepsilon \in N_\varepsilon$ and (3.38) implies

$$\|u_\varepsilon\| \ge \sqrt[4]{\frac{1 - \widetilde{\varepsilon} C_2}{C_2 C_\varepsilon (S^*)^3}} > 0, \quad \text{for all } u_\varepsilon \in N_\varepsilon.$$

Hence any limit point of a sequence in the Nehari manifold is different from zero.

Now, we shall prove that E_ε is bounded from below on N_ε, that is, there exists $M > 0$ such that $E_\varepsilon(u_\varepsilon) \ge -M$ for all $u_\varepsilon \in N_\varepsilon$. Otherwise, there exists $\{u_n\} \subset N_\varepsilon$ such that

$$E_\varepsilon(u_n) < -n \quad \text{for all } n \in N. \tag{3.39}$$

From (3.4), we have $E_\varepsilon(u_n) \ge \|u_n\|^2/4 - C_3 \|u_n\|^6$. This and (3.39) imply that $\|u_n\| \to +\infty$. Let $w_n = u_n \|u_n\|^{-1}$, there is $w \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} w_n \rightharpoonup w & \text{weakly in } H^1(\mathbb{R}^3), \\ w_n \to w & \text{a.e. in } \mathbb{R}^3, \\ w_n \to w & \text{strongly in } L^2_{\rm loc}(\mathbb{R}^3), \end{cases}$$

as $n \to \infty$. Note that $E'_\varepsilon(u_n) = 0$ for $u_n \in N_\varepsilon$, as in the proof of Lemma 3.3, we obtain that $\|u_n\| \to +\infty$ is impossible. Then, E_ε is bounded from below on N_ε. So, we may define $\overline{c} = \inf\{E_\varepsilon(u_\varepsilon),\ u_\varepsilon \in N_\varepsilon\}$, and $\overline{c} \ge -M$. Let $\{\overline{u}_n\} \subset N_\varepsilon$ be such that $E_\varepsilon(\overline{u}_n) \to \overline{c}$ as $n \to \infty$. Following almost the same procedures as proofs of Lemmas 3.3–3.5, we can show that $\{\overline{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and it has a convergence subsequence which strongly converges to $u_\varepsilon \in H^1(\mathbb{R}^3) \setminus \{0\}$. Then $E_\varepsilon(u_\varepsilon) = \overline{c}$ and $E'_\varepsilon(u_\varepsilon) = 0$. Therefore, $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ is a ground state solution of system (1.1). □

4. Proof of Theorem 1.3

Here we shall study the behavior of the solution $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ obtained via Theorem 1.1.

Proof of Theorem 1.3. From Lemma 2.6, with $f_\varepsilon = f = K(x)u^2$ for all $\varepsilon > 0$, we can easily check that when $\varepsilon \to 0$ we have

$$E_\varepsilon(u) \to E_0(u) \quad \text{for all } u \in H^1(\mathbb{R}^3). \tag{4.1}$$

From Lemma 3.2, we know that there exists $v \in H^1(\mathbb{R}^3)$ such that $E_0(v) < 0 = E_0(0)$. By (4.1), we deduce that there exists $\varepsilon^* > 0$ small enough such that

$E_\varepsilon(v) < E_0(0)$ for $\varepsilon \in (0, \varepsilon^*)$. Since E_ε attains its minimum on $H^1(\mathbb{R}^3)$ at u_ε, we obtain that

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(v) < E_0(0) \quad \text{for all } u \in H^1(\mathbb{R}^3). \tag{4.2}$$

First of all, we claim that $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$. Indeed, by (4.2), there exists $\widetilde{c} \in \mathbb{R}$ independent of ε such that $E_\varepsilon(u_\varepsilon) \le \widetilde{c}$. Now to prove $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$, assume by contradiction that there exists a subsequence, denoted by $\{u_\varepsilon\}$, satisfying $E_\varepsilon(u_\varepsilon) \le \widetilde{c}$ and $\|u_\varepsilon\| \to \infty$.

Set $\|u_\varepsilon\| := \alpha_\varepsilon \to \infty$. Define $w_\varepsilon = u_\varepsilon \|u_\varepsilon\|^{-1} = \alpha_\varepsilon^{-1} u_\varepsilon$. Clearly, $\{w_\varepsilon\}$ is bounded and $\|w_\varepsilon\| = 1$ in $H^1(\mathbb{R}^3)$. An argument similar to the one used in the proof of Lemma 3.3, we can obtain contradiction, so $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore there exists u_0 such that, up to a subsequence, we have

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\mathbb{R}^3), \\ u_\varepsilon \to u_0 & \text{a.e. in } \mathbb{R}^3, \\ u_\varepsilon \to u_0 & \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^3) (p \in [2,6]), \end{cases}$$

as $\varepsilon \to 0$. Since $(u_\varepsilon, \phi_{\varepsilon,K}(u_\varepsilon))$ is the ground state solution of system (1.1), by Proposition 2.5 and (2.7), for $\psi \in C_c^\infty(\Omega)$ and a compact Ω such that $\operatorname{supp}\psi \subset \Omega$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla u_\epsilon \cdot \nabla \psi \, dx + \int_{\mathbb{R}^3} u_\varepsilon \psi \, dx \\ &\qquad + \int_\Omega K(x) \phi_{\varepsilon,K}(u_\varepsilon) u_\varepsilon \psi \, dx - \int_\Omega a(x) f(u_\varepsilon) \psi \, dx = 0. \end{aligned} \tag{4.3}$$

Since $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla \psi \, dx + \int_{\mathbb{R}^3} u_\varepsilon \psi \, dx \to \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \psi \, dx + \int_{\mathbb{R}^3} u_0 \psi \, dx \tag{4.4}$$

as $\varepsilon \to 0^+$.

Next, we shall prove that $K(x) u_\varepsilon^2 \rightharpoonup K(x) u_0^2$ weakly in $L^{6/5}(\mathbb{R}^3)$. In fact, let $\xi \in L^6(\mathbb{R}^3) = (L^{6/5}(\mathbb{R}^3))'$, then $K(x)\xi \in L^{3/2}(\mathbb{R}^3)$. Consider the subset of $\mathbb{R}^3$, $A_\lambda := \{x \mid |K(x)\xi| > \lambda\}$ and a compact subset Ω_0 of A_λ suitably chosen later. By the Hölder inequality, imbedding theorem and $u_\varepsilon \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$, we write

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)(u_\varepsilon - u_0)^2 \xi \, dx &= \int_{\mathbb{R}^3 - A_\lambda} K(x)(u_\varepsilon - u_0)^2 \xi \, dx \\ &\quad + \int_{A_\lambda - \Omega_0} K(x)(u_\varepsilon - u_0)^2 \xi \, dx + \int_{\Omega_0} K(x)(u_\varepsilon - u_0)^2 \xi \, dx \\ &\le \lambda \|u_\varepsilon - u_0\|_2^2 + \|K(x)\xi\|_{L^{3/2}(A_\lambda - \Omega_0)} \|u_\varepsilon - u_0\|_6^2 \\ &\quad + \|K(x)\xi\|_{L^{3/2}(\Omega_0)} \|u_\varepsilon - u_0\|_{L^6(\Omega_0)}^2 \\ &\le \lambda C_{14} + C_{15} \|K(x)\xi\|_{L^{3/2}(A_\lambda - \Omega_0)} + \|K(x)\xi\|_{L^{3/2}(\Omega_0)} \|u_\varepsilon - u_0\|_{L^6(\Omega_0)}^2. \end{aligned} \tag{4.5}$$

For a given arbitrary $\delta > 0$, we fix first λ such that $\lambda C_{14} < \delta/3$. Next we choose a compact subset $\Omega_0 \subset A_\lambda$ such that $C_{15}\|K(x)\xi\|_{L^{3/2}(A_\lambda-\Omega_0)} < \delta/3$ and subsequence of $\{u_\varepsilon\}$, still denoted by $\{u_\varepsilon\}$, $\|K(x)\xi\|_{L^{3/2}(\Omega_0)}\|u_\varepsilon - u_0\|^2_{L^6(\Omega_0)} < \delta/3$. Together with (4.5), we obtain

$$\int_{\mathbb{R}^3} K(x)(u_\varepsilon - u_0)^2\xi\,dx \to 0 \tag{4.6}$$

as $\varepsilon \to 0$. Since $u_0^2\xi^2 \in L^{3/2}(\mathbb{R}^3)$, by the same method, we can prove

$$\int_{\mathbb{R}^3} u_0^2\xi^2(u_\varepsilon - u_0)^2\,dx \to 0 \tag{4.7}$$

as $\varepsilon \to 0$. By the Hölder inequality, (4.6) and (4.7), we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)u_\varepsilon^2\xi\,dx - \int_{\mathbb{R}^3} K(x)u_0^2\xi\,dx = \int_{\mathbb{R}^3} K(x)(u_\varepsilon^2 - u_0^2)\xi\,dx\\
&\quad= \int_{\mathbb{R}^3} K(x)(u_\varepsilon - u_0)^2\xi\,dx + 2\int_{\mathbb{R}^3} K(x)(u_\varepsilon - u_0)u_0\xi\,dx\\
&\quad\le \int_{\mathbb{R}^3} K(x)(u_\varepsilon - u_0)^2\xi\,dx + 2\|K(x)\|_2\bigg(\int_{\mathbb{R}^3} u_0^2\xi^2(u_\varepsilon - u_0)^2\,dx\bigg)^{1/2} \to 0
\end{aligned}$$

as $\varepsilon \to 0$. We infer that

$$\int_{\mathbb{R}^3} K(x)u_\varepsilon^2\xi\,dx \to \int_{\mathbb{R}^3} K(x)u_0^2\xi\,dx$$

as $\varepsilon \to 0$. Therefore $K(x)u_\varepsilon^2 \rightharpoonup K(x)u_0^2$ weakly in $L^{6/5}(\mathbb{R}^3)$.

Since $K(x)u_\varepsilon^2 \rightharpoonup K(x)u_0^2$ weakly in $L^{6/5}(\mathbb{R}^3)$, by Lemma 2.6, we obtain $\phi_{\varepsilon,K}(u_\varepsilon) \rightharpoonup \phi_{0,K}(u_0)$ weakly in $L^6(\mathbb{R}^3)$. So, for all $\varrho \in L^{6/5}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \phi_{\varepsilon,K}(u_\varepsilon)\varrho\,dx \to \int_{\mathbb{R}^3} \phi_{0,K}(u_0)\varrho\,dx. \tag{4.8}$$

Furthermore, since $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, by the Sobolev imbedding theorem, we have

$$u_\varepsilon \to u_0 \quad \text{strongly in } L^p(\Omega)(p \in [2,6]). \tag{4.9}$$

Clearly, $u_0\psi \in L^{6/5}(\mathbb{R}^3)$. Together with $K \in L^2(\mathbb{R}^3)$, by the Hölder inequality, (4.9) and (4.8), we infer that

$$\begin{aligned}
&\int_\Omega K(x)\phi_{\varepsilon,K}(u_\varepsilon)u_\varepsilon\psi\,dx - \int_\Omega K(x)\phi_{0,K}(u_0)u_0\psi\,dx\\
&\qquad\le \int_\Omega |K(x)||\phi_{\varepsilon,K}(u_\varepsilon)||u_\varepsilon - u_0||\psi|\,dx\\
&\qquad\quad + \int_\Omega |K(x)||\phi_{\varepsilon,K}(u_\varepsilon) - \phi_{0,K}(u_0)||u_0\psi|\,dx\\
&\qquad\le C_{16}\|K(x)\|_2\|\phi_{\varepsilon,K}(u_\varepsilon)\|_6\|u_\varepsilon - u_0\|_{6,\Omega}\\
&\qquad\quad + C_{17}\int_{\mathbb{R}^3} |\phi_{\varepsilon,K}(u_\varepsilon) - \phi_{0,K}(u_0)||u_0\psi|\,dx \to 0.
\end{aligned} \tag{4.10}$$

By (f1)–(f3), (3.6) and (3.3), it is easy to prove that

$$\lim_{\varepsilon\to 0}\int_\Omega a(x)f(u_\varepsilon)\psi\,dx=\int_\Omega a(x)f(u_0)\psi\,dx. \tag{4.11}$$

If not, there exists $\delta_0>0$, for all $\varepsilon_0>0$, exists $\varepsilon\in(0,\varepsilon_0)$ such that

$$\bigg|\int_\Omega a(x)f(u_\varepsilon)\psi\,dx-\int_\Omega a(x)f(u_0)\psi\,dx\bigg|\geq\delta_0.$$

That is, there exists $\delta_0>0$ and $\varepsilon_n\in(0,\varepsilon_0)$ for $\varepsilon_n\to 0$ such that

$$\bigg|\int_\Omega a(x)f(u_{\varepsilon_n})\psi\,dx-\int_\Omega a(x)f(u_0)\psi\,dx\bigg|\geq\delta_0 \tag{4.12}$$

and subsequence of u_{ε_n}, still denoted by u_{ε_n}, is convergent in $L^p(\Omega)$ ($p\in[2,6]$). By (4.9) and the uniqueness of limit, $u_{\varepsilon_n}\to u_0$ strongly in $L^p(\Omega)$ ($p\in[2,6]$). Thus, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}a(x)(f(u_{\varepsilon_n})-f(u_0))\psi\,dx=0.$$

This contradicts (4.12). So, (4.11) holds.

Consequently, letting $\varepsilon\to 0$ in (4.3) and according to (4.4), (4.10) and (4.11), u_0 satisfies

$$\int_{\mathbb{R}^3}\nabla u_0\cdot\nabla\psi\,dx+\int_{\mathbb{R}^3}u_0\psi\,dx+\int_{\mathbb{R}^3}K(x)\phi_{0,K}(u_0)u_0\psi\,dx-\int_{\mathbb{R}^3}a(x)f(u_0)\psi\,dx=0$$

for all $\psi\in C_c^\infty(\Omega)$. We conclude that u_ε converges weakly in $H^1(\mathbb{R}^3)$ to u_0. Thus, $\phi_{\varepsilon,K}(u_\varepsilon)\rightharpoonup\phi_{0,K}(u_0)$ in $D^{1,2}(\mathbb{R}^3)$ by Lemma 2.6, where $(u_0,\phi_{0,K}(u_0))$ is a solution of system (1.1) with $\varepsilon=0$. □

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