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COMPUTATIONAL TOPOLOGY OF EQUIPARTITIONS BY HYPERPLANES

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Abstract. We compute a primary cohomological obstruction to the existence of an equipartition for j mass distributions in $\mathbb{R}^d$ by two hyperplanes in the case $2d-3j=1$. The central new result is that such an equipartition always exists if $d=6\cdot 2^k+2$ and $j=4\cdot 2^k+1$ which for $k=0$ reduces to the main result of the paper P. Mani-Levitska et al., Topology and combinatorics of partitions of masses by hyperplanes, Adv. Math. 207 (2006), 266–296. The theorem follows from a Borsuk–Ulam type result claiming the non-existence of a $\mathbb{D}_8$-equivariant map $f\colon S^d\times S^d\to S(W^{\oplus j})$ for an associated real $\mathbb{D}_8$-module W. This is an example of a genuine combinatorial geometric result which involves $\mathbb{Z}/4$-torsion in an essential way and cannot be obtained by the application of either Stiefel–Whitney classes or cohomological index theories with $\mathbb{Z}/2$ or $\mathbb{Z}$ coefficients. The method opens a possibility of developing an "effective primary obstruction theory" based on G-manifold complexes, with applications in geometric combinatorics, discrete and computational geometry, and computational algebraic topology.

1. Introduction

1.1. Computational topology. Algebraic topology as a tool "useful for solving discrete geometric problems of relevance to computing and the analysis of algorithms" was in [30] isolated as one of important themes of the emerging

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'applied and computational topology'. Since the appearance of the review [30], the field as a whole has undergone a rapid development, the progress has been made in many open problems, and the *configuration space/test map* paradigm [30, Section 14.1] has maintained its status as one of central schemes for applying equivariant topological methods to combinatorics and discrete geometry.

Our objective is twofold: (a) to report a progress (Theorems 2.1 and 2.2) on the 'equipartitions of masses by hyperplanes' problem, which has been after [20] one of benchmark problems for applying topological methods in computational geometry, (b) to explore a new scheme for developing 'effective primary obstruction theory' based on G-manifold complexes (see Section B.5 in Appendix II for an outline).

1.2. The equipartition problem. An old problem in combinatorial geometry is to determine when j measurable sets in $\mathbb{R}^d$ admit an *equipartition* by a collection of h hyperplanes; if this is possible we say that the triple (d, j, h) is admissible. In this generality the problem was formulated by Grünbaum in [8] but the problem clearly stems from the classical results of Banach and Steinhaus [24] and Stone and Tukey [25] on the "ham sandwich theorem".

Among the highlights of the theory of hyperplane partitions of measurable sets (measures) are Hadwiger's equipartition of a single mass distribution in $\mathbb{R}^3$ by three hyperplanes [11], results of Ramos [20] who proved that $(5, 1, 4), (9, 3, 3), (9, 5, 2)$ are admissible triples, and a result of Mani-Levitska et al. [17] who established a 2-equipartition for 5 measures in $\mathbb{R}^8$. More recent is the proof of the existence of a 4-equipartition in $\mathbb{R}^4$ for measures that admit some additional symmetries (Živaljević [31]) and a result of Matschke [19] describing a general reduction procedure for admissible triples.

For an account of known results, history of the problem, and an exposition of equivariant topological methods used for its solution, the reader is referred to [17]. The landmark paper of E. Ramos [20] is a valuable source of information and a link with earlier results on the discrete and computational geometry of half-space range queries (D. Dobkin, H. Edelsbrunner, M. Paterson, A. Yao, F. Yao). The chapter "Topological Methods" in CRC Handbook of Discrete and Computational Geometry [30], provides an overview of the general *configuration space/test map*-scheme for applying equivariant topological methods on problems of geometric combinatorics and discrete and computational geometry.

DEFINITION 1.1. Suppose that $\mathcal{M} = \{\mu_1, \dots, \mu_j\}$ is a collection of j *continuous mass distributions* (measures) defined in $\mathbb{R}^d$, meaning that each μ_j is a finite, positive, σ-additive Borel measure, absolutely continuous with respect to Lebesgue measure. If $\mathcal{H} = \{H_i\}_{i=1}^k$ is a collection of k hyperplanes in $\mathbb{R}^d$ in general position, the connected components of the complement $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ are called (open) k-orthants. The definition can be clearly extended to the case of degenerate collections $\mathcal{H}$, when some of the k-orthants are allowed to be empty. A collection $\mathcal{H}$ is an *equipartition*, or more precisely a k-equipartition for $\mathcal{M}$ if

$$\mu_i(O) = \mu_i(\overline{O}) = \frac{1}{2^k}\,\mu_i(\mathbb{R}^d)$$

for each of the measures $\mu_i \in \mathcal{M}$ and for each k-orthant O associated to $\mathcal{H}$.

A triple (d, j, k) is called *admissible* if for each collection $\mathcal{M}$ of j continuous measures on $\mathbb{R}^d$ there exists an equipartition of $\mathcal{M}$ by k hyperplanes. It is not difficult to see that if (d, j, k) is admissible than $(d+1, j, k)$ is also admissible ([1]) which motivates the following general problem.

PROBLEM 1.2. The general measure equipartition problem is to determine or estimate the function

$$\Delta(j, k) := \min\{d \mid (d, j, k) \text{ is admissible}\}$$

or equivalently to find the minimum dimension d such that for each collection $\mathcal{M}$ of j continuous measures on $\mathbb{R}^d$, there exists an equipartition of $\mathcal{M}$ by k hyperplanes.

2. New results

Theorem 2.1 is our central new result about equipartitions of masses by two hyperplanes. For $k = 0$ it reduces to the result that the triple $(8, 5, 2)$ is admissible, which is the main result of [17]. The reader is referred to references [20], [17], [19] for an analysis explaining the special role of equipartition problems associated to the triples of the form $(d, j, 2)$ where $2d - 3j = 1$. We emphasize that the admissibility of all triples listed in Theorem 2.1 cannot be established either by the parity count results of Ramos [20], or by the use of both Stiefel-Whitney characteristic classes and the ideal-valued cohomology index theory with $\mathbb{Z}/2$-coefficients.

THEOREM 2.1. *Each collection of $j = 4 \cdot 2^k + 1$ measures in $\mathbb{R}^d$ where $d = 6 \cdot 2^k + 2$ admits an equipartition by two hyperplanes. In light of the lower bound $\Delta(j, 2) \geq 3j/2$, established by Ramos in* [20]*, this result implies that for each integer $k \geq 0$,*

$$\Delta(4 \cdot 2^k + 1, 2) = 6 \cdot 2^k + 2. \tag{2.1}$$

Theorem 2.1 is deduced from the following Borsuk–Ulam type result about maps equivariant with respect to the dihedral group actions.

THEOREM 2.2. *There does not exist a $\mathbb{D}_8$-equivariant map*

$$f\colon S^d \times S^d \to S(W^{\oplus j}) \tag{2.2}$$

([1]) More general reduction results for admissible triples can be found in [19] and [20].

where $\mathbb{D}_8$ *is the dihedral group of order eight,* $d = 6 \cdot 2^k + 2, j = 4 \cdot 2^k + 1$ *for some integer* $k \geq 0$*, and* W *is the representation space of the real* 3*-dimensional* $\mathbb{D}_8$*-representation described in Section* 4.2*. Moreover, a first obstruction to the existence of* (2.2) *lies in the* (*special*) *equivariant cohomology group* $\mathcal{H}^{2d-1}_{\mathbb{D}_8}(S^d \times S^d, \mathcal{Z}) \cong \mathbb{Z}/4$*, described in Section* B (*see Definition* B.3 *and Remark* B.8) *and evaluated in Sections* 7 *and* 8*, where* $\mathcal{Z} = H_{2d-2}(S(W^{\oplus j}); \mathbb{Z})$ *and* $2d - 3j = 1$*. The obstruction vanishes unless*

$$d = 6 \cdot 2^k + 2 \quad \textit{and} \quad j = 4 \cdot 2^k + 1$$

when it turns out to be equal to $2X$ *where* X *is a generator of the group* $\mathbb{Z}/4$*.*

The reader is referred to Section 10 for an outline and overall strategy of proofs of Theorems 2.1 and 2.2. A broader perspective on the method applied in the proof of Theorem 2.2, emphasizing its relevance for computational obstruction theory, is offered in Section B (Subsection B.5).

Caveat: We emphasize that the special equivariant cohomology groups $\mathcal{H}^*_G(X; M)$, used in Theorem 2.2, are in general different from the usual equivariant cohomology groups of a G-CW complex X (as described in [3]). Nevertheless in many cases they are easier to compute and still may contain non-zero classes which can detect a non-trivial obstruction for the existence of an equivariant map. For their definition and main properties the reader is referred to Section B.

Remark 2.3. The fact that the obstruction $2X \in \mathbb{Z}/4$ is divisible by 2 is precisely the reason why Theorem 2.2 is not accessible by the methods based on $\mathbb{Z}/2$-coefficients (parity count [20], Stiefel-Whitney classes, $\mathbb{D}_8$-equivariant index theory with $\mathbb{Z}/2$-coefficients). As the elaborate spectral sequences calculations [1] demonstrate, the methods of $\mathbb{D}_8$-equivariant index theory with $\mathbb{Z}$-coefficients are not sufficient either. This may be somewhat accidental since, as shown in [6] and [4], the $\mathbb{Z}/4$-torsion is often present in related cohomology calculations.

Remark 2.4. In light of the reduction procedure of Matschke [19], who proved the inequality $\Delta(j,k) \leq \Delta(j+1,k) - 1$, it is interesting to test if Theorem 2.1 generates some new admissible triples aside from those implied by the inequality (2.1). The answer is no, since the inequality $\Delta(2^{k+1}, 2) \leq 3 \cdot 2^k$ was established already by Ramos in [20].

Remark 2.5. In the cases when (by Theorem 2.2) the first obstruction vanishes, it is still possible that the secondary obstruction for the existence of the map (2.2) is non-zero. Both the calculation of the secondary obstruction and the detection of new admissible triples (d, j, h) is an interesting open problem.

3. Related results and background information

Results about partitions of measures by hyperplanes have found numerous applications. The survey [30], which appeared in 2004, is a good source of information about the results obtained before its publication. However new applications have emerged in the meantime so we include a brief review of some of the most interesting recent developments illustrating the relevance and importance of hyperplane partitions for different areas of mathematics.

3.1. Polyhedral partitions of measures. Equipartitions of measures by k-orthants (Definition 1.1) are a special case of equipartitions into polyhedral regions.

Very interesting polyhedral partitions are introduced by Gromov in [7]. His *spaces of partitions* [7, Section 5] are defined as the configuration spaces of labelled binary trees T_d of height d, with $2^d - 1$ internal nodes N_d and 2^d external nodes L_d (leaves of the tree T_d). More explicitly a labelled binary tree $(T_d, \{H_\nu\}_{\nu \in N_d})$ has an oriented hyperplane H_ν associated to each of the internal nodes $\nu \in N_d$ of T_d. The left (respectively right) outgoing edge, emanating from $\nu \in N_d$ is associated the positive half-space H_ν^+ (respectively the negative half-space H_ν^-) determined by H_ν.

Each of the leaves $\lambda \in L_d$ is the end point of the unique maximal path π_λ in the tree T_d. Each of the maximal paths π_λ is associated a polyhedral region Q_λ defined as the intersection of all half-spaces associated to edges of the path π_λ. The associated partition $\{Q_\lambda\}_{\lambda \in L_d}$ depends continuously on the chosen labels (hyperplanes) and defines an element of the associated 'space of partitions'.

These and related configuration spaces were used in [7] for the proof a general Borsuk–Ulam type theorem ($c_\bullet$-Corollary 5.3 on page 188) and utilized by Gromov for his proof of the *Waist of the Sphere Theorem*.

Very interesting 'Voronoi polyhedral partitions' of measures were recently introduced in [12]. Far reaching results about polyhedral equipartitions of measures were along these lines obtained by Soberon [22], Karasev [13], and Aronov and Hubard [12].

3.2. Polynomial measure partitions theorems. Theorem 2.1, being a relative of the Ham Sandwich Theorem, has some standard consequences and extensions. One of them is the Polynomial Ham Sandwich Theorem [25], which has recently found striking applications to some old problems of discrete and computational geometry, see [10], [23] and the references in these papers. These breakthroughs have generated a lot of interest and enthusiasm, see the reviews of J. Matoušek [18], M. Sharir [21] and T. Tao [26].

Theorem 2.1, as well as other results about hyperplane measure partitions, have immediate polynomial versions. They can be obtained by the usual Veronese map $\mathbb{R}^d \hookrightarrow \mathbb{R}^D$, or some of its variations. It remains to be seen if some of these polynomial measure partition results can be used as a natural tool which can replace the standard polynomial ham sandwich theorem in some applications.

4. Preliminaries, definitions, notation

4.1. Manifold complexes.

DEFINITION 4.1. A space X is called a *manifold pre-complex* if it is either a compact topological manifold (with or without boundary) or if it is obtained by gluing a compact topological manifold with boundary to a manifold pre-complex via a continuous map of the boundary.

This definition appeared in [14] (Chapter 9) where manifold pre-complexes are referred to as *nice spaces*. One can *mutatis mutandis* modify the Definition 4.1 by allowing different kinds of "manifolds". For example X is a pseudomanifold pre-complex (orientable manifold pre-complex, complex manifold pre-complex, etc.), if the constituent "manifolds" are pseudomanifolds (orientable manifolds, complex manifolds, etc.).

A manifold pre-complex should be seen as a straightforward generalization of a (finite) CW-complex. However a CW-complex has a natural filtration (and an associated rank function) so for this reason we slightly modify the definition and introduce *manifold complexes*.

DEFINITION 4.2. A space X with a finite filtration

$$X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n \tag{4.1}$$

is called a *manifold complex* (orientable manifold complex, etc.) if

(a) X_0 is a finite set of points (0-dimensional manifold);
(b) For each $k \leq n$, $X_k = X_{k-1} \bigcup_{\phi} Y_k$ where Y_k is a compact k-dimensional manifold with boundary and $\phi\colon \partial Y_k \to X_{k-1}$ is a continuous map.

As a variation on a theme we introduce manifold complexes with an action of a finite group G.

DEFINITION 4.3. Let G be a finite group. A G-space X which is also a manifold complex in the sense of Definition 4.2 is called a G-manifold complex if G preserves the filtration (4.1).

4.2. Dihedral group $\mathbb{D}_8$. For basic notation and standard facts about group actions the reader is referred to [3]. A representation space V for a given G-representation $\rho\colon G \to GL(V)$ is also referred to as a (real or complex) G-module. $S(V)$ is the unit sphere in an orthogonal (or unitary) G-representation space V. $X * Y$ is the join of two spaces X and Y, and a standard fact is that if U

and V are two orthogonal G-modules, $S(U \oplus V)$ and $S(U) * S(V)$ are isomorphic as G-spaces.

$\mathbb{D}_8$ is the dihedral group of order 8. $\Lambda = \mathbb{Z}[\mathbb{D}_8]$ is the integral group ring of $\mathbb{D}_8$ and Λx denotes the free, one-dimensional Λ-module generated by x. In this paper x is often a fundamental class of an orientable $\mathbb{D}_8$-pseudomanifold with boundary.

As the group of symmetries of the square $Q = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq |x|,\ |y| \leq 1\}$ the dihedral group $\mathbb{D}_8$ has three distinguished involutions α, β and γ where

$$\alpha(x, y) = (-x, y), \quad \beta(x, y) = (x, -y), \quad \gamma(x, y) = (y, x). \tag{4.2}$$

A standard presentation of $\mathbb{D}_8$ is

$$\mathbb{D}_8 = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \alpha\gamma = \gamma\beta,\ \beta\gamma = \gamma\alpha \rangle.$$

The real two-dimensional representation $\rho\colon \mathbb{D}_8 \to O(2)$ arising from the action on the square is denoted by U. Let $W := U \oplus \lambda$ where λ is the one-dimensional (real) $\mathbb{D}_8$-representation, such that

$$\alpha \cdot z = -z, \quad \beta \cdot z = -z, \quad \gamma \cdot z = z. \tag{4.3}$$

Interpreting $\mathbb{D}_8$ as a Sylow 2-subgroup of the symmetric group S_4 we see that the $\mathbb{D}_8$-module W is isomorphic to the restriction of the reduced permutation representation of S_4 to the dihedral group.

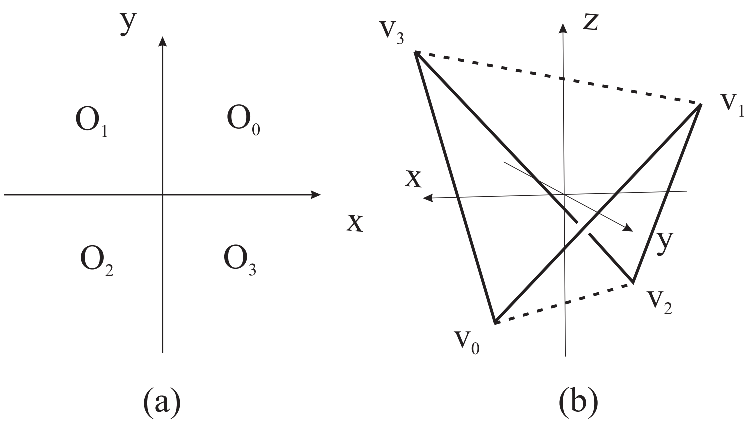


FIGURE 1. $\mathbb{D}_8$-module W as a permutation representation

More explicitly, as shown in Figure 1, the permutations associated to the basic involutions α, β and γ are:

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix}. \tag{4.4}$$

5. The topology of the equipartition problem

The problem of deciding if for a given collection $\{\mu_1, \dots, \mu_j\}$ of measures in $\mathbb{R}^d$ there exists an equipartition by an ordered pair (H_0, H_1) of (oriented) hyperplanes, can be reduced to a topological problem via the usual *Configuration Space/ Test Map Scheme*, see [30, Section 14.1] or [17, Section 2.3]. Here is a brief outline of this construction.

The configuration space, or the space of all candidates for the equipartition, is the space of all ordered pairs (H_0, H_1) of oriented hyperplanes. After a suitable compactification this space can be identified as $S^d \times S^d$. The natural group of symmetries for the equipartition problem is the dihedral group $\mathbb{D}_8$ and the associated action on $S^d \times S^d$ is given by formulas (4.2).

The importance of the representation W stems from the fact that it naturally arises [17, Section 2.3.3] as the associated *Test Space* for a single (probabilistic) measure μ. Indeed for each ordered pair (H_0, H_1) of oriented hyperplanes there is an associated collection of hyperorthants O_0, O_1, O_2, O_3 (Figure 1(a)). Then $\mu(O_\nu)$, $\nu = 0, 1, 2, 3$ are naturally interpreted as the *barycentric coordinates* of a point $v = \mu(O_0)v_0 + \mu(O_1)v_1 + \mu(O_2)v_2 + \mu(O_3)v_3 \in W$ (Figure 1(b)) and the action of $\mathbb{D}_8$ on $S^d \times S^d$ induces an action on these barycentric coordinates which is precisely the action on W as a $\mathbb{D}_8$-module described in Section 4.2. Assume that the barycenter of the tetrahedron is the origin, i.e. $(1/4)(v_0 + v_1 + v_2 + v_3) = 0 \in W$. The map

$$F_\mu \colon S^d \times S^d \to W, \quad (H_0, H_1) \mapsto \mu(O_0)v_0 + \mu(O_1)v_1 + \mu(O_2)v_2 + \mu(O_3)v_3$$

has the property that (H_0, H_1) is an equipartition for μ if and only if (H_0, H_1) is a zero of F_μ. More generally, $z := (H_0, H_1)$ is an equipartition for the collection $\{\mu_1, \dots, \mu_j\}$ of probability measures if and only if $z = (H_0, H_1)$ is a common zero of the associated test maps F_{μ_j},

$$F_{\mu_1}(z) = F_{\mu_2}(z) = \dots = F_{\mu_j}(z) = 0.$$

Summarizing we have the following proposition.

PROPOSITION 5.1. *A triple $(d, j, 2)$ is admissible if each $\mathbb{D}_8$-equivariant map*

$$F \colon S^d \times S^d \to W^{\oplus j}$$

has a zero, or equivalently if there does not exist a $\mathbb{D}_8$-equivariant map

$$f \colon S^d \times S^d \to S(W^{\oplus j}). \tag{5.1}$$

6. Standard admissible filtration of $S^n \times S^n$

In order to prove the non-existence of an equivariant map (5.1) we apply the equivariant obstruction theory in the form outlined in Section B. The first step is a construction of an appropriate filtration on $S^n \times S^n$ which is admissible in the sense of Definition B.3. Less formally, we turn $S^n \times S^n$ into a $\mathbb{D}_8$-manifold complex in the sense of Definition 4.3 by allowing orientable $\mathbb{D}_8$-manifolds with corners and mild singularities.

For $i = 1, \dots, n+1$ define $\pi_i \colon S^n \times S^n \to \mathbb{R}^2$, $\pi_i(x, y) := (x_i, y_i)$ as the restriction of the obvious projection map. The maps π_i are clearly $\mathbb{D}_8$-equivariant and the images $\mathrm{Image}(\pi_i) = \{(x_i, y_i) \mid -1 \leq x_i, y_i \leq +1\} =: Q_i$ are squares which are here referred to as "$\mathbb{D}_8$-screens", Figure 2. The screens Q_i admit a $\mathbb{D}_8$-invariant triangulation which is the starting point for the construction of an admissible filtration on $S^n \times S^n$.

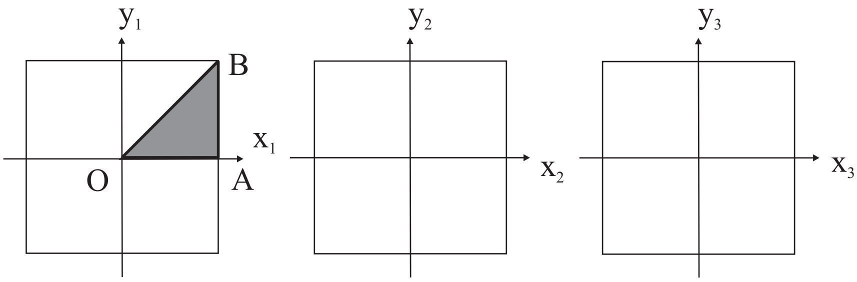


FIGURE 2. $\mathbb{D}_8$-screens for $S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$

The filtration can be informally described as follows. The manifold $S^n \times S^n$ is fibered over the first screen Q_1 with a generic fiber homeomorphic to $S^{n-1} \times S^{n-1}$. The fiber $S^{n-1} \times S^{n-1}$ itself is fibered over the second screen Q_2 with $S^{n-2} \times S^{n-2}$ as a generic fiber, etc. This priority order of screens together with their minimal $\mathbb{D}_8$-invariant triangulations are used to define a version of "lexicographic filtration" of $S^n \times S^n$. For the intended application of the obstruction theory methods from Section B it will be sufficient to give a precise description only for the first three terms of this filtration.

The π_1-preimage $X := \pi_1^{-1}(\Delta_{OAB})$ of the triangle $\Delta_{OAB} \subset Q_1$ is (the closure of) a fundamental domain of the $\mathbb{D}_8$-action on $S^n \times S^n$. It is described by the inequalities $0 \leq y_1 \leq x_1 \leq 1$ and as a subset of $S^n \times S^n$ it is an orientable manifold with boundary. This manifold has corners and possibly singularities of high codimension, however the associated fundamental class $x \in H_{2n}(X, \partial X)$ is well defined. The geometric boundary of X is

$$\partial X = \pi_1^{-1}(\partial \Delta_{OAB}) = X_0' \cup X_1' \cup X_2' = \pi_1^{-1}(OA) \cup \pi_1^{-1}(OB) \cup \pi_1^{-1}(AB).$$

If $n \geq 2$ homologically significant are only X_0' and X_1' since X_2' has codimension n in $S^n \times S^n$ and they contribute to the (homological) boundary evaluated in dimension $2n-1$. Since $\mathrm{Stab}(X_0') = \langle \beta \rangle$ we subdivide and define $X_0 = X_0' \cap \{y_2 \geq 0\}$ as the associated fundamental domain. Similarly, $\mathrm{Stab}(X_1') = \langle \gamma \rangle$ and $X_1 = X_1' \cap \{y_2 \leq x_2\}$, cf. Figure 3 and the subsequent tree-like diagram.

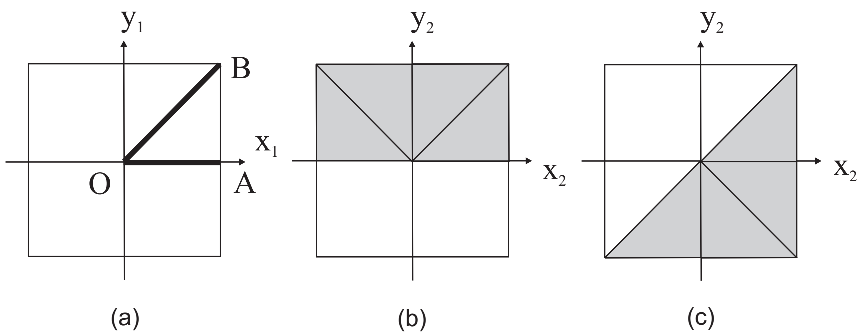


FIGURE 3. Admissible filtration for $S^n \times S^n$

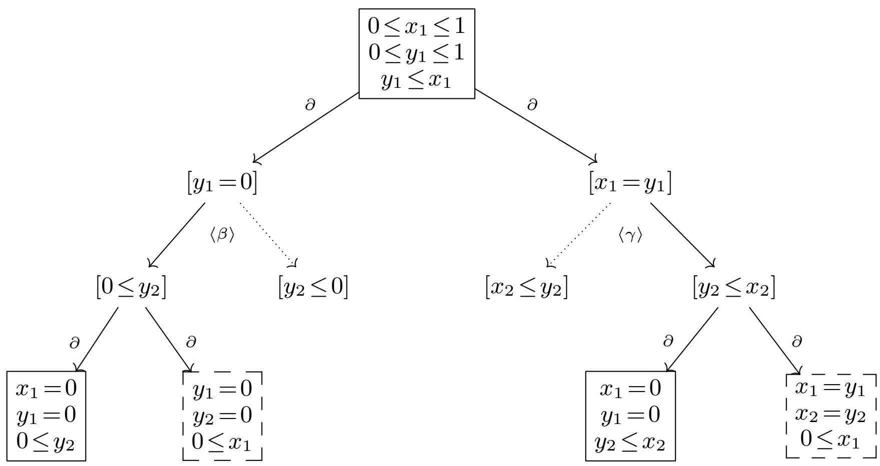
Assuming that $n \geq 8$, we continue the "subdivide and take the boundary"-procedure, focusing only on the homologically significant part of the boundary.

$$\begin{aligned}\partial X_0 &= Z_0' \cup T_0' = \{x_1 = y_1 = 0,\ y_2 \geq 0\} \cup \{x_1 \geq 0,\ y_1 = y_2 = 0\},\\ \partial X_1 &= Z_0'' \cup T_1' = \{x_1 = y_1 = 0,\ y_2 \leq x_2\} \cup \{x_1 = y_1,\ x_1 \geq 0,\ x_2 = y_2\}.\end{aligned}$$

The sets Z_0' and Z_0'' can be further subdivided as follows (Figure 3):

$$\begin{aligned}Z_0' &= Z_0 \cup \alpha Z_0 \cup \gamma Z_0 \cup \alpha\gamma Z_0,\\ Z_0'' &= Z_0 \cup \beta Z_0 \cup \beta\gamma Z_0 \cup \alpha\beta\gamma Z_0,\end{aligned}$$

where $Z_0 := \{x_1 = y_1 = 0,\ 0 \leq y_2 \leq x_2 \leq 1\}$. Finally, $T_0' = T_0 \cup \beta T_0$ and $T_1' = T_1 \cup \gamma T_1$ where $T_0 = T_0' \cap \{0 \leq y_3 \leq 1\}$ and $T_1 = T_1' \cap \{y_3 \leq x_3\}$.



6.1. The fundamental domain – geometric boundary procedure. As a summary of the construction we observe that the "take the boundary, then subdivide"-procedure produces an admissible filtration

$$S^n \times S^n = F_{2n} \supset F_{2n-1} \supset F_{2n-2} \supset \dots \tag{6.1}$$

where $F_{2n-1} := \bigcup_{g\in\mathbb{D}_8} g(\partial X)$ and $F_{2n-2} := \bigcup_{g\in\mathbb{D}_8} g(Z_0 \cup T_0 \cup T_1 \cup X_2)$.

For the intended application the explicit description (6.1) is sufficient. However, it is clear how one can continue the construction by using screens of higher order. Observe that the 'tree of fundamental domains' is formally generated by two types of branching while the root of the tree is a fundamental domain of the manifold.

7. Fragment of the chain complex for $S^n \times S^n$

The sets X, X_0, X_1, Z_0, T_0, T_1 described in the previous section are connected manifolds with boundary (with corners and possibly with mild singularities in codimension ≥ 2). They all can be oriented in which case the corresponding fundamental classes are denoted by x, x_0, x_1, z_0, t_0, t_1. These classes are naturally interpreted as the generators of $\mathbb{D}_8$-modules $H_k(F_k, F_{k-1}; \mathbb{Z})$ for $k = 2n, 2n-1, 2n-2$.

The orientation character of the $\mathbb{D}_8$-manifold $S^n \times S^n$ is given by

$$(\alpha, \beta, \gamma) = ((-1)^{n-1}, (-1)^{n-1}, (-1)^n). \tag{7.1}$$

From (7.1) and the analysis of geometric boundaries given in Section 6 one deduces the following relations:

$$\partial x = (1 + (-1)^n\beta)x_0 + (1 + (-1)^{n-1}\gamma)x_1, \tag{7.2}$$

$$\partial x_0 = (1 + (-1)^n\alpha + (-1)^{n-1}\gamma - \alpha\gamma)z_0 + (1 + (-1)^{n-1}\beta)t_0, \tag{7.3}$$

$$\partial x_1 = -(1 + (-1)^n\beta - \beta\gamma + (-1)^{n-1}\alpha\beta\gamma)z_0 + (1 + (-1)^n\gamma)t_1. \tag{7.4}$$

The top dimensional fragment of the associated chain complex is,

$$\Lambda x \xrightarrow{B} \Lambda x_0 \oplus \Lambda x_1 \xrightarrow{A} \Lambda z_0 \oplus \Lambda t_0 \oplus \Lambda t_1 \longrightarrow \cdots \tag{7.5}$$

The boundary homomorphisms are described by (7.2) so, for example, if n is even,

$$A^t = \begin{bmatrix} 1+\alpha-\gamma-\alpha\gamma & 1-\beta & 0 \\ -(1+\beta-\beta\gamma-\alpha\beta\gamma) & 0 & 1+\gamma \end{bmatrix} B = \begin{bmatrix} 1+\beta \\ 1-\gamma \end{bmatrix}. \tag{7.6}$$

REMARK 7.1. The reader may use the $\mathbb{D}_8$-screens, introduced in Section B.2 and depicted in Figures 2 and 3, as a useful bookkeeping device for checking the formulas (7.2). For example the term $(1 + (-1)^n\alpha + (-1)^{n-1}\gamma - \alpha\gamma)z_0$, in the middle equation, corresponds to the decomposition of the shaded rectangle, shown in Figure 3(b), into four triangles. Being in the second screen, this rectangle corresponds to a region in $S^{n-1} \times S^{n-1}$ whose orientation, following (7.1), transforms by the rule $(\alpha, \beta, \gamma) = (-1)^n, (-1)^n, (-1)^{n-1}$. This explains the sign in $(-1)^n\alpha$, etc.

8. Evaluation of the cohomology group $\mathcal{H}^{2n-1}_{\mathbb{D}_8}(S^n \times S^n; \mathcal{Z})$

The first obstruction to the existence of an equivariant map (5.1) lies in the group $\mathcal{H}^{2n-1}_{\mathbb{D}_8}(S^n \times S^n; \mathcal{Z})$ where $\mathcal{Z} \cong H_{3j-1}(S(W^{\oplus j}); \mathbb{Z})$ is the orientation character of the sphere $S(W^{\oplus j})$. We remind the reader that these groups are defined (Section B) as functors of $\mathbb{D}_8$-spaces with admissible filtrations. Also note that the condition $2d - 3j = 1$ (Section 2) allows us to assume that j is an odd integer.

Proposition 8.1. *Let $\mathcal{M} = \mathcal{Z}$ be the orientation character of the sphere $S(W^{\oplus j})$ where j is an odd integer. Then,*

$$\mathcal{H}^{2n-1}_{\mathbb{D}_8}(S^n \times S^n; \mathcal{Z}) \cong \begin{cases} \mathbb{Z}/4 & \text{if } n \text{ is even;} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd.} \end{cases} \tag{8.1}$$

Moreover, a generator of $\mathbb{Z}/4$ can be described as the cocycle

$$\phi\colon \Lambda x_0 \oplus \Lambda x_1 \to \mathcal{Z}, \qquad \phi(x_0) = 1, \phi(x_1) = -1. \tag{8.2}$$

Proof. The orientation character $\mathcal{Z}$ of the $\mathbb{D}_8$-sphere $S(W)$ is easily determined from the signs of permutations (4.4) and reads as follows

$$(\alpha, \beta, \gamma) = (+1, +1, -1). \tag{8.3}$$

Since j is an odd integer the same answer is obtained for the orientation character of the sphere $S(W^{\oplus j})$. Assume that n is even. By applying the functor $\mathrm{Hom}(\,\cdot\,, \mathcal{Z})$ to the chain complex (7.5) we obtain the complex

$$\mathbb{Z} \xleftarrow{B_1} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{A_1} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \tag{8.4}$$

where,

$$A_1 = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}, \qquad B_1^t = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

From here we deduce that

$$\mathcal{H}^{2n-1}_{\mathbb{D}_8}(S^n \times S^n; \mathcal{Z}) \cong \mathbb{Z}/4 \tag{8.5}$$

where a generator is the cocycle $\phi\colon \Lambda x_0 \oplus \Lambda x_1 \to \mathcal{Z}$, $\phi(x_0) = 1$, $\phi(x_1) = -1$. The second half of (8.1) is established by a similar calculation. □

9. Evaluation of the obstruction

In light of Propositions 5.1 and 8.1 (isomorphisms (8.1)) our primary concern are admissible triples $(d, j, 2)$ such that d is an even integer; this is precisely the case when the obstruction group is $\mathbb{Z}/4$. Hence, throughout this section we assume that $(d, j) = (6k + 2, 4k + 1)$ for some integer $k \geq 0$.

As emphasized in Section B.4 the evaluation of the obstruction class θ can be linked with the computation of mapping degrees of carefully chosen maps. We begin with a map which plays an auxiliary role in these calculations.

9.1. The degree of the multiplication of monic polynomials. Let $\mathcal{P}_m := \{p(x) = a_0+a_1x+\ldots+a_mx^m \mid a_i \in \mathbb{R}\}$ be the vector space of polynomials of degree at most m with coefficients in the field of real numbers. Let

$$\mu_{m,n}\colon \mathcal{P}_m \times \mathcal{P}_n \to \mathcal{P}_{m+n}$$

be the multiplication of polynomials. We focus our attention on the (affine) space $\mathcal{P}^0_m := \{p(x) = a_0 + a_1x + \ldots + a_{m-1}x^{m-1} + x^m \mid a_i \in \mathbb{R}\}$ of monic polynomials of degree m and the associated multiplication map

$$\mu^0_{m,n}\colon \mathcal{P}^0_m \times \mathcal{P}^0_n \to \mathcal{P}^0_{m+n}. \tag{9.1}$$

Our objective is to evaluate the mapping degree of the map $\mu^0_{m,n}$, say as the algebraic count of the number of points in the pre-image $f^{-1}(z)$ of a regular point $z \in \mathcal{P}^0_{m+n}$. We begin with a preliminary proposition which guarantees that the degree is well defined.

PROPOSITION 9.1. *The multiplication* (9.1) *of monic polynomials is a proper map of manifolds.*

PROOF. Assume that $A \subset \mathcal{P}^0_m$ and $B \subset \mathcal{P}^0_n$ are sets of polynomials such that $A \cdot B := \{p \cdot q \mid p \in A,\ q \in B\}$ is bounded as a set of polynomials in $\mathcal{P}^0_{m+n} \cong \mathbb{R}^{m+n}$. We want to conclude that both A and B individually are bounded sets of polynomials. This is easily deduced from the following claim.

CLAIM. If $A \subset \mathcal{P}^0_n$ is bounded set of polynomials then the set $\mathrm{Root}(A) := \{z \in \mathbb{C} \mid p(z) = 0 \text{ for some } p \in A\}$ is also bounded. Conversely, if $\mathrm{Root}(A)$ is a bounded, A is also a bounded set of polynomials.

PROOF OF THE CLAIM. The implication $\Leftarrow$ follows from Viète's formulas, while the opposite implication $\Rightarrow$ follows from the inequality

$$|\lambda| \leq \mathrm{Max}\left\{1, \sum_{j=0}^{n-1} |a_j|\right\},$$

where λ is a root of the polynomial $a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x_n$. $\square$

The next step needed for computation of the mapping degree of the map (9.1) is the evaluation of the differential $d\mu^0_{m,n}$. The tangent space $T_p(\mathcal{P}^0_m)$ at the monic polynomial $p \in \mathcal{P}^0_m$ is naturally identified with the space $\mathcal{P}_{m-1}$ of all polynomials of degree at most $m-1$.

LEMMA 9.2. *Given monic polynomials $p \in \mathcal{P}_m^0$ and $q \in \mathcal{P}_n^0$ and the polynomials $u \in \mathcal{P}_{m-1}$, $v \in \mathcal{P}_{n-1}$, playing the role of the associated tangent vectors, the differential $d\mu_{m,n}^0 = d\mu$ is evaluated by the formula*

$$d\mu_{(p,q)}(u,v) = \frac{d}{dt}(p+tu)(q+tv)\Big|_{t=0} = pv + uq. \tag{9.2}$$

Let us determine the matrix of the map $d\mu_{(p,q)}^0$ in suitable bases of the associated tangent spaces $T_{(p,q)}(\mathcal{P}_m^0 \times \mathcal{P}_n^0) \cong \mathcal{P}_{m-1} \times \mathcal{P}_{n-1}$ and $T_{pq}(\mathcal{P}_{m+n}^0) \cong \mathcal{P}_{m+n-1}$. A canonical choice of basis for $\mathcal{P}_{m-1}^0$ is $u_0 = 1, u_1 = x, \dots, u_{m-1} = x^{m-1}$ with similar choices $v_0 = 1, v_1 = x, \dots, v_{n-1} = x^{n-1}$ and $w_0 = 1,\ w_1 = x, \dots, w_{p+q-1} = x^{m+n-1}$ for $\mathcal{P}_{n-1}^0$ and $\mathcal{P}_{m+n-1}^0$, respectively. Formula (9.2) applied to this basis gives

$$d\mu_{(p,q)}(0,v_j) = d\mu_{(p,q)}(0,x^j) = x^j p(x), \quad d\mu_{(p,q)}(u_i,0) = d\mu_{(p,q)}(x^i,0) = x^i q(x).$$

We conclude from here that the determinant of this matrix is equal to the *resultant* (9.3) of two polynomials!

$$\mathcal{R}(p,q) = \mathrm{Det}\begin{bmatrix} a_0 & a_1 & \dots & a_{m-1} & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{m-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{m-1} \\ b_0 & b_1 & \dots & b_{n-1} & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}. \tag{9.3}$$

In particular we can use classical formulas for $\mathcal{R}(p,q)$, see [5, Chapter 12], among them the formula

$$\mathcal{R}(p,q) = \prod_{i,j}(\alpha_i - \beta_j) \tag{9.4}$$

where α_i are roots of p and β_j are roots of q respectively, counted with the appropriate multiplicities.

LEMMA 9.3. *Suppose that $p(x) = a_0 + a_1x + \dots + a_{m-1}x^{m-1} + x^m$ and $q(x) = b_0 + b_1x + \dots + b_{n-1}x^{n-1} + x^n$ are two polynomials with real coefficients such that the corresponding roots $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_n$ are all distinct and non-real, $\{\alpha_i\}_{i=1}^m \cup \{\beta_j\}_{j=1}^n \subset \mathbb{C} \setminus \mathbb{R}$. Then the resultant of polynomials p and q is real and positive, $\mathcal{R}(p,q) > 0$.*

PROOF. By assumption all roots of $p(x)$ (respectively $q(x)$) can be divided in conjugate pairs $\alpha, \overline{\alpha}$ (respectively $\beta, \overline{\beta}$). These two pairs contribute to the product (9.4) the factor

$$(\alpha-\beta)(\overline{\alpha}-\overline{\beta})(\alpha-\overline{\beta})(\overline{\alpha}-\beta)=A\overline{A}B\overline{B}>0. \qquad \square$$

Proposition 9.4. *Suppose that $m=2k$ and $n=2l$ are even integers. The degree of the map $\mu^0_{m,n}\colon \mathcal{P}^0_m\times\mathcal{P}^0_n\to\mathcal{P}^0_{m+n}$ is*

$$\deg(\mu^0_{m,n})=\binom{k+l}{k}. \tag{9.5}$$

Proof. We compute the degree $\deg(\mu^0_{m,n})$ by an algebraic count of the number of points in the pre-image $(\mu^0_{m,n})^{-1}(\rho)$ where the polynomial $\rho\in\mathcal{P}^0_{m+n}$ is a regular value of the map $\mu^0_{m,n}$.

Assume that $\rho=\rho_1\ldots\rho_{k+l}$ is a product of pairwise distinct, irreducible, quadratic (monic) polynomials ρ_i. Equivalently ρ does not have real roots and all its roots are pairwise distinct. Note that such a polynomial can be easily constructed by prescribing its roots, for example it can be found in any neighbourhood of the polynomial x^{m+n}. The inverse image $(\mu^0_{m,n})^{-1}(\rho)$ is,

$$(\mu^0_{m,n})^{-1}(\rho)=\{(p,q)\in\mathcal{P}^0_m\times\mathcal{P}^0_n\mid p\cdot q=\rho\}.$$

It follows from Lemma 9.3 that $\mathcal{R}(p,q)>0$ for each pair of polynomials in the inverse image $(\mu^0_{m,n})^{-1}(\rho)$. In particular ρ is a regular value of the map $\mu^0_{m,n}$ and each pair (p,q) such that $p\cdot q=\rho$ contributes $+1$ to the degree. From here we deduce that

$$\deg(\mu^0_{m,n})=\binom{k+l}{k} \qquad \square$$

The multiplication map $\mu_{m,n}(p,q)=p\cdot q$ is non-degenerate in the sense that $\mu_{m,n}(p,q)=0$ implies that either $p=0$ or $q=0$. As a consequence it induces a map

$$\mu'_{m,n}\colon \mathcal{P}'_m\times\mathcal{P}'_n\to\mathcal{P}'_{m+n}$$

where $\mathcal{P}'_m:=\mathcal{P}_m\setminus\{0\}$ is the set of non-zero polynomials.

Proposition 9.5. *Suppose that $m=2k$ and $n=2l$ are even integers. The degree of the map $\mu'_{m,n}\colon \mathcal{P}'_m\times\mathcal{P}'_n\to\mathcal{P}'_{m+n}$ is*

$$\deg(\mu'_{m,n})=\pm 2\cdot\binom{k+l}{k}. \tag{9.6}$$

Proof. Let $\mathcal{P}^0_d:=\{a_0+\ldots+a_dt^d\in\mathcal{P}_d\mid a_d=1\}$ be the hyperplane of monic polynomials, viewed as the tangent space at t^d to the euclidean sphere $S^{d-1}=\{p\in\mathcal{P}_d\mid \|p\|=1\}$ in the space of polynomials $\mathcal{P}_d$. The degree $\deg(\mu'_{m,n})$ can be evaluated as the degree of the map

$$\widehat{\mu}=\widehat{\mu}_{m,n}\colon S^m\times S^n\to S^{m+n} \tag{9.7}$$

where $\widehat{\mu}(p,q) = (p\cdot q)/\|p\cdot q\|$ is the multiplication of polynomials composed with the radial projection. The radial projection also induces a bijection of $\mathcal{P}_d^0$ with the open hemisphere $S_+^d := \{p \in \mathcal{P}_d \mid \|p\| = 1,\ a_d > 0\}$. Similarly $-\mathcal{P}_d^0$ is radially projected on the negative (open) hemisphere S_-^d.

Following the proof of Proposition 9.4, let us choose for a regular value ρ of the map $\mu_{m,n}^0$ a polynomial very close to the north pole t^{m+n} of the sphere S^{m+n}. We observe that each decomposition $p\cdot q = \rho$, contributing to the degree of $\mu_{m,n}^0$, defines two points, (p,q) and $(-p,-q)$ in the preimage $\hat{\mu}^{-1}(\rho)$. Since the map $(x,y)\mapsto(-x,-y)$ preserves the orientation of the manifold $S^m\times S^n$, we observe that both (p,q) and $(-p,-q)$ contribute to the degree of $\hat{\mu}_{m,n}$ with the same sign which is independent of the choice of the decomposition $p\cdot q=\rho$. As a consequence, $\deg(\hat{\mu}_{m,n}) = 2\cdot\deg(\mu_{m,n}^0)$ which in light of (9.5) completes the proof of the proposition. □

Corollary 9.6. *Suppose that $f\colon S^m\times S^m \to S(\lambda^{\oplus(2m+1)})$ is a $\mathbb{D}_8$-equivariant map where $m=2k$ is an even integer. Then*

$$\deg(f) \equiv 4\binom{2k-1}{k-1} \pmod 8.$$

Proof. By Corollary A.2 (Section A) it is sufficient to exhibit a single map with the indicated degree. The multiplication map $\mu_{m,m}\colon \mathcal{P}_m\times\mathcal{P}_m \to \mathcal{P}_{2m}$ is $\mathbb{D}_8$-equivariant if $\mathcal{P}_{2m}\cong\lambda^{\oplus(2m+1)}$ as $\mathbb{D}_8$-modules. Hence, the result follows from Proposition 9.5. □

Example 9.7. The $\mathbb{D}_8$-equivariant map $\mu\colon S^2\times S^2\to S^4$ associated to the multiplication of quadratic polynomials $\mu'_{2,2}\colon \mathcal{P}'_2\times\mathcal{P}'_2 \to \mathcal{P}'_4$ has degree ±4, consequently the degree of any $\mathbb{D}_8$-equivariant map is an integer of the form $8k+4$.

9.2. Evaluation of the obstruction class θ. The following proposition is a key result for evaluation of the primary obstruction $\theta\in\mathbb{Z}/4$ to the existence of a $\mathbb{D}_8$-equivariant map (5.1).

Proposition 9.8. *Suppose that*

$$\phi\colon S^{d-1}\times S^{d-1}\to S(W^{\oplus j}) \tag{9.8}$$

is a $\mathbb{D}_8$-equivariant map where $(d,j)=(6k+2,4k+1)$ for some integer $k\geq 0$. Then,

$$\deg(\phi)\equiv 4\binom{2k-1}{k-1} \pmod 8. \tag{9.9}$$

Proof. By Corollary A.3 it is sufficient to exhibit a particular $\mathbb{D}_8$-equivariant map (9.8) which satisfies the congruence (9.9). Let us construct a $\mathbb{D}_8$-equivariant map

$$\Phi\colon \mathbb{R}^d\times\mathbb{R}^d \to W^{\oplus j} \tag{9.10}$$

with the property that if $\Phi(p,q) = 0$ then either $p = 0$ or $q = 0$. By decomposing the real $\mathbb{D}_8$-modules $\mathbb{R}^d \times \mathbb{R}^d \cong U^{\oplus d} \cong U^{\oplus j} \oplus U^{\oplus(d-j)}$ and $W^{\oplus j} \cong U^{\oplus j} \oplus \lambda^{\oplus j}$ (Section 4.2) we observe that it is sufficient to construct a $\mathbb{D}_8$-equivariant map

$$\Phi'\colon \mathbb{R}^{d-j} \times \mathbb{R}^{d-j} \to \lambda^{\oplus j} \tag{9.11}$$

which is non-degenerate in the sense that if $\Phi'(p,q) = 0$ then either $p = 0$ or $q = 0$. Let $m := 2k$ so $(d,j) = (3m+2, 2m+1) = (6k+2, 4k+1)$. Identify $\mathbb{R}^{d-j}$ with the space of real polynomials of degree less or equal to $d - j - 1 = m$ and $\lambda^{\oplus j}$ with the space of real polynomials of degree $\leq j - 1 = 2m$. Then the multiplication of polynomials defines a non-degenerate (symmetric) bilinear form $\Phi'(p,q) = p \cdot q$ which is an example of a $\mathbb{D}_8$-equivariant map (9.11) with the desired properties.

Summarizing, using the identifications $\mathbb{R}^d \cong \mathcal{P}_{2m} \oplus \mathcal{P}_m$ and $W^{\oplus j} \cong (\mathcal{P}_{2m})^{\oplus 3}$ with the corresponding vector spaces of polynomials, we observe that the map Φ has the following explicit form

$$\Phi(p,q) = \Phi(p', p''; q', q'') = (p', q', p''q'') \tag{9.12}$$

where $p''q'' = \mu(p'', q'')$ is the polynomial multiplication. The degree $\deg(\Phi)$ can be calculated again, as in the proof of Proposition 9.5, by the reduction to the case of monic polynomials p'' and q''. By choosing the regular value of the map (9.12) in the form $(0,0,\rho)$, where ρ is a regular value for the multiplication of monic polynomials we observe that Proposition 9.8 is an immediate consequence of Proposition 9.4. $\square$

The following proposition is the central result of this section and the ultimate goal of all earlier computations. Note that the manifold $S^{d-1} \times S^{d-1}$ (Proposition 9.8) can be naturally identified as a subset of the $(2d-2)$-dimensional element F_{2d-2} of the filtration (6.1) via the identification

$$S^{d-1} \times S^{d-1} = (S^d \times S^d) \cap \{x_1 = y_1 = 0\} = \bigcup_{g \in \mathbb{D}_8} g(Z_0).$$

PROPOSITION 9.9. *Suppose that $\Delta = 2d - 3j = 1$ where $(d,j) = (3m+2, 2m+1)$ and $m = 2k$ is an even integer. Then the first obstruction to the existence of a $\mathbb{D}_8$-equivariant map* (5.1)*, evaluated as an element of $\mathbb{Z}/4$, is equal to*

$$\theta = 2\binom{2k-1}{k-1} \pmod 4. \tag{9.13}$$

PROOF. By (8.2) a generator of the obstruction group is the cocycle ϕ such that $\phi(x_0) = 1$, $\phi(x_1) = -1$ where x_0 and x_1 are the (relative) fundamental classes of the pseudomanifolds X_0 and X_1.

Following the calculations (and notation) from Section 6, the homologically relevant part of the geometric boundary of the pseudomanifold X_0 has the following representation:

$$\partial X_0 = Z_0' \cup T_0' = (Z_0 \cup \alpha Z_0 \cup \gamma Z_0 \cup \alpha\gamma Z_0) \cup (T_0 \cup \beta T_0). \tag{9.14}$$

The set $M := Z_0' \cup T_0'$ is, up to a closed subset of high codimension, a closed oriented manifold. In light of (B.8) the obstruction θ can be evaluated as the degree $\deg(f)$ where $f = \psi|_M$ is the restriction to M of an arbitrary $\mathbb{D}_8$-equivariant map $\psi\colon F_{2d-2} \to S(U^{\oplus j} \oplus \lambda^{\oplus j})$. Such a map clearly exists since $\dim(F_{2d-2}) = \dim(S(W^{\oplus j}))$. Interpreting the degree as an algebraic count of points in the preimage of a regular value, we observe that

$$\deg(\psi) = \deg(\psi_{|Z_0'}) + \deg(\psi_{|T_0'}).$$

The computation of the degrees on the right is facilitated by the equivariance of ψ and existing $\mathbb{D}_8$-symmetries of X_0 and T_0. In particular, the fact that X_0 is "a half of the manifold" $S^{d-1} \times S^{d-1} \subset F_{2d-2}$, together with Proposition 9.8, implies that

$$\deg(\psi_{|Z_0'}) = 2\binom{2k-1}{k-1} \pmod 4. \tag{9.15}$$

Similarly, $\deg(\psi_{|T_0'}) = 0$ follows from the fact that β acts on $T_0' = T_0 \cup \beta T_0$ by changing its orientation (equation (7.2)), while it keeps the orientation of $S(W^{\oplus j})$ fixed (equation (8.3)). This observation completes the proof of the proposition. □

10. Summary of proofs of main results

PROOF OF THEOREM 2.1. By Proposition 5.1, Theorem 2.1 is an immediate consequence of Theorem 2.2. □

PROOF OF THEOREM 2.2. By Proposition B.6 the existence of an equivariant map (2.2) implies the existence of the chain map (B.6) where C_* is the chain complex associated to an admissible filtration in the sense of Definition B.3.

In order to facilitate calculations we choose the standard filtration on $X = S^n \times S^n$ described in Section 6 and calculate the relevant fragment of the associated chain complex $C_* = C_*(S^n \times S^n)$ in Section 7.

By Proposition B.7 the first obstruction θ to the existence of the chain map (B.6) lies in the cohomology group (B.7) which is in our case (see (B.4) and Remark B.8) the group $\mathcal{H}^{2n-1}_{\mathbb{D}_8}(S^n \times S^n; \mathcal{Z})$. This group is evaluated in Proposition 8.1 (Section 8) and found to be isomorphic to $\mathbb{Z}/4$.

The obstruction class θ is evaluated in Proposition 9.9 following the description of the associated cocycle given in (B.8). We use the idea, described in greater detail in Section B.4, that in some situations we have some freedom in choosing the map f_n for evaluating the obstruction cocycle $\theta(f_n)$ by the formula (B.8).

As an element of the group $\mathbb{Z}/4$ the obstruction class θ is according to (9.13) equal to

$$\theta = 2\binom{2k-1}{k-1}.$$

It is non-zero if and only if $\binom{2k-1}{k-1}$ is an odd integer which is the case if and only if $k = 2^l$. It follows that in the case when d is even the triple $(d, j, 2)$ is admissible if for some integer $l \geq 0$,

$$d = 3 \cdot 2^{l+1} + 2 \quad \text{and} \quad j = 2 \cdot 2^{l+1} + 1. \qquad \square$$

Appendix A. Mapping degree of equivariant maps

There is a general principle, see Kushkuley and Balanov [16], equation (0.1) in Section 0.3., relating the mapping degrees of two G-equivariant maps

$$f, g \colon M^n \to S(V)$$

where G is a finite group, M^n is a not necessarily free, closed, oriented G-manifold, and $S(V)$ is the unit sphere in a real, $(n+1)$-dimensional G-vector space V. The principle says that there exists a relation

$$\pm(\deg(f) - \deg(g)) = \sum_{j=1}^{k} a_j |G/H_j| \tag{A.1}$$

for some integers a_j, where $\mathcal{H} = \{H_1, \dots, H_k\}$ is a list of isotropy groups corresponding to orbit types of M^n, provided the *orientation characters* of M and V are the same in the sense that each $g \in G$ either changes orientations of both M and V, or keeps them both unchanged.

In some (favorable) situations the "local degrees" a_j vanish if $H_j \neq \{e\}$ is a non-trivial subgroup of G, in which case the equation (A.1) implies the congruence

$$\deg(f) \equiv \deg(g) \pmod{|G|}. \tag{A.2}$$

Here we record one of these favorable situations applying to the case of the dihedral group $\mathbb{D}_8$ of order 8 acting on the manifold $M = S^m \times S^m$.

THEOREM A.1. *Let V be a real $\mathbb{D}_8$-vector space of dimension $(2m+1)$ such that the isotropy space V_γ, corresponding to the element $\gamma \in \mathbb{D}_8$, has dimension $\geq m+2$. Assume that the representation V has the same orientation character as the space $S^m \times S^m$. Then for each pair $f, g \colon S^m \times S^m \to S(V)$ of $\mathbb{D}_8$-equivariant maps, the associated mapping degrees satisfy the following congruence relation*:

$$\deg(f) \equiv \deg(g) \pmod{8}. \tag{A.3}$$

Proof. The action of $\mathbb{D}_8$ on $S^m \times S^m$ is free away from the two spheres $S_\gamma = \{(x,y) \in S^m \times S^m \mid x = y\}$ and $S_{\alpha\beta\gamma} = \{(x,y) \in S^m \times S^m \mid x = -y\}$.

Let f_1 and g_1 be the restrictions of f (respectively g) on their union $T = S_\gamma \cup S_{\alpha\beta\gamma}$. The space T is $\mathbb{D}_8$-invariant and our initial objective is to show that there exists a $\mathbb{D}_8$-equivariant homotopy $F_1 \colon T \times [0,1] \to V$ between f_1 and g_1.

We apply the Proposition I.7.4 from [3, p. 52] which says that it is sufficient to construct a $\mathbb{Z}/2$-equivariant homotopy $F_1' \colon S_\gamma \times [0,1] \to V_\gamma$ between the restrictions $f_1' = f_1|S_\gamma$ and $g_1' = g_1|S_\gamma$, where $\mathbb{Z}/2 = N(\langle\gamma\rangle)/\langle\gamma\rangle$ is the Weyl group of $\langle\gamma\rangle$. Since this $\mathbb{Z}/2$-action on S_γ is free, the existence of the homotopy F_1' follows immediately from the assumption:

$$\dim(S_\gamma) = m \leq \dim(S(V_\gamma)) - 1.$$

The homotopy F_1 can be extended to a $\mathbb{D}_8$-equivariant homotopy

$$F \colon S^m \times S^m \to D(V)$$

between f and g where $D(V) = \mathrm{Cone}(S(V))$ is the unit ball in V. Moreover, since the action of $\mathbb{D}_8$ on $(S^m \times S^m) \setminus (S_\gamma \cup S_{\alpha\beta\gamma})$ is free, we can assume that F is smooth away from $S_\gamma \cup S_{\alpha\beta\gamma}$ and transverse to $0 \in V$.

The set $Z(F) := F^{-1}(0)$ is finite, G-invariant and a union of free orbits. For each $x \in Z(F)$ choose an open ball $O_x \ni x$ such that $O := \bigcup_{x \in Z(F)} O_x$ is G-invariant and $O_x \cap O_y \neq \emptyset$ for $x \neq y$. Let $S_x := \partial(O_x) \cong S^{2m}$ be the boundary of O_x.

Let $N := (M \times [0,1]) \setminus O$, $M_0 := M \times \{0\}$ and $M_1 := M \times \{1\}$. By construction, there is a relation among (properly oriented) fundamental classes

$$[M_1] - [M_0] = \sum_{x \in Z(F)} [S_x] \tag{A.4}$$

inside the homology group $H_{2m+1}(N, \mathbb{Z})$. The map

$$F_* \colon H_{2m+1}(N, \mathbb{Z}) \to H_{2m+1}(V \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$$

maps the left hand side of the relation (A.4) to the difference of degrees, $\deg(f) - \deg(g)$. The right hand side is mapped to an element divisible by 8 since by assumption the orientation characters of manifolds $M \times [0,1]$ and V are the same. □

Corollary A.2. *Suppose that $f, g \colon S^m \times S^m \to S(\lambda^{\oplus(2m+1)})$ are $\mathbb{D}_8$-equivariant maps where $m = 2k$ is an even integer. Then*

$$\deg(f) \equiv \deg(g) \pmod 8.$$

Proof. The orientation character of both $S^m \times S^m$ and $S(\lambda^{\oplus(2m+1)})$ is given by the formula $(\alpha, \beta, \gamma) = (-1, -1, +1)$, see (4.3) in Section 4.2 and (7.1) in Section 7. Since the whole space $\lambda^{\oplus(2m+1)}$ is fixed by γ the result is an immediate consequence of Theorem A.1. □

COROLLARY A.3. *Suppose that*

$$f, g\colon S^{d-1} \times S^{d-1} \to S(W^{\oplus j}) \tag{A.5}$$

are $\mathbb{D}_8$*-equivariant maps where* $(d,j) = (6k+2, 4k+1)$ *for some integer* $k > 0$. *Then,*

$$\deg(f) \equiv \deg(g) \pmod 8.$$

PROOF. In light of (7.1) and (8.3) the orientation character of both $S^{d-1} \times S^{d-1}$ and $W^{\oplus j}$ is given by $(\alpha, \beta, \gamma) = (+1, +1, -1)$. The dimension D of the isotropy space $W_\gamma^{\oplus j}$ is $2j = 8k+2$ so the dimension requirement $D \geq d+1$ from Theorem A.1 reduces to $8k+2 \geq 6k+3$ which is satisfied if $k > 0$. □

Appendix B. Obstructions, filtrations and chain complexes

B.1. The obstruction exact sequence. For a review of equivariant obstruction theory, which includes an exposition of G-CW-complexes and the G-cellular approximation theorem, the reader is referred to [3, Chapter II].

One of the central results in the area is the following *obstruction exact sequence.*

THEOREM B.1. *Suppose that* X *is a free* G-CW*-complex and that* Y *is* n*-simple* G*-space for a fixed integer* $n \geq 1$. *Then there exists an exact sequence*

$$[X^{(n+1)}, Y]_G \to \mathrm{Image}\{[X^{(n)}, Y]_G \to [X^{(n-1)}, Y]_G\} \to \mathcal{H}_G^{n+1}(X, \pi_n(Y)) \tag{B.1}$$

In many applications $Y = S(V)$ is a G-sphere $S(V) \cong S^n$ for some real G-module V. In that case Theorem B.1 has the following important corollary.

COROLLARY B.2. *Suppose that* X *is a free* G-CW*-complex. Let* $Y = S(V) = S^n$ *be an* n*-dimensional* G*-sphere associated to a real* G*-representation* V $(n \geq 2)$. *Then* (B.1) *reduces to*

$$[X^{(n+1)}, S(V)]_G \to \{*\} \to \mathcal{H}_G^{n+1}(X, \mathcal{Z}) \tag{B.2}$$

where $\mathcal{Z}$ *is the orientation character of* $S(V)$ *and* $\{*\}$ *is a singleton.*

The exactness of the sequence (B.2) means that there exists a single element of the obstruction group $\mathcal{H}_G^{n+1}(X, \mathcal{Z})$ (the image of $*$) which is zero if and only if there exists a G-equivariant map $f\colon X^{(n+1)} \to S(V)$.

B.2. Admissible filtrations. Our objective is to extend the applicability of basic obstruction theory, as outlined in Section B.1, by introducing more general filtrations which do not necessarily arise from a G-CW-structure on X. The reader may keep in mind the G-manifold complexes introduced in Section 4.1 as a guiding example of such filtrations.

Definition B.3. Let X be a not necessarily free G-space which admits a G-invariant triangulation (CW-structure) turning X into a simplicial complex (CW-complex) of dimension $d \geq n+1$. Let $X^{(k)}$ be the associated k-skeleton. A finite filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n \subset X_{n+1} \subset \ldots \subset X_d = X \tag{B.3}$$

is called *admissible* if the following condition is satisfied,

(a) X_k is a G-invariant subcomplex of $X^{(k)}$ for each k.

Let $C_*(X)$ be the G-chain complex associated to the filtration (B.3) where $C_k(X) := H_k(X_k, X_{k-1}; \mathbb{Z})$. The associated cohomology groups with coefficients in a $\mathbb{Z}[G]$-module M are sometimes referred to as *special equivariant cohomology groups* and denoted by

$$\mathcal{H}^*_G(X; M). \tag{B.4}$$

An admissible filtration (B.3) is said to be *free in dimension k* if,

(b) $C_k = C_k(X) := H_k(X_k, X_{k-1}; \mathbb{Z})$ is a free Λ-module, where $\Lambda := \mathbb{Z}[G]$ is the group ring of the group G.

Remark B.4. The reader should keep in mind that the cohomology groups (B.4) are functors of a filtered space X, not the space alone. This is in agreement with the approach in [3, p. 112] where the corresponding equivariant cohomology groups depend on a given G-CW-structure.

Remark B.5. The structure of a simplicial complex on a G-space X plays an auxiliary role and in intended applications one starts directly with a filtration (B.3), tacitly assuming that it can be "triangulated". The most natural is the situation when X is a G-manifold complex where the constituent manifolds are semialgebraic sets (as in Section 6) and this condition is automatically satisfied. The condition (b) is evidently not necessarily satisfied if X is a free G-space with an admissible filtration.

We assume that the target G-space Y is also filtered by a filtration

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset Y_{n+1} \subset \ldots \subset Y_\nu = Y \tag{B.5}$$

arising from some, not necessarily free, G-CW-structure on Y. Let $D_* = (\{D_k\}_0^\nu, \partial)$ be the associated cellular chain complex, $D_k := H_k(Y_k, Y_{k-1})$.

The following proposition allows us to reduce the problem of the existence of G-equivariant maps between X and Y, to the question of the existence of chain maps between the associated chain complexes of $\Lambda = \mathbb{Z}[G]$ modules.

PROPOSITION B.6. *Suppose that X is a d-dimensional G-space with an* admissible filtration (B.3) (*Definition* B.3). *Suppose that Y is a G-CW-complex and let* (B.5) *be its associated filtration by skeletons. Then if there exists a G-equivariant map $f\colon X \to Y$, there exists also a chain map $f_*\colon C_*(X) \to D_*(Y)$ of the associated, augmented chain complexes*

$$\text{(B.6)}\qquad \begin{array}{ccccccccccccccc} \cdots \to C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \to \cdots \to & C_1 & \xrightarrow{\partial} & C_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \cong} & & \\ \cdots \to D_{n+1} & \xrightarrow[\partial]{} & D_n & \xrightarrow[\partial]{} & D_{n-1} & \to \cdots \to & D_1 & \xrightarrow[\partial]{} & D_0 & \xrightarrow[\partial]{} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

PROOF. If there exists a G-equivariant map $f\colon X \to Y$ then by the *cellular approximation theorem* (Theorem 2.1 in [3, Section II.2]) there exists a cellular map $g\colon X \to Y$ which is G-homotopic to f. It follows that $g(X_k) \subset g(X^{(k)}) \subset Y^{(k)} = Y_k$ which in turn implies the existence of the chain map (B.6). □

B.3. Algebraic description of the obstruction. Proposition B.6 allows us to reduce the topological problem of the existence of equivariant maps to an algebraic problem of finding a chain map. This in turn leads to algebraic counterparts of Theorem B.1 and Corollary B.2.

PROPOSITION B.7. *Suppose that $C_* := \{C_k\}_{k=-1}^d$ and $D_* := \{D_k\}_{k=-1}^d$ are finite chain complexes of $\Lambda = \mathbb{Z}[G]$ modules where $C_{-1} \cong D_{-1} \cong \mathbb{Z}$. Suppose that the chain map $F_{n-1} := (f_j)_{j=-1}^{n-1}\colon \{C_k\}_{k=-1}^{n-1} \to \{D_k\}_{k=-1}^{n-1}$ exists and is fixed in advance ($n+1 \leq d$). Suppose that F_{n-1} can be extended one step further, i.e. that there exists a homomorphism $f_n\colon C_n \to D_n$ such that $\partial \circ f_n = f_{n-1} \circ \partial$. Then the obstruction to the existence of a chain map* (B.6), *$F_{n+1} := (f_j)_{j=-1}^{n+1}\colon \{C_k\}_{k=-1}^{n+1} \to \{D_k\}_{k=-1}^{n+1}$, extending the chain map F_{n-1} (with the modification of f_n if necessary) is a well defined element θ of the cohomology group*

$$H^{n+1}(C_*; H_n(D_*)) = H_{n+1}(\mathrm{Hom}(C_*, H_n(D_*))). \tag{B.7}$$

Moreover, θ is represented by the cocycle

$$\theta(f_n)\colon C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f_n} Z_n(D_*) \xrightarrow{\pi} H_n(D_*). \tag{B.8}$$

The vanishing of θ is not only necessary but also sufficient for the existence of the chain map F_{n+1} (B.6) *if C_n and C_{n+1} are free (or projective) modules.*

PROOF. The homomorphism $f_n\partial\colon C_{n+1} \to D_n$ has the following properties:

(1) $\mathrm{Image}(f_n\partial) \subset Z_n(D_*) := \mathrm{Ker}(D_n \xrightarrow{\partial} D_{n-1})$.

(2) $\mathrm{Image}(f_n\partial) \subset B_n(D_*) := \mathrm{Image}(D_{n+1} \xrightarrow{\partial} D_n)$ if and only if there exists f_{n+1} such that $\partial \circ f_{n+1} = f_n \circ \partial$ (provided C_{n+1} is a projective module).

In other words, if $\theta(f_n) = \pi \circ f_n \circ \partial$ is the homomorphism

$$\theta(f_n)\colon C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f_n} Z_n(D_*) \xrightarrow{\pi} H_n(D_*)$$

then f_{n+1} exists if and only if $\theta(f_n) = 0$.

(3) $\theta(f_n) \in \mathrm{Hom}(C_*, H_n(D_*))$ is a cocycle.

If $\theta := [\theta(f_n)]$ is the associated cohomology class let us show that it has the properties claimed in the proposition.

Suppose that the chain map F_{n+1} exists where f_n is replaced by a homomorphism $f_n'\colon C_n \to D_n$ such that $f_{n+1} \circ \partial = \partial \circ f_n'$ and $f_n' \circ \partial = \partial \circ f_{n-1}$ (in particular $\theta(f_n') = 0$). Let $h_1 = f_n - f_n'\colon C_n \to D_n$. Since $\partial \circ f_n = \partial \circ f_n'$ we observe that $\mathrm{Image}(h_1) \subset Z_n(D_*)$. Let $h = \pi \circ h_1 \colon C_n \to H_n(D_*)$. It follows that

$$\theta(f_n) = \theta(f_n) - \theta(f_n') = \theta(h_1) = \pi \circ h_1 \circ \partial = h \circ \partial = \delta(h)$$

which implies that $[\theta(f_n)] = 0 \in H^{n+1}(C_*; H_n(D_*))$.

Conversely, if $\theta = 0$ then $\theta(f_n)$ is a coboundary, i.e. $\theta(f_n) = \delta(h) = h \circ \partial$ for some homomorphism $h\colon C_n \to H_n(D_*)$. Since the module C_n is projective, there exists a homomorphism $h_1\colon C_n \to Z_n(D_*)$ such that $h = \pi \circ h_1$, hence

$$\theta(f_n) = \pi \circ f_n \circ \partial = \pi \circ h_1 \circ \partial \colon C_{n+1} \to H_n(D_*). \tag{B.9}$$

Let $f_n' := f_n - h_1$. Since $\mathrm{Image}(h_1) \subset Z_n(D_*)$ we observe that $\partial \circ f_n' = \partial \circ f_n$ which implies that $\partial \circ f_n' = f_{n-1} \circ \partial$.

From the equality $\theta(f_n') = \pi \circ f_n' \circ \partial = 0$ we deduce that $\mathrm{Image}(f_n' \circ \partial) \subset B_n(D_*)$. Since C_{n+1} is a projective module, there exists a homomorphism $f_{n+1}\colon C_{n+1} \to D_{n+1}$ such that $\partial \circ f_{n+1} = f_n' \circ \partial$. This completes the construction of the chain map F_{n+1}. □

Remark B.8. If the chain complex $C_* = C_*(X)$ arises from a fixed admissible filtration (B.3) (Definition B.3) then the group $H^{n+1}(C_*; M)$ (where $M = H_n(D_*)$) is nothing but the (special) equivariant cohomology group $\mathcal{H}_G^{n+1}(X; M)$ which appears in line (B.4).

Example B.9. The first obstruction θ to the existence of a chain map between the following chain complexes lies in the group $H^2(C_*; H_1(D_*)) \cong \mathbb{Z}/4$. Assuming

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2] \arrow[r, "1-\omega"] \arrow[d, "f_3"] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "2(1+\omega)"] \arrow[d, "f_2"] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "1-\omega"] \arrow[d, "f_1"] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "1-\omega"'] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{B.10}$$

that both f_0 and f_1 are the identity maps a simple calculation shows that $\theta = 2$.

Remark B.10. The cyclic group $\mathbb{Z}/4 = \{1, \omega, \omega^2, \omega^3\}$ acts on the sphere $S^3 = S^1 * S^1 \subset \mathbb{C}^2$ by rotating each of the circles S^1 through the angle of $90°$ with the Lens space $L(4,1) = S^3/(\mathbb{Z}/4)$ as the quotient. The partial quotient S^3/H where $H = \{1, \omega^2\}$ is the projective space $\mathbb{R}P^3$ which inherits a $\mathbb{Z}/2$ action form the group $(\mathbb{Z}/4)/H$. It is not difficult to see that the first line of (B.10) is an associated chain complex obtained as a quotient from the standard $\mathbb{Z}/4$-invariant CW-structure on S^3. In light of Proposition B.6 the non-triviality of the obstruction θ, as calculated in Example B.9, guarantees that there does not exist a $\mathbb{Z}/2$-equivariant map $f : (\mathbb{R}P^3)^{(2)} \to S^1$, where S^1 has the antipodal action and $(\mathbb{R}P^3)^{(2)}$ is the 2-skeleton of $\mathbb{R}P^3$.

Example B.11. Suppose we want to show that there does not exist a $\mathbb{Z}/2$-equivariant map $f\colon S^{n+1} \to S^n$ (Borsuk–Ulam theorem) by the methods of this paper. We initially choose an admissible filtration $\{F_k\}_{k=0}^{n+1}$ of S^{n+1} (in the sense of Definition B.3) by defining $F_0 = \ldots = F_{n-1} = S^0$, $F_n = S^n$ and $F_{n+1} = S^{n+1}$, where $S^0 \subset S^n$ are $\mathbb{Z}/2$-invariant subspheres of S^{n+1}. Then $C_n \cong H_n(S^n, S^0; \mathbb{Z}) \cong \mathbb{Z}$ and easy calculation shows that the associated obstruction θ is 0, meaning that this filtration is not well adopted for this problem. If we modify the filtration by choosing F_{n-1} to be a $\mathbb{Z}/2$-invariant sphere $F_{n-1} = S^{n-1}$ (where $S^0 \subset S^{n-1} \subset S^n$) then a direct calculations shows that $\theta \neq 0$.

Remark B.12. The last part of Proposition B.7, claiming that θ is a complete obstruction provided C_n and C_{n+1} are free modules, is a motivation for isolating admissible filtrations free in selected dimensions, cf. condition (b) in Definition B.3. Note that the freeness of C_n and C_{n+1} is a condition that can be satisfied even if X is not a free G-space, e.g. if the corresponding set of fixed points X^H has a codimension ≥ 2 for each subgroup $H \neq \{e\}$.

B.4. Heuristics for evaluating the obstruction θ. In many cases the chain map $F_{n-1} = (f_j)_{j=-1}^{n-1}$, which in Proposition B.7 serves as an input for calculating the obstruction θ, is unique up to a chain homotopy. This happens for example when D_* is a chain complex associated to a G-sphere Y of dimension n.

In this case one can inductively build a 'ladder of maps' (B.6) in order to find a representative of the chain homotopy class of the chain map $[F_{n-1}]$ needed for the evaluation of θ. A very instructive explicit example of this calculation can be found in [15].

In practise one can bypass these calculations by constructing and using instead an arbitrary G-equivariant map $\phi_n\colon X_n \to Y$. In the case of G-manifold complexes (Section 4.1) the group $C_n = H_n(X_n, X_{n-1}; \mathbb{Z})$ is generated by the (relative) fundamental classes of manifolds with boundary and the calculation of the map $\pi \circ f_n : C_n \to H_n(D_*) \cong H_n(Y) \cong H_n(S^n)$ in (B.8) is reduced to the calculation of the corresponding mapping degrees.

Examples of calculations which essentially follow this procedure can be found in [28, Section 4].

B.5. Computational topology and effective obstruction theory. The problem of calculating topological obstructions to the existence of equivariant maps is identified in [2] as one of the questions of great relevance for computational topology. The focus is naturally on those features of the obstruction problem where topology and computational mathematics interact in an essential way. This brings forward algorithmic aspects of the question emphasizing explicit procedures suitable for semiautomatic and/or large scale calculations. Here we recapitulate and briefly summarize some of the general ideas used in the proof of Theorem 2.2 which may be of some independent interest in the development of the effective obstruction theory.

(1) One works with manifold G-complexes (Section 4.1) which are more general and often more economical than G-CW-complexes.
(2) Given a G-space (manifold) X, the associated G-manifold complex arises through the iteration of the 'fundamental domain - geometric boundary' procedure (see the diagram in Section 6).
(3) The fact that the generators of G-modules are fundamental classes (Section B.4) allows us to evaluate boundaries and chain maps as the mapping degrees (Sections 7–9).
(4) The emphasis in the basic set-up of the obstruction theory (Appendix, Section B) is on 'admissible filtrations' (Proposition B.6) and chain complexes, rather than spaces, (Proposition B.7).

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References

[1] P.V.M. Blagojević and G.M. Ziegler, *The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes*, Topology Appl. **158** (2011), no. 12, 1326–1351.
[2] P.V.M. Blagojević, S.T. Vrećica and R.T. Živaljević, *Computational topology of equivariant maps from spheres to complements of arrangements*, Trans. Amer. Math. Soc. **361** (2009), 1007–1038.
[3] T. tom Dieck, *Transformation Groups*, de Gruyter Stud. Math., Vol. 8, Springer, Berlin, 1987.

([2]) Combinatorics in Geometry, Topology, and Algebra.

[4] C. Domínguez, J. González and P. Landweber, *The integral cohomology of configuration spaces of pairs of points in real projective spaces*, Forum Math. Vol. 25, Issue 6 (November 2013), 1217–1248.

[5] I.M. Gelfand, A.V. Zelevinsky and M.M. Kapranov, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[6] J. González and P. Landweber, *Cohomology groups of configuration spaces of pairs of points in real projective spaces*, Morphismos, **14**, No. 2 (2010), 17–59.

[7] M. Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13** (2003), 178–215.

[8] B. Grünbaum, *Partitions of mass–distributions and convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261.

[9] L. Guth, *The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture*, Acta Math. **205** (2010), no. 2, 263–286.

[10] L. Guth and N.H. Katz, *On the Erdös distinct distance problem in the plane*, Ann. of Math. **181** (2015), 155–190.

[11] H. Hadwiger, *Simultane Vierteilung zweier Körper*, Arch. Math. (Basel) **17** (1966), 274–278.

[12] A. Hubard and B. Aronov, *Convex equipartitions of volume and surface area*, arXiv: 1010.4611v3 [math.MG].

[13] R.N. Karasev, *Equipartition of several measures*, arXiv:1011.4762v6 [math.MG].

[14] M. Kreck, *Differential Algebraic Topology*, Graduate Studies in Mathematics, vol. 110, American Mathematical Society 2010.

[15] I. Kriz, A correction to "*Equivariant Cohomology and Lower Bounds for Chromatic Numbers*", Trans. Amer. Math. Soc. **352** (2000), 1951–1952.

[16] A. Kushkuley and Z.I. Balanov *Geometric Methods in Degree Theory for Equivariant Maps*, Lecture Notes in Math. vol. 1632, Springer, Berlin 1996.

[17] P. Mani-Levitska, S. Vrećica and R. Živaljević, *Topology and combinatorics of partitions of masses by hyperplanes*, Adv. Math. **207** (2006), 266-296.

[18] J. Matoušek, *The Dawn of an Algebraic Era in Discrete Geometry*? EuroCG 2011, Morschach, Switzerland, March 28–30, 2011, `http://kam.mff.cuni.cz/∼matousek/mor.pdf`.

[19] B. Matschke, *A note on masspartitions by hyperplanes*, arXiv:1001.0193v1 [math.CO], 31 Dec 2009.

[20] E.A. Ramos, *Equipartitions of mass distributions by hyperplanes*, Discrete Comput. Geom. **15** (1996), 147–167.

[21] M. Sharir, *From joints to distinct distances and beyond*: *The dawn of an algebraic era in combinatorial geometry*, `http://www.cs.tau.ac.il/ michas/algober.pdf`.

[22] P. Soberón, *Balanced convex partitions of measures in* $\mathbb{R}^d$, Mathematika **58** (2012), no. 1, 71–76.

[23] J. Solymosi and T. Tao, *An incidence theorem in higher dimensions*, Discrete Comput. Geom. **48**, Issue 2, 255–280, arXiv:1103.2926v5 [math.CO].

[24] H. Steinhaus, *A note on the ham sandwich theorem*, Mathesis Polska **9** (1938), 26–28.

[25] A.H. Stone and J.W. Tukey, *Generalized sandwich theorems*, Duke Math. J. **9** (1942), 356–359.

[26] T. Tao, `http://terrytao.wordpress.com/?s=polynomial+ham+sandwich`

[27] S.T. Vrećica and R.T. Živaljević, *Chessboard complexes indomitable*, J. Combin. Theory Ser. A **118** (2011), no. 7, 2157–2166.

[28] R. Živaljević, *User's guide to equivariant methods in combinatorics* II, Publ. Inst. Math. Belgrade **64** (78) (1998), 107–132.

[29] ______, *The Tverberg–Vrećica problem and the combinatorial geometry on vector bundles*, Israel J. Math **111** (1999), 53–76.

[30] ______, *Topological methods*, Chapter 14 in Handbook of Discrete and Computational Geometry (J.E. Goodman, J. O'Rourke, eds.), Chapman & Hall/CRC 2004, 305–330.

[31] ______, *Equipartitions of measures in* $\mathbb{R}^4$, Trans. Amer. Math. Soc. **360** (2008), 153–169.

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