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LOCAL CONNECTIVITY OF LIMIT SETS

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ABSTRACT. This paper examines the dynamics of a continuous flow on a compact surface of genus greater than one with an orbit whose ω-limit set is locally connected. We show that if the orbit's lift to the Poincare disk limits to a rational point, then its ω-limit set contains a simple closed invariant curve that is not null homotopic. We also find sufficient conditions for the orbit's lift to stay a bounded distance from a geodesic with the same limiting point.

1. Introduction. When the ω-limit set is locally connected, the dynamics of a continuous flow on a compact surface M is linked to the existence of invariant simple closed curves on M. The focus of this paper will be when M is orientable and has genus $g > 1$. Theorems 5.1 and 5.2 state that such a curve exists if there is a positive orbit on M satisfying the following conditions: (a) its lift to the Poincare disk limits to a rational point and (b) either its ω-limit set is locally connected or the set of fixed points in its ω-limit set is totally disconnected. These theorems were proved by Markley for the torus in [**5**].

Note that if condition (a) fails, then $\omega(x)$ might be a Denjoy minimal set. If condition (b) fails, then $\omega(x)$ might look like the topologist's sine curve. In both cases the results no longer hold.

Markley also showed in [**5**] that if a positive orbit of a continuous flow on the torus has a lift $\mathcal{O}^+(\tilde{x})$ to the plane that goes to infinity, i.e., $|\tilde{x}t| \to \infty$ as $t \to \infty$, and its ω-limit set contains a moving point, then $\mathcal{O}^+(\tilde{x})$ will lie between two parallel lines. This result only holds for the torus. In [**6**] Markley and the author gave an example of a continuous flow on a compact surface of genus 2 with a positive orbit whose ω-limit set contains a nonperiodic orbit along with a simple closed curve of fixed points. The orbit does not wrap down on this simple closed curve in the usual way, and its lift to the Poincaré disk does not stay a bounded distance from a hyperbolic ray with the same limiting point on the unit circle. Theorem 6.5 shows that it was no accident that the

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ω-limit set in our example was not locally connected. It states that if a positive orbit on M has an unbounded lift and its ω-limit set is locally connected, contains a moving point and has empty interior, then its lift stays a bounded distance from some rational hyperbolic ray, and the ω-limit set of the positive orbit contains an invariant simple closed curve that is not null homotopic.

2. Main definitions. A *flow* or *continuous real action* on M is a continuous mapping $\phi : M \times \mathbf{R} \to M$ where $\mathbf{R}$ is the reals, such that $\phi(\phi(x,t),s) = \phi(x,t+s)$ and $\phi(x,0) = x$ for all $x \in M$ and $s,t \in \mathbf{R}$. For convenience we will often follow the convention of writing xt for $\phi(x,t)$. The set of *fixed points* of ϕ is $F = \{x \in M : xt = x \text{ for all } t \in \mathbf{R}\}$. If $x \notin F$, then we say x is a *moving point*. The *orbit* of x is defined by $\mathcal{O}(x) = \{xt : t \in \mathbf{R}\}$. The *positive orbit* of x is defined by $\mathcal{O}^+(x) = \{xt : t \geq 0\}$. The *$\omega$-limit set* of x is defined by $\omega(x) = \cap_{t\geq 0}\overline{\mathcal{O}^+(xt)}$, and the *$\alpha$-limit set* is defined similarly.

Let ϕ be a flow on M. A *local cross section* $\sum$ of ϕ at a point $x \in M$ is a closed subset $\sum$ of M containing x such that the map $(x,t) \to xt$ is a homeomorphism of $\sum \times [-\varepsilon, \varepsilon]$ onto the closure of an open neighborhood V of x for some $\varepsilon > 0$. If x is a moving point, then a local cross section exists at x [**8**]. When M is a compact connected surface, $\sum$ is a closed arc [**10**].

Throughout this paper M will be a compact surface of genus $g > 1$. Thus the universal cover $\tilde{M}$ of M is the Poincare disk: the open unit disk with the hyperbolic metric d_h. The flow on M lifts to a unique flow $\tilde{\phi}$ on $\tilde{M}$ such that the covering projection $\pi : \tilde{M} \to M$ is a homomorphism of flows, i.e., $\pi(\tilde{\phi}(\tilde{x},t)) = \phi(\pi(\tilde{x}),t)$, and every covering transformation T of $\tilde{M}$ is an automorphism of the flow $\tilde{\phi}$. Moreover, $\pi(\tilde{x}) \in F$ if and only if $\tilde{x} \in \tilde{F}$, where $\tilde{F}$ denotes the fixed points of $\tilde{\phi}$. These results are consequence of the homotopy lifting theorem and can be found in [**4**].

Furthermore, the group of covering transformations is a discrete group of hyperbolic linear fractional transformations Γ and M is homeomorphic to the quotient space $\tilde{M}/\Gamma$. Each $T \in \Gamma$ has exactly two fixed points. These lie on the unit circle denoted by S_∞; one fixed point is attracting and the other is repelling. Following Aranson in [**2**], the set of all fixed points of Γ will be called the set of *rational points*.

Let $\tilde{x} \in \tilde{M}$ and let $\mathcal{U}$ be the closed unit disk with Euclidean metric d, so $\mathcal{U} = \tilde{M} \cup S_\infty$. The following definitions can be found in [**1**] and [**2**]. We say that $\mathcal{O}^+(\tilde{x})$ is *unbounded* if $\overline{\lim}_{t\to\infty} d_h(\tilde{x}, \tilde{x}t) = \infty$. If $\lim_{t\to\infty} d_h(\tilde{x}, \tilde{x}t) = \infty$, then its limit set does not belong to $\tilde{M}$. To study the asymptotic behavior of $\mathcal{O}^+(\tilde{x})$ we can extend the lifted flow to $\mathcal{U}$ by taking S_∞ to be fixed points of $(\tilde{M}, \tilde{\phi})$. We say that $\mathcal{O}^+(\tilde{x})$ is the *type of a rational h-ray* if it satisfies the following two conditions. It limits to a rational point σ of S_∞, i.e., $d(\tilde{x}t, \sigma) \to 0$ as $t \to \infty$ and, for any hyperbolic ray R that also limits to σ, $K > 0$ exists such that $d_h(\tilde{x}t, R) < K$ for all $t \geq 0$. (We assume the reader has a rudimentary knowledge of the hyperbolic geometry of $\tilde{M}$ from a book like Katok's [**3**].)

We will use the following notation found in [**5**] for segments of curves and orbits. If C is a simple curve, hence homeomorphic to an interval, and a and b lie on C, then $(a, b)_C$ will denote the open segment of C between a and b. If $s, \tau \in \mathbf{R}$, then $[xs, x\tau]_\phi$ will denote $\{xt : s \leq t \leq \tau\}$ or $\{xt : \tau \leq t \leq s\}$, according to whether $s < \tau$ or $\tau < s$. Then $[a, b]_C$ and $(xs, x\tau)_\phi$ have the obvious meanings.

3. Curves in $\omega(x)$.

Let ϕ be a continuous flow on M and let $x \in M$. We next prove some results about curves in $\omega(x)$ when it has empty interior.

Lemma 3.1. *Let γ be a curve in $\omega(x)$, say $\gamma : [0,1] \to \omega(x)$ and γ is one-to-one. If* $\mathrm{Int}\,(\omega(x)) = \varnothing$*, then given $0 < \tau < 1$, there exists $\alpha > 0$ such that $\phi(\gamma(\tau), t) \in \gamma$ for $|t| < \alpha$.*

Proof. We may as well assume that $\gamma(\tau) \notin F$. Let $\sum$ be a local cross section of length ε at $\gamma(\tau)$. There exists $\delta > 0$ such that $(\gamma(\tau - \delta), \gamma(\tau + \delta))_\gamma$ is contained in the interior of $\sum \times [-\varepsilon, \varepsilon]$. The projection of $(\gamma(\tau - \delta), \gamma(\tau + \delta))_\gamma$ onto $\sum$ is an interval, including a single point, which contains $\gamma(\tau)$. Since $\mathrm{Int}\,(\omega(x)) = \varnothing$, this interval must equal $\gamma(\tau)$. Thus $(\gamma(\tau - \delta), \gamma(\tau + \delta))_\gamma \subset \mathcal{O}(\gamma(\tau))$. Since γ is one-to-one it follows that $\alpha > 0$ exists such that $\phi(\gamma(\tau), t) \in \gamma$ for $|t| < \alpha$. $\square$

Proposition 3.2. *Let γ be a curve in $\omega(x)$, say $\gamma : [0,1] \to \omega(x)$ and γ is one-to-one. If* $\operatorname{Int}(\omega(x)) = \varnothing$, *then given* $0 < \tau < 1$, *either*

1. *There exists $t_0 \in [0,1]$ such that $\phi(\gamma(\tau), t) \to \gamma(t_0)$ as $t \to \infty$, and $\gamma(t_0) \in F$, or*

2. *$[\gamma(\tau), \gamma(1)]_\gamma \subset \mathcal{O}^+(\gamma(\tau))$ or $[\gamma(0), \gamma(\tau)]_\gamma \subset \mathcal{O}^+(\gamma(\tau))$.*

Proof. We may assume that $\gamma(\tau) \notin F$. Let $V = \{t > 0 : (\gamma(\tau), \phi(\gamma(\tau), t))_\phi \subset \gamma\}$. By Lemma 3.1 V is open and nonempty. Note that if $s \in V$, then for large n, we have $\phi(\gamma(\tau), s - (1/n)) \in \gamma$. Since γ is compact, it follows that $\phi(\gamma(\tau), s) \in \gamma$.

If $\sup(V) = \infty$, then $\mathcal{O}^+(\gamma(\tau)) \subset \gamma$. Hence $\lim_{t\to\infty} \phi(\gamma(\tau), t) = \gamma(t_0)$ for some $t_0 \in [0,1]$ and $\gamma(t_0) \in F$.

If $\rho = \sup(V) < \infty$, then $\phi(\gamma(\tau), \rho) = \gamma(\hat{t})$ for some $\hat{t}$ where either $0 \le \hat{t} < \tau$ or $\tau < \hat{t} \le 1$. It suffices to show that $\hat{t} = 0$ or $\hat{t} = 1$. We will proceed by contradiction, that is, we will assume that either $0 < \hat{t} < \tau$ or $\tau < \hat{t} < 1$. The proofs of both cases are analogous and we will give the proof for the latter case. Since $\gamma(\hat{t}) \notin F$, we can apply Lemma 3.1 to find $\alpha > 0$ such that $\phi(\gamma(\hat{t}), s) \in \gamma$ for $\hat{t} \le s \le \hat{t} + \alpha$. But for such s we have $\phi(\gamma(\hat{t}), s) = \phi(\phi(\gamma(\tau), \rho), s) = \phi(\gamma(\tau), \rho + s)$, which contradicts ρ being the supremum of V. Thus $\hat{t} = 1$ and hence $[\gamma(\tau), \gamma(1)]_\gamma \subset \mathcal{O}^+(\gamma(\tau))$. $\square$

An immediate consequence of Proposition 3.2 is the following.

Corollary 3.3. *Let γ be a curve in $\omega(x)$, say $\gamma : [0,1] \to \omega(x)$ and γ is one-to-one. If* $\operatorname{Int}(\omega(x)) = \varnothing$, *$\gamma(0) \in F$ and $\gamma(1) \in F$, then given $0 < \tau < 1$, $\mathcal{O}(\gamma(\tau)) \subset \gamma$ and γ is invariant.*

4. Locally connected ω-limit sets. Let $\tilde{\omega}(x)$ denote $\pi^{-1}(\omega(x))$, and let C_α, $\alpha \in \Lambda$, denote the components of $\tilde{\omega}(x)$. The following proposition will be used to prove Theorems 5.1 and 5.2.

Proposition 4.1. *Let ϕ be a continuous flow on M; $x \in M$; $\tilde{x} \in \{\pi^{-1}(x)\}$; and let C_α be a component of $\tilde{\omega}(x)$. If $\omega(x)$ is locally connected,* $\operatorname{Int}(\omega(x)) = \varnothing$ *and $\mathcal{O}^+(\tilde{x})$ is unbounded, then $\{\pi^{-1}(y)\} \cap C_\alpha$*

contains more than one point for all $y \in \omega(x)$ *and* $\{T \in \Gamma : TC_\alpha = C_\alpha\} \neq \{I\}$.

The proof of the proposition will be given at the end of this section, following the development of some preliminary material and lemmas. Throughout this section we will assume that $\omega(x)$ is locally connected and has empty interior. Recall that a complete metric space which is locally connected and connected is path connected [**7**]. Since the ω-limit sets of a continuous flow on a compact Hausdorff space are compact and connected, it follows that $\omega(x)$ is path connected if it is locally connected. Also note that, if $\omega(x)$ is locally connected, then $\tilde{\omega}(x)$ is locally connected.

Lemma 4.2. *There exists* $\delta > 0$ *such that* $d_h(C_\alpha, C_\beta) > \delta$ *for all* $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$.

Proof. Suppose not. Then for every $n > 0$, distinct components C_n and $C_{n'}$ of $\tilde{\omega}(x)$ exist such that $d_h(C_n, C_{n'}) < 1/n$. Hence sequences $\{y_n\}$ and $\{y_{n'}\}$, $y_n \in C_n$, $y_{n'} \in C_{n'}$ exist such that $d_h(y_n, y_{n'}) < 1/n$.

Let $\mathcal{D}$ be a fundamental domain in $\tilde{M}$. For every $n, T_n \in \Gamma$ exists such that $T_n(y_n) \in \mathcal{D}$. Taking a subsequence if necessary, we can assume that $T_n(y_n) \to y \in \mathcal{D}$. Note that $T_n(y_{n'}) \to y$, $\pi(y_{n'}) \to \pi(y)$ and $\pi(y) \in \omega(x)$. Also note that since $\tilde{\omega}(x)$ is invariant under Γ, $T_n(C_n)$ and $T_n(C_{n'})$ are components of $\tilde{\omega}(x)$.

Choose $\varepsilon > 0$ such that $\pi|_{B_\varepsilon(y)}$ is a homeomorphism onto a neighborhood of $\pi(y)$ where $B_\varepsilon(y)$ is the open ε-ball about y in the Hausdorff metric. Let $U = B_\varepsilon(y) \cap \tilde{\omega}(x)$. ($U$ is open in the subspace topology.) Since $\tilde{\omega}(x)$ is locally connected, a connected neighborhood V of y exists which is contained in U. Observe that $V = V' \cap \tilde{\omega}(x)$ for some open set $V' \subset B_\varepsilon(y)$. There exists N such that $T_N(C_N) \cap V' \neq \varnothing$ and $T_N(C_{N'}) \cap V' \neq \varnothing$. Since $T_N(C_N) \subset \tilde{\omega}(x)$ and $T_N(C_{N'}) \subset \tilde{\omega}(x)$, it follows that $T_N(C_N) \cap V \neq \varnothing$ and that $T_N(C_{N'}) \cap V \neq \varnothing$. Hence $V \subset T_N(C_N)$ and $V \subset T_N(C_{N'})$, implying $C_N = C_{N'}$ which contradicts our assumption. Thus $\delta > 0$ exists such that $d_h(C_\alpha, C_\beta) > \delta$ for all $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$. $\square$

The next result shows that a lifted orbit will eventually stay close to one component.

Corollary 4.3. *Suppose $\delta > 0$ is given by Lemma* 4.2. *Given ε, $0 < \varepsilon < \delta/2$, $\tau \geq 0$ exists such that for every $\tilde{x} \in \{\pi^{-1}(x)\}$ there is a unique component C_α of $\tilde{\omega}(x)$ satisfying $d_h(\tilde{x}t, C_\alpha) < \varepsilon$ for all $t \geq \tau$.*

Proof. There exists $\tau \geq 0$ such that $d_h(\tilde{x}t, \tilde{\omega}(x)) < \varepsilon$ for all $t \geq \tau$ and $\tilde{x} \in \{\pi^{-1}(x)\}$. There exists a component C_α of $\tilde{\omega}(x)$ such that $d_h(\tilde{x}\tau, C_\alpha) < \varepsilon$. This component is unique by Lemma 4.2.

We will proceed by contradiction. Suppose that $t_0 > \tau$ exists such that $d_h(\tilde{x}t_0, C_\alpha) \geq \varepsilon$. By continuity a time $s, \tau < s \leq t_0$ exists such that $d_h(\tilde{x}s, C_\alpha) = \varepsilon$. By Lemma 4.2, a unique component C_β of $\tilde{\omega}(x)$, $C_\beta \neq C_\alpha$, exists such that $d_h(\tilde{x}s, C_\beta) < \varepsilon$. Thus $d_h(C_\alpha, C_\beta) \leq d_h(\tilde{x}s, C_\beta) + d_h(\tilde{x}s, C_\alpha) < \varepsilon + \varepsilon < \delta$, contradicting Lemma 4.2. □

Lemma 4.4. *The path components and the components of $\tilde{\omega}(x)$ coincide.*

Proof. Since each component of $\tilde{\omega}(x)$ is closed and, by Lemma 4.2, the components are locally finite, the union of any collection of the components is closed. Thus each component C_α is open in $\tilde{\omega}(x)$ since its complement, the union of all the components except C_α is closed. It follows by Lemma 4.2 that each component of $\tilde{\omega}(x)$ is locally connected.

Since each component is connected, locally connected, and a complete metric space, each component is path connected. Because each component is a maximal connected set, the components and path components must coincide. □

Proof of Proposition 4.1. Let C_α be a component of $\tilde{\omega}(x)$. Note that $\pi(C_\alpha) = \omega(x)$. Since $\omega(x)$ is compact, connected, locally connected, and metrizable, by the Hahn-Mazurkiewicz theorem, a continuous and onto function $\beta : [0,1] \to \omega(x)$ exists. Let $\tilde{\beta} : [0,1] \to C_\alpha$ be a lift of β. Next, arguing by contradiction, we will show that π is not one-to-one on C_α.

Suppose π is one-to-one on C_α. Observe that $\tilde{\beta}$ must map $[0,1]$ onto C_α and hence C_α is compact. Let $\varepsilon > 0$. By Corollary 4.3

$\tau \geq 0$ and $T \in \Gamma$ exist such that $d_h(\tilde{x}t, TC_\alpha) < \varepsilon$ for all $t \geq \tau$. Since $\mathcal{O}^+(\tilde{x})$ is unbounded and TC_α is compact, $t_0 > \tau$ exist such that $d_h(\tilde{x}t_0, TC_\alpha) > \varepsilon$, which contradicts Corollary 4.3. Thus π is not one-to-one on C_α. Hence C_α must contain two equivalent points, i.e., $T \in \Gamma \setminus I$ exists such that $T(C_\alpha) \cap C_\alpha \neq \varnothing$. If two path components intersect they must be equal. Thus $T(C_\alpha) = C_\alpha$ and hence $\{\pi^{-1}(y)\} \cap C_\alpha$ contains more than one point for all $y \in \omega(x)$. □

5. Invariant simple closed curves. We now prove the two main results about the existence of invariant simple closed curves that are not null homotopic. Recall that M is a compact orientable surface of genus $g > 1$.

Theorem 5.1. *Let ϕ be a continuous flow on M, $x \in M$ and $\tilde{x} \in \{\pi^{-1}(x)\}$. If $\mathcal{O}^+(\tilde{x})$ limits to a rational point of S_∞ and $\omega(x)$ is locally connected, then $\omega(x)$ contains an invariant simple closed curve that is not null homotopic.*

Proof. Since M is a compact Hausdorff surface, $\omega(x)$ is connected and compact and hence path connected because it is locally connected by hypothesis. Let C_α be a component of $\tilde{\omega}(x)$. By Lemma 4.4, C_α is a path component of $\tilde{\omega}(x)$.

Since $\omega(x)$ is compact and invariant, it must contain a minimal set. By Theorem 1 of [**2**], $\omega(x)$ cannot contain a strictly recurrent point since σ^+, the limiting point of $\mathcal{O}^+(\tilde{x})$, is rational. Moreover, the unboundedness of $\mathcal{O}^+(\tilde{x})$ and the Poincare Bendixson theorem imply that $\omega(x)$ cannot contain a null homotopic periodic orbit. Hence $\omega(x)$ must contain a fixed point or a periodic orbit that is not null homotopic. Thus it suffices to restrict our attention to the case where $\omega(x) \cap F \neq \varnothing$.

Furthermore, Int$\,(\omega(x))$ must be empty because x cannot be strictly recurrent. Thus the results of the previous section can be used in this proof.

Let $a \in \omega(x) \cap F$. By Proposition 4.1, $C_\alpha \cap \{\pi^{-1}(y)\}$ contains more than one point for all $y \in \omega(x)$. Let $\tilde{a}_1$ and $\tilde{a}_2$ be lifts of a lying in C_α, and let β be a simple path in C_α between $\tilde{a}_1$ and $\tilde{a}_2$.

We can extract a curve from β whose projection onto M is simple as follows. Let $A = \{\sigma \in [0,1]$: there exist $\sigma' \in (\sigma, 1]$ and $T \in \Gamma$, $T \neq I$, such that $T(\beta(\sigma)) = \beta(\sigma')\}$. The set A is a nonempty closed curve subset of $[0,1]$ such that $T(\beta(\sigma)) = \beta(\sigma')\}$. The set A is a nonempty closed subset of $[0,1]$ that does not contain 1. Let $\sigma_0 = \sup(A)$ and note $\sigma_0 \in A$. Let $\gamma = \beta|_{[\sigma_0, \sigma_0']}$. Then $\pi \circ \gamma$ is a simple curve on M that is not null homotopic.

We now argue by contradiction to show that $\beta(\sigma_0) \in \tilde{F}$. Suppose $\beta(\sigma_0) \notin \tilde{F}$. Applying Lemma 3.1 to $\beta(\sigma_0)$ and $\beta(\sigma_0')$ we find $\alpha_0 > 0$ and $\alpha_0' > 0$ such that $\phi(\beta(\sigma_0), t) \in \beta$ for $0 < t < \alpha_0$ and that $\phi(\beta(\sigma_0'), t) \in \beta$ for $0 < t < \alpha_0'$. Let $\alpha = \min(\alpha_0, \alpha_0')$. There exist σ_1 and σ_1', $\sigma_0 < \sigma_1 < 1$ and $\sigma_0' < \sigma_1' < 1$ such that $\phi(\beta(\sigma_0), (\alpha/2)) = \beta(\sigma_1)$ and that $\sigma(\beta(\sigma_0'), (\alpha/2)) = \beta(\sigma_1')$. There exists $T \in \Gamma$ such that $T\beta(\sigma_0) = \beta(\sigma_0')$. We have $T\beta(\sigma_1) = T\phi(\beta(\sigma_0), (\alpha/2)) = \phi(T\beta(\sigma_0), (\alpha/2)) = \phi(\beta(\sigma_0'), (\alpha/2)) = \beta(\sigma_1')$. Hence $\sigma_1 \in A$ which contradicts σ_0 being the supremum of A. Thus $\beta(\sigma_0) \in \tilde{F}$.

Since $\beta(\sigma_0') = T\beta(\sigma_0)$ we also have that $\beta(\sigma_0') \in \tilde{F}$. By Corollary 3.3, γ is invariant and therefore $\pi \circ \gamma$ is also invariant. □

Theorem 5.2. *Let ϕ be a continuous flow on M, $x \in M$ and $\tilde{x} \in \{\pi^{-1}(x)\}$. If $\mathcal{O}^+(\tilde{x})$ is the type of a rational h-ray and $\omega(x) \cap F$ is totally disconnected, then $\omega(x)$ is locally connected and contains an invariant simple closed curve that is not null-homotopic.*

Proof. By Theorem 5.1 it suffices to show just that $\omega(x)$ is locally connected. We will make use of the following theorem of Markley found in [**5**].

Theorem. *Let π be a continuous flow on an open subset W of $\mathbf{R}^2$ and let $w \in W$. If $\mathcal{O}^+(w)$ is bounded and $\omega(w) \cap F$ is totally disconnected, then $\omega(w)$ is locally connected.*

Let σ^+ be the limiting point of $\mathcal{O}^+(\tilde{x})$. Let $T \in \Gamma$ be the transformation which fixes σ^+ and let A denote the axis of T. Since $\mathcal{O}^+(\tilde{x})$ is the type of an h-ray, equidistant curves E_1 and E_2 from A exist such that $\mathcal{O}^+(\tilde{x})$ lies in the region between E_1 and E_2.

Let $[T]$ be the cyclic group $\{T^n : n \in \mathbf{Z}\}$. Covering maps $\pi_1 : \tilde{M}/[T] \to M$ and $\pi_2 : \tilde{M} \to \tilde{M}/[T]$ exist such that $\pi = \pi_1 \circ \pi_2$. Note that $\tilde{M}/[T]$ is a cylinder and that $\pi_2(E_1)$ and $\pi_2(E_2)$ are simple closed homotopic curves on the cylinder that are not null homotopic. Since $\omega(x) \cap F$ is totally disconnected, $\pi_1^{-1}(\omega(x) \cap F)$ is also totally disconnected. Let Z denote the open region of the cylinder $\tilde{M}/[T]$ between the curves $\pi_2(E_1)$ and $\pi_2(E_2)$. Note that Z is homeomorphic to an annulus in $\mathbf{R}^2$. Let $y = \pi_2(\tilde{x})$. By the aforementioned theorem of Markley, $\omega(y)$ is locally connected. Thus $\pi_1(\omega(y)) = \omega(x)$ is locally connected. $\square$

6. Asymptotic behavior and $\omega(x)$. In this section we will assume that $\omega(x)$ is locally connected and has empty interior in M. In addition we will assume that $\omega(x) \not\subset F$ and $\mathcal{O}^+(\tilde{x})$ is unbounded. Let $\sum$ be a section at a moving point $p \in \omega(x)$, and let $\sum^a$ and $\sum^b$ denote the two sides of p on $\sum$. Let $\tilde{x} \in \{\pi^{-1}(x)\}$.

Since $\omega(x)$ is locally connected, $\omega(x) \cap \sum$ must be finite because the interior of $\omega(x)$ is empty. It follows that if x is recurrent it must be periodic.

The following dichotomy occurs: either $\tilde{p} \in \omega(\tilde{x})$ with $\pi(\tilde{p}) = p$ exists or does not. We first suppose that $\tilde{p} \in \omega(\tilde{x})$ exists such that $\pi(\tilde{p}) = p$. (In Lemma 6.3 we will show that this case cannot occur.) Let $\tilde{\Sigma}$ denote the lift of $\sum$ containing $\tilde{p}$. By Poincare-Bendixson theory, relabeling if necessary, we can assume without loss of generality that $\mathcal{O}^+(\tilde{x}) \cap \tilde{\Sigma}^b = \varnothing$, and that $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ exists such that $\tilde{x}t_n \in \tilde{\Sigma}^a$, $\{\tilde{x}t_n\}$ converges monotonely to $\tilde{p}$ and, if $\tilde{x}t \in \tilde{\Sigma}^a$, then $t = t_n$ for some n. Without loss of generality $t_0 = 0$.

Lemma 6.1. *If there exists $\tilde{p} \in \omega(\tilde{x})$ such that $\pi(\tilde{p}) = p$, then $\mathcal{O}^+(\tilde{x})$ crosses only one lift of $\sum^a$.*

Proof. We will argue by contradiction. Suppose $\tau > 0$ and $S \in \Gamma$ exist with $S \neq I$, such that $\tilde{x}\tau \in S\tilde{\Sigma}^a$. There exists $t_j \in \{t_n\}$ with $t_j < \tau < t_{j+1}$. Without loss of generality we may assume that $(\tilde{x}t_j, \tilde{x}\tau)_{\tilde{\phi}} \cap T\tilde{\Sigma}^a = \varnothing$ for all $T \in \Gamma$.

Let J be the simple curve defined by $J = \cup_{n\in\mathbf{Z}} S^n([\tilde{x}t_j, \tilde{x}\tau]_{\tilde{\phi}} \cup$

$(\tilde{x}t_j, S^{-1}\tilde{x}\tau)_{\tilde{\Sigma}^a})$. The curve J divides $\tilde{M}$ into two invariant regions: J^+ which is positively invariant and J^- which is negatively invariant. Observe that $\mathcal{O}^+(\tilde{x}\tau) \subset J^+$ and $\mathcal{O}^-(\tilde{x}t_j) \subset J^-$.

Since $\tilde{p} \in \omega(\tilde{x})$ it follows that $\tilde{p} \in J^+$. Let G be the Jordan curve defined by $G = [\tilde{x}t_j, \tilde{x}t_{j+1}]_{\tilde{\phi}} \cup (\tilde{x}t_j, \tilde{x}t_{j+1})_{\tilde{\Sigma}^\alpha}$. Note that $G \cap J^- = \varnothing$ since $\mathcal{O}^+(\tilde{x}\tau) \subset J^+$. Hence $\mathrm{Int}\,(G) \subset J^+$. Since $\mathcal{O}^+(\tilde{x})$ is unbounded, $\mathrm{Ext}\,(G)$ is positively invariant and $\mathrm{Int}\,(G)$ is negatively invariant. Thus $\mathcal{O}^-(\tilde{x}t_j) \subset \mathrm{Int}\,(G)$ and hence $\tilde{x}(t_j - \varepsilon) \in \mathrm{Int}\,(G) \subset J^+$ for $\varepsilon > 0$ which contradicts $\mathcal{O}^-(\tilde{x}t_j) \subset J^-$. Thus $\mathcal{O}^+(\tilde{x})$ crosses only one lift of $\sum^a$. $\square$

Lemma 6.2. *If $\mathcal{O}^+(\tilde{x})$ crosses only one lift of $\sum^a$, then any path between two different points on two different lifts of $\mathcal{O}^+(x)$ must contain a point of $\tilde{\omega}(x)$.*

Proof. It suffices to show that this holds for a path between two distinct lifts of x because $\tilde{\omega}(x)$ is invariant. Let $\tilde{x}_1$ and $\tilde{x}_2$ be distinct lifts of x, say $T\tilde{x}_1 = \tilde{x}_2$. Let β be a path between $\tilde{x}_1$ and $\tilde{x}_2$. By hypothesis a lifted section $\tilde{\Sigma}$ of $\sum^a$ exists such that $\tilde{x}_1 \in \tilde{\Sigma}$ and $\mathcal{O}^+(\tilde{x}_1) \cap H\tilde{\Sigma} = \varnothing$ for all $H \in \Gamma \setminus I$. Let $\tilde{p}$ be the lift of p contained on $\tilde{\Sigma}$. Let G_n be the Jordan curve defined by $G_n = [\tilde{x}_1 t_n, \tilde{x}_1 t_{n+1}]_{\tilde{\phi}} \cup (\tilde{x}_1 t_n, \tilde{x}_1 t_{n+1})_{\tilde{\Sigma}}$. Since $\mathcal{O}^+(\tilde{x}_1)$ is unbounded it follows by Poincare-Bendixson theory that $\tilde{p} \in \mathrm{Ext}\,(G_n)$ for all $n \geq 0$. Moreover, $\tilde{x}_1 \in \mathrm{Int}\,(G_n)$ for all $n \geq 1$.

Suppose $G_N \cap \beta = \varnothing$ for some $N > 1$. Since β is connected, either $\beta \subset \mathrm{Int}\,(G_N)$ or $\beta \subset \mathrm{Ext}\,(G_N)$. If $\beta \subset \mathrm{Int}\,(G_N)$, then $\tilde{x}_2 \in \mathrm{Int}\,(G_N)$. Since $\mathcal{O}^+(\tilde{x}_2)$ is unbounded, $\tilde{x}_2\tau \in \tilde{\Sigma}$ for some $\tau > 0$. Hence $T^{-1}\tilde{x}_2\tau = \tilde{x}_1\tau \in T^{-1}\tilde{\Sigma}$, which is a contradiction. If $\beta \subset \mathrm{Ext}\,(G_N)$, then $\tilde{x}_1 \in \mathrm{Ext}\,(G_N)$ which is also a contradiction. Therefore $G_n \cap \beta \neq \varnothing$ for all $n \geq 1$. Now one can show that a sequence $\{\tau_n\}$, $t_n < \tau_n < t_{n+1}$, exists such that $\tilde{x}_1\tau_n \in \beta$. Since β is compact $\{\tilde{x}_1\tau_n\}$ has a convergent subsequence which limits to a point $z \in \mathcal{B}$. Clearly $z \in \tilde{\omega}(x)$. $\square$

Lemma 6.3. *There does not exist a point $\tilde{p} \in \omega(\tilde{x})$ with $\pi(\tilde{p}) = p$, that is, $\tilde{p} \notin \omega(\tilde{x})$ for all $\tilde{p} \in \{\pi^{-1}(p)\}$.*

Proof. We will argue by contradiction. Suppose $\tilde{p} \in \omega(\tilde{x})$ exists with $\pi(\tilde{p}) = p$. By Lemma 6.1, $\mathcal{O}^+(\tilde{x})$ crosses only one lift of $\sum^a$. We will show that if this occurs then $\omega(x)$ is not locally connected at some point.

Since $\mathcal{O}^+(\tilde{x})$ is unbounded, it follows that $\{\tau_n\}$, $\tau_n \to \infty$ as $n \to \infty$, exists such that $d_h(\tilde{x}\tau_n, \tilde{p}) \to \infty$. Let $K > 1$ so that, taking a subsequence if necessary, $d_h(\tilde{x}\tau_j, \tilde{x}\tau_m) > K$ for all $j \neq m$. Consider $\{\pi(\tilde{x}\tau_n)\}$. Taking a subsequence if necessary, this sequence converges to some point $q \in \omega(x)$. Note that $q \neq p$. (If $q = p$, then $\tilde{x}\tau_n \in T\tilde{\Sigma}^a(-\varepsilon, \varepsilon)$ for large n and some $T \in \Gamma$, which is impossible by Lemma 6.1.) Let U be a neighborhood of some lift $\tilde{q}$ of q. Choose U so that $\operatorname{diam}(U) < (K/2)$, $\pi|_U$ is a homeomorphism, $\{\pi^{-1} \sum^a\} \cap U = \varnothing$ and $\{\pi^{-1}(x)\} \cap U = \varnothing$.

Because the covering transformations are isometries, an infinite number of distinct lifts of $\mathcal{O}^+(x)$ intersect U. (If there exists $H \in \Gamma$ such that $H\tilde{x}\tau_j$ and $H\tilde{x}\tau_k$ are both in U for some $j \neq k$, then $d_h(H\tilde{x}\tau_j, H\tilde{x}\tau_k) = d_h(\tilde{x}\tau_j, \tilde{x}\tau_k) > K$.) Hence there exists $\{\tilde{x}_n\}$ with $\pi(\tilde{x}_n) = x$ and $\tilde{x}_n \neq \tilde{x}_m$ for $n \neq m$ such that $\tilde{x}_n\tau_n \to \tilde{q}$.

Let $S_\varepsilon(\tilde{q})$ denote the circle of radius ε at $\tilde{q}$. If $a, b \in S_\varepsilon(\tilde{q})$, then $(a, b)_{S_\varepsilon(\tilde{q})}$ will denote the clockwise arc of the circle between a and b.

There exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(\tilde{q}) \subset U$. Let $\mathcal{A} = \{y | (\varepsilon/2) < d_h(y, \tilde{q}) < \varepsilon\}$. We will show that $z \in \mathcal{A} \cap \tilde{\omega}(x)$ exists where $\tilde{\omega}(x)$ is not locally connected. Without loss of generality $\tilde{x}_n\tau_n \in \mathcal{B}_{\varepsilon/4}(\tilde{q})$. Hence there exist sequences $\{s_n\}$ and $\{\lambda_n\}$ such that $\lambda_n < s_n < \tau_n$; $d_h(\tilde{x}_n\lambda_n, \tilde{q}) = \varepsilon$; $d_h(\tilde{x}_n s_n, \tilde{q}) = \varepsilon/2$; and $K_n := \{\tilde{x}_n t : \lambda_n < t < s_n\} \subset \mathcal{A}$ for all $n \geq N$.

Because $\pi|_U$ is a homeomorphism, we must have that $\lambda_n \to \infty$ as $n \to \infty$. By continuity, passing to a subsequence if necessary, a sequence $\{t'_n\}$, $\lambda_n < t'_n < s_n$, exists such that $d_h(\tilde{x}_n t'_n, \tilde{q}) = 3\varepsilon/4$. By taking a subsequence if necessary, $\tilde{x}_n t'_n \to z$ where $\pi(z) \in \omega(x)$.

We may assume without loss of generality that $\{\tilde{x}_n\lambda_n\}$ is a clockwise monotone sequence on $S_\varepsilon(\tilde{q})$, i.e., $(\tilde{x}_{n-1}\lambda_{n-1}, \tilde{x}_n\lambda_n, \tilde{x}_{n+1}\lambda_{n+1})$ is a clockwise triple on $S_\varepsilon(\tilde{q})$ for all n. This implies that $\{\tilde{x}_n s_n\}$ is also a clockwise monotone sequence on $S_{\varepsilon/2}(\tilde{q})$.

Let V_n denote the interior of the region bounded by K_n, K_{n+1}, $(\tilde{x}_n s_n, \tilde{x}_{n+1}s_{n+1})_{S_{\varepsilon/2}(\tilde{q})}$ and $(\tilde{x}_n\lambda_n, \tilde{x}_{n+1}\lambda_{n+1})_{S_\varepsilon(\tilde{q})}$. Note that $V_j \cap$

$V_k = \varnothing$ for $j \neq k$. Let W be any neighborhood of z. There exists $M > 0$ such that $V_n \cap W \neq \varnothing$ for all $n \geq M$.

Let $y \in K_M \cap S_{(3/4)\varepsilon}(\tilde{q})$. For each $n > M$, times a_n, b_n exist with $\lambda_n < a_n \leq b_n < s_n$, such that as one moves clockwise from y along $S_{(3/4)\varepsilon}(\tilde{z})$, $\tilde{x}_n a_n (\tilde{x}_n b_n)$ is the first (last) occurrence of K_n crossing $S_{(3/4)\varepsilon}(\tilde{z})$. Let $\beta_n = (\tilde{x}_n b_n, \tilde{x}_{n+1} a_{n+1})_{S_{(3/4)\varepsilon}(\tilde{q})}$. Note that $\beta_n \subset V_n$.

Since $z \in S_{(3/4)\varepsilon}(\tilde{q})$, there exists a closed arc $\mathcal{E}$ of $S_{(3/4)\varepsilon}(\tilde{q})$ of length δ such that $z \in \mathcal{E} \subset W$. There exists $M > 0$ such that $\beta_n \subset \mathcal{E}$ for all $n \geq M$. By Lemma 6.2, $\beta_n \cap \tilde{\omega}(x) \neq \varnothing$ for all n. Thus $\tilde{\omega}(x)$ is not locally connected at z because each component of $\tilde{\omega}(x) \cap W$ must be contained in a single V_n and a sequence of distinct components of $\tilde{\omega}(x) \cap W$ accumulates at z. Since we are assuming that $\omega(x)$ is locally connected we have reached a contradiction. Thus $\tilde{p} \notin \omega(\tilde{x})$ for all $\tilde{p} \in \{\pi^{-1}(p)\}$. $\square$

It follows by Lemma 6.3 and the fact that $\mathcal{O}^+(\tilde{x})$ is unbounded that $\mathcal{O}^+(\tilde{x})$ must cross infinitely many distinct lifts of $\sum^a$ (again, relabeling $\sum^a$ and $\sum^b$ if necessary).

Lemma 6.4. *There exists $\tau > 0$ such that $\mathcal{O}^+(\tilde{x}\tau)$ crosses any lift of $\sum^a$ at most once.*

Proof. By hypothesis times $0 \leq \tau_0 < \tau_1$, a lift $\tilde{\Sigma}$ of $\sum^a$ and a transformation $T \in \Gamma$, $T \neq I$, exist such that $\tilde{x}\tau_0 \in \tilde{\Sigma}$ and $\tilde{x}\tau_1 \in T\tilde{\Sigma}$. Without loss of generality we may assume that $(\tilde{x}\tau_0, \tilde{x}\tau_1)_{\tilde{\phi}} \cap \{\pi^{-1} \sum^a\} = \varnothing$. Let J be the simple curve defined by $J = \cup_{n \in \mathbf{Z}} T^n([\tilde{x}\tau_0, \tilde{x}\tau_1]_{\tilde{\phi}} \cup (\tilde{x}\tau_0, T^{-1}\tilde{x}\tau_1)_{\tilde{\Sigma}})$. Observe that $\mathcal{O}^+(\tilde{x}\tau_1) \subset J^+$.

We will proceed by contradiction. Suppose times s_0, s_1 exist, where $\tau_1 < s_0 < s_1$ and $S \in \Gamma$ such that $\tilde{x}s_0 \in S\tilde{\Sigma}$ and $\tilde{x}s_1 \in S\tilde{\Sigma}$. Without loss of generality $(\tilde{x}s_0, \tilde{x}s_1)_{\tilde{\phi}} \cap S\tilde{\Sigma} = \varnothing$. Let G be the Jordan curve defined by $G = [\tilde{x}s_0, \tilde{x}s_1]_{\tilde{\phi}} \cup (\tilde{x}s_0, \tilde{x}s_1)_{S\tilde{\Sigma}}$. Since $\mathcal{O}^+(\tilde{x})$ is unbounded, $\mathcal{O}^+(\tilde{x}s_1) \subset \mathrm{Ext}\,(G)$, $\mathrm{Int}\,(G)$ is negative invariant and $\mathrm{Ext}\,(G)$ is positively invariant. Note that $G \subset J^+$ and hence $\mathrm{Int}\,(G) \subset J^+$.

Let $\varepsilon > 0$. Since $\mathrm{Int}\,(G)$ is negatively invariant $\tilde{x}(\tau_0 - \varepsilon) \in \mathcal{O}^-(\tilde{x}s_1) \subset \mathrm{Int}\,(G)$. But we also have that $\tilde{x}(\tau_0 - \varepsilon) \in J^-$. Hence $\mathrm{Int}\,(G) \cap J^- \neq \varnothing$, which contradicts $\text{Int}(G) \subset J^+$. Therefore if $\tau > \tau_1$, then $\mathcal{O}^+(\tilde{x}\tau)$ crosses any lift of $\sum^a$ at most once. □

Theorem 6.5. *Let ϕ be a continuous flow on M, $x \in M$ and $\tilde{x} \in \{\pi^{-1}(x)\}$. If $\omega(x)$ is locally connected, Int$(\omega(x)) = \varnothing$ and $\omega(x) \not\subset F$, then either* (a) *$\mathcal{O}^+(\tilde{x})$ is bounded or* (b) *$\mathcal{O}^+(\tilde{x})$ is the type of a rational h-ray and $\omega(x)$ contains an invariant simple closed curve that is not null homotopic.*

Proof. We may assume that $\mathcal{O}^+(\tilde{x})$ is unbounded and x is not periodic since otherwise the result immediately follows. Let $\sum$ be a section at a moving point $p \in \omega(x)$ and let $\sum^a$ and $\sum^b$ denote the two sides of p on $\sum$. By Lemma 6.4 relabeling if necessary, $\tau > 0$ exists such that $\mathcal{O}^+(\tilde{x}\tau)$ crosses any lift of $\sum^a$ at most once. Thus times $\tau_1 < \tau_2$, a lift $\tilde{\Sigma}$ of $\sum^a$ and a transformation $T \in \Gamma$, $T \neq I$, exist such that $\tilde{x}\tau_1 \in \tilde{\Sigma}$, $\tilde{x}\tau_2 \in T\tilde{\Sigma}$ and $T^{-1}\tilde{x}\tau_2 \in (\tilde{x}\tau_1, \tilde{p})_{\tilde{\Sigma}}$. It is easy to check that we may also assume that τ_1 and τ_2 satisfy $(\tilde{x}\tau_1, \tilde{x}\tau_2)_{\tilde{\phi}} \cap S\tilde{\Sigma} = \varnothing$ for all $S \in \Gamma$.

As in Lemma 6.1 let J be the simple curve defined by $J = \cup_{n\in\mathbf{Z}} T^n([\tilde{x}\tau_1, \tilde{x}\tau_2]_{\tilde{\phi}} \cup (\tilde{x}\tau_1, T^{-1}\tilde{x}\tau_2)_{\tilde{\Sigma}})$ and let a^+ and a^- denote the attracting and repelling fixed points of T, respectively.

Let $\tilde{p}$ denote the lift of p contained on $\tilde{\Sigma}$. Since we can choose τ, arbitrarily large, we can apply Corollary 4.3 and Lemma 4.4 to conclude that $\tilde{p}$ and $T\tilde{p}$ lie in the same path component C_α of $\tilde{\omega}(x)$. Let $\gamma \subset C_\alpha$ be a simple path between $\tilde{p}$ and $T\tilde{p}$ ($\pi \circ \gamma$ may not be simple). Since $T^{-1}\tilde{x}\tau_2 \in (\tilde{x}\tau_1, \tilde{p})_{\tilde{\Sigma}}$, we have that $\tilde{p}$ and $T\tilde{p}$ lie in J^+.

Observe that if $\mathcal{O}^+(\tilde{x}\tau_1) \cap T^n\gamma \neq \varnothing$ for some $n \in \mathbf{Z}$, then $x \in \omega(x)$ and so x is recurrent. This is impossible since Int$(\omega(x)) = \varnothing$, $\omega(x)$ is locally connected and x is neither periodic nor fixed. Thus $\mathcal{O}^+(\tilde{x}\tau_1) \cap T^n\gamma = \varnothing$, and hence $\gamma \subset J^+$. Let Q be the Jordan curve determined by $\gamma, [\tilde{x}\tau_1, \tilde{x}\tau_2]_{\tilde{\phi}}$, $(\tilde{x}\tau_1, \tilde{p})_{\tilde{\Sigma}}$, and $(\tilde{x}\tau_2, T\tilde{p})_{T\tilde{\Sigma}}$. Let $\hat{Q} = \cup_{n\in\mathbf{Z}}(T^n \text{Int}\, Q)$. Note that $\hat{Q}$ is positively invariant and $\mathcal{O}^+(\tilde{x}\tau_2) \subset \hat{Q}$.

Note that $\mathcal{O}^+(\tilde{x}\tau_2)$ must leave $T^n(\text{Int}\, Q)$ for each n since $\mathcal{O}^+(\tilde{x})$ is unbounded. Moreover, once $\mathcal{O}^+(\tilde{x}\tau_2)$ leaves $T^n(\text{Int}\, Q)$ it cannot return. Hence for each N, $\mathcal{O}^+(\tilde{x}\tau_2)$ is eventually in $\cup_{n=N}^{\infty} T^n(\text{Int}\, Q)$. Thus $\omega(\tilde{x}\tau_2) \subset \cap_N(\overline{\cup_{n=N}^{\infty} T^n(\text{Int}\, Q)}) = \{a^+\}$. Hence $\tilde{x}t \to a^+$ at $t \to \infty$. Since $\mathcal{O}^+(\tilde{x}\tau_1) \subset \hat{Q}$, we have that $\mathcal{O}^+(\tilde{x})$ is the type of a rational h-ray.

By Theorem 5.1, $\omega(x)$ must contain an invariant simple closed curve that is not null homotopic. $\square$

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