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POSITIVE SOLUTIONS OF NONLINEAR FOCAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper investigates the nth order ordinary differential equation: $-x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)})$ with focal boundary value conditions. We give some estimation results to deal with the term $x^{(n-1)}$ which appeared in f. By using the fixed point index theory, we obtain the existence of positive and multiple positive solutions.

1. Introduction. In this paper we consider the existence of positive solutions and multiple positive solutions of

$$-x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)}), \quad 0 < t < 1, \tag{1.1}$$

with focal boundary value conditions

$$\begin{aligned} &x^{(r_i-1)}(0) = 0, \quad 1 \le i \le k; \\ &x^{(s_j-1)}(1) = 0, \quad 1 \le j \le n-k, \end{aligned} \tag{1.2}$$

where $\{r_1, \dots, r_k\}$ and $\{s_1, \dots, s_{n-k}\}$ form a disjoint partition of $\{1, 2, \dots, n\}$ such that $r_1 < \dots < r_k$ and $s_1 < \dots < s_{n-k}$. For each $0 \le i \le n-1$, define

$$\sigma_i = \operatorname{card}\{j \mid s_j > i\} + 1. \tag{1.3}$$

We assume throughout that

(i) $f \in C[I \times K, R_+]$ where $I = [0, 1]$, $R_+ = [0, +\infty)$ and $K = (-1)^{\sigma_0} R_+ \times (-1)^{\sigma_1} R_+ \times (-1)^{\sigma_2} R_+ \times \dots \times (-1)^{\sigma_{n-2}} R_+ \times R^1$.

(ii) $\{r_{k-1}, r_k\} \ne \{n-1, n\}$, $\{s_{n-k-1}, s_{n-k}\} \ne \{n-1, n\}$.

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(iii) For $t \in I$, $x_i \in (-1)^{\sigma_{i-1}} R_+$, $1 \le i \le n-1$, $|x_n| \ge c > 0$,

$$f(t, x_1, x_2, \dots, x_{n-1}, x_n) \le a(t, x_1, \dots, x_{n-1}) + b(t, x_1, \dots, x_{n-1})|x_n|, \tag{1.4}$$

where a, b are nonnegative continuous functions.

Denote $C(I)$ the Banach space of continuous functions on I with norm $\|x\|_C = \max_{0 \le t \le 1} |x(t)|$. Let $P = \{x \in C(I) \mid x(t) \ge 0$ for $t \in I\}$; then P is a cone of $C(I)$ (see, e.g., [**3**], [**8**]). Denote $L^1(I) = \{x : I \to R^1 \mid \int_0^1 |x(t)|\, dt < +\infty\}$ and $L^1(I, u) = \{x : I \to R^1 \mid \int_0^1 u(t)|x(t)|\, dt < +\infty\}$ where $u(t) \in P \setminus \{\theta\}$ is given. Then $L^1(I)$ and $L^1(I, u)$ are Banach spaces with norm $\|x\|_{L^1} = \int_0^1 |x(t)|\, dt$ for $x \in L^1(I)$ and $\|x\|_{L^1(I,u)} = \int_0^1 u(t) \cdot |x(t)|\, dt$ for $x \in L^1(I, u)$, respectively.

Many recently published papers are devoted to solutions of $-x^{(n)} = f(t, x)$ with two-point boundary value conditions (see, for example, [**1**], [**4**]–[**7**], [**10**], [**11**]). For the case of second order boundary value problems (BVPs), the above equation describes various phenomena, such as nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems where only positive solutions are meaningful (see, e.g., [**1**], [**2**]).

Eloe and Henderson [**5**], [**6**] investigated solutions for $(k, n-k)$ conjugate BVPs, Agarwal, etc., [**1**] and Kaufmann [**10**] studied nonlinear focal BVPs and Lou [**11**] studied (1.1) for $n = 2$ in abstract spaces, all of them considered only the case f was independent of $x', x'', \dots, x^{(n-1)}$, and the Green's functions for corresponding linear problems played important roles in them. By using their methods it seems difficult to obtain the existence of (multiple) *positive* solutions for *general* focal BVPs such as (1.2). Eloe and Zhang [**7**] investigated the second order BVPs and required $\partial f/\alpha x$ and $\partial^2 f/\partial x^2$ to be continuous, and their methods are difficult to be applied to obtain positive solutions ($> 0, \not\equiv 0$). This paper improves the results in [**1**], [**5**]–[**7**], [**10**], [**11**] and lots of other recently published papers. First we deal with (1.1) instead of $-x^{(n)} = f(t, x)$. We obtain the existence results by calculating the fixed point index instead of using the monotone iterative method and the methods as in [**1**], [**5**]–[**7**]. Second we will discuss both superlinear and sublinear cases, and the nonlinearity of f is expressed by the first eigenvalue of the corresponding linear problem rather than 0 and ∞, hence it is difficult to apply the widely-used Krasnosel'skii's theorem (see, e.g, [**1**], [**6**], [**10**]). Third, since we discuss BVPs with boundary value condition (1.2), we do not know the concrete expression of Green's function, we can only use the properties of it (see Lemmas 2.1–2.3 below).

2. Preliminaries. It is well known that finding solutions of (1.1)–(1.2) in $C^n(I)$ is equivalent to finding solutions of

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds, \tag{2.1}$$

in $C(I)$ where $G(t,s)$ is the Green's function for $-x^{(n)} = 0$ with boundary value conditions (1.2). Eloe [**4**] has shown that, for $0 \le i \le n-2$,

$$(-1)^{\sigma_i} \frac{\partial^i}{\partial t^i} G(t,s) > 0 \quad \text{on } (0,1) \times (0,1), \tag{2.2}$$

where σ_i is given by (1.3) as well as the fact that

$$K(t,s) = \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) \tag{2.3}$$

is the Green's function for $-y'' = 0$ satisfying

$$y(0) = y'(1) = 0, \quad \text{if } r_k = n-1, \quad s_{n-k} = n, \tag{2.4}$$

or

$$y'(0) = y(1) = 0, \quad \text{if } r_k = n, \quad s_{n-k} = n-1. \tag{2.5}$$

Hence,

$$K(t,s) = \begin{cases} t & 0 \le t \le s \le 1, \\ s & 0 \le s \le t \le 1. \end{cases} \quad \text{or} \quad K(t,s) = \begin{cases} 1-s & 0 \le t \le s \le 1, \\ 1-t & 0 \le s \le t \le 1. \end{cases} \tag{2.6}$$

in the case of (2.4) or (2.5), respectively. By careful analysis one can get the following properties of $G(t,s)$:

(P1) $G(t,s)$ is the $(n-1)$th polynomial with respect to t and s on Ω_1, Ω_2 where $\Omega_1 = \{(t,s) \in I \times I \mid s \le t\}$, $\Omega_2 = \{(t,s) \in I \times I \mid t \le s\}$.

(P2) $(\partial^i/\partial t^i)G(t,s)$, $(\partial^i/\partial s^i)G(t,s)$, $0 \le i \le n-2$, are continuous on $I \times I$.

(P3) $(\partial^{n-1}/\partial t^{n-1})G(t,s)$ and $(\partial^{n-1}/\partial s^{n-1})G(t,s)$ are bounded and continuous on Ω_1, Ω_2, respectively.

By (2.2), $(-1)^{\sigma_0}G(t,s) > 0$ for $t,s \in (0,1)$. Without loss of generality, we assume hereinafter that σ_0 is even, i.e.,

$$G(t,s) > 0, \quad t,s \in (0,1). \tag{2.7}$$

Define a linear operator $G : C(I) \to C(I)$ by

$$Gh(t) = \int_0^1 G(s,t)h(s)\,ds.$$

Then G is a positive linear completely continuous operator and the spectral radius of $G : r(G) \equiv r_1$ satisfies $r_1 > 0$ (see Nussbaum [**12**]). Therefore, by the Krein-Rutman theorem (cf. [**12**]), $p_1 \in P \setminus \{\theta\}$ exists with $\|p_1\|_C = 1$ such that

$$Gp_1(t) = \int_0^1 G(s,t)p_1(s)\,ds = r_1 \cdot p_1(t). \tag{2.8}$$

By (2.7), (2.8) and (P1)–(P3), it is not difficult to prove that

Lemma 2.1. *A constant $\delta_1 > 0$ exists such that*

$$p_1(s) \ge \delta_1 G(t,s), \quad \forall\, t,s \in I. \tag{2.9}$$

It is easy to see that (2.1) is equivalent to

$$\begin{cases} x_1(t) = \int_0^1 G_1(t,s)f(s,x_1(s),x_2(s),\dots,x_n(s))\,ds \\ \quad \equiv A_1(x_1,x_2,\dots,x_n)(t), \\ x_2(t) = \int_0^1 G_2(t,s)f(s,x_1(s),x_2(s),\dots,x_n(s))\,ds \\ \quad \equiv A_2(x_1,x_2,\dots,x_n)(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n(t) = \int_0^1 G_n(t,s)f(s,x_1(s),x_2(s),\dots,x_n(s))\,ds \\ \quad \equiv A_n(x_1,x_2,\dots,x_n)(t), \end{cases} \tag{2.10}$$

where $G_1(t,s) \equiv G(t,s)$, $G_i(t,s) = (\partial^{i-1}/\partial t^{i-1})G(t,s)$, $2 \le i \le n$.

By (2.2) and the Krein-Rutman theorem, $p_i \in P\setminus\{\theta\}$ and $\|p_i\|_C = 1$, $i = 2,3,\dots,n-1$, exist such that

$$r_i p_i(s) = \int_0^1 (-1)^{\sigma_{i-1}} G_i(t,s) p_i(t)\,dt, \tag{2.11}$$

where r_i, $i = 2,3,\dots,n-1$, denote the spectral radii of the linear integral operators with kernel $(-1)^{\sigma_{i-1}}G_i(t,s)$, respectively. Clearly, $p_i(t) > 0$, $0 < t < 1$, and in a similar way as establishing (2.9), we can get, for $i = 2,3,\dots,n-1$,

$$p_i(s) \ge \delta_i \cdot (-1)^{\sigma_{i-1}} G_i(t,s), \quad \forall\, t,s \in I, \tag{2.12}$$

for some $\delta_i > 0$.

Lemma 2.2. *There exist $\alpha_i > 0$, $i = 1,2,\dots,n-1$, such that for any $x_i \in (-1)^{\sigma_{i-1}}P$, $i = 1,2,\dots,n-1$, $x_n \in L^1(I)$,*

$$\text{a)}\quad \|A_i(x_1,x_2,\dots,x_n)\|_{L^1} \ge \alpha_i \|A_i(x_1,x_2,\dots,x_n)\|_C, \tag{2.13}$$

$$\text{b)}\quad \|A_i(x_1,x_2,\dots,x_n)\|_{L^1} \le \|A_{i-1}(x_1,x_2,\dots,x_n)\|_C. \tag{2.14}$$

Proof. By (2.8), (2.9), (2.11) and (2.12), we have

$$\begin{aligned}
&\|A_i(x_1,x_2,\dots,x_n)\|_{L^1(I,p_i)}\\
&\qquad = \int_0^1 |A_i(x_1,x_2,\dots,x_n)(t)| \cdot p_i(t)\,dt\\
&\qquad = \int_0^1 p_i(t)\,dt \bigg| \int_0^1 G_i(t,s) f(s,x_1(s),\dots,x_n(s))\,ds \bigg|\\
&\qquad = (-1)^{\sigma_{i-1}} \int_0^1 f(s,x_1(s),\dots,x_n(s))\,ds \int_0^1 p_i(t) G_i(t,s)\,dt\\
&\qquad = r_i \int_0^1 p_i(s) f(s,x_1(s),\dots,x_n(s))\,ds\\
&\qquad \ge r_i \delta_i (-1)^{\sigma_i - 1} \int_0^1 G_i(\tau,s) f(s,x_1(s),\dots,x_n(s))\,ds
\end{aligned}$$

$$= r_i \delta_i A_i(x_1, x_2, \dots, x_n)(\tau), \quad \forall \tau \in I.$$

Hence,

$$\begin{aligned} \|A_i(x_1, \dots, x_n)\|_{L^1} &\geq \|A_i(x_1, \dots, x_n)\|_{L^1(I, p_i)} \\ &\geq r_i \delta_i \|A_i(x_1, \dots, x_n)\|_C, \end{aligned}$$

a) is proved.

b) Since $(-1)^{\sigma_{i-1}} G_i(t,s) > 0$ for $t, s \in (0,1)$ and $i = 1, 2, \dots, n-1$, we have

$$\begin{aligned} \int_0^1 |G_i(t,s)|\, dt &= \left| \int_0^1 G_i(t,s)\, dt \right| \\ &= |G_{i-1}(1,s) - G_{i-1}(0,s)| \leq |G_{i-1}(1,s)|. \end{aligned}$$

Hence,

$$\begin{aligned} \|A_i(x_1, x_2, \dots, x_n)\|_{L^1} &= \int_0^1 \left| \int_0^1 G_i(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \right| dt \\ &\leq \int_0^1 |f(s, x_1(s), \dots, x_n(s))|\, ds \cdot \int_0^1 |G_i(t,s)|\, dt \\ &\leq \int_0^1 |G_{i-1}(1,s) f(s, x_1(s), \dots, x_n(s))|\, ds \\ &= \left| \int_0^1 G_{i-1}(1,s) f(s, x_1(s), \dots, x_n(s))\, ds \right| \\ &= |A_{i-1}(x_1, x_2, \dots, x_n)(1)| \\ &\leq \|A_{i-1}(x_1, x_2, \dots, x_n)\|_C. \end{aligned}$$

b) is proved.

Lemma 2.3. *For any $x_i \in (-1)^{\sigma_{i-1}} P$, $1 \leq i \leq n-1$, $x_n \in L^1(I)$,*

$$\|A_n(x_1, x_2, \dots, x_x)\|_{L^1} \leq 2 \|A_{n-1}(x_1, x_2, \dots, x_n)\|_{L^1}. \tag{2.15}$$

Proof. We first show that

$$\int_0^1 |G_n(t,s)|\, dt \leq 2 \int_0^1 |G_{n-1}(t,s)|\, dt. \tag{2.16}$$

In fact, in the case of (2.4) by (2.3) and (2.6),

$$\int_0^1 |G_{n-1}(t,s)|\,dt = \int_0^1 K(t,s)\,dt = s - \frac{s^2}{2},$$
$$\int_0^1 |G_n(t,s)|\,dt = \int_0^1 \left|\frac{\partial}{\partial t}K(t,s)\right| dt = s.$$

Hence (2.16) holds. In the case of (2.5), the proof is similar. Consequently,

$$\begin{aligned}
&\|A_n(x_1,x_2,\ldots,x_n)\|_{L^1}\\
&\quad= \int_0^1 \left|\int_0^1 G_n(t,s) f(s,x_1(s),\ldots,x_n(s))\,ds\right| dt\\
&\quad\le \int_0^1 |f(s,x_1(s),\ldots,x_n(s))|\,ds \cdot \int_0^1 |G_n(t,s)|\,dt\\
&\quad\le 2\int_0^1 dt \int_0^1 |G_{n-1}(t,s)|\cdot|f(s,x_1(s),\ldots,x_n(s))|\,ds\\
&\quad= 2\int_0^1 \left|\int_0^1 G_{n-1}(t,s) f(s,x_1(s),\ldots,x_n(s))\,ds\right| dt\\
&\quad= 2\|A_{n-1}(x_1,\ldots,x_n)\|_{L^1},
\end{aligned}$$

i.e., (2.15) holds.

Let $\mathcal{E} = \underbrace{C(I)\times C(I)\times\cdots\times C(I)}_{n-1}\times L^1(I)$, then $\mathcal{E}$ is a Banach space with norm $\|X\|_{\mathcal{E}} = \sum_{i=1}^{n-1}\|x_i\|_C + \|x_n\|_{L^1}$ for any $X(t) = (x_1(t),\ldots,x_n(t)) \in \mathcal{E}$. Obviously, (2.10) can be rewritten as

$$X(t) = AX(t) \equiv (A_1X(t), A_2X(t), \ldots, A_nX(t)).$$

Set

$$\mathcal{K}_0 = \{(x_1,\ldots,x_n) \in \mathcal{E} \mid x_i \in (-1)^{\sigma_{i-1}}P(1 \le i \le n-1),\ x_n \in L^1(I)\}.$$

By (1.4), (2.2) and (P2), $A_i(x_1,\ldots,x_n) \in (-1)^{\sigma_{i-1}}P(1\le i\le n-1)$ for $X \in \mathcal{K}_0$ and, by (2.15), $A_n(x_1,\ldots,x_n) \in L^1(I)$ for $X \in \mathcal{K}_0$, that is, A maps $\mathcal{K}_0$ into $\mathcal{K}_0$. Therefore, the fixed points of A in $\mathcal{K}_0$ are equivalent to positive solutions of (1.1)–(1.2).

Set

$$\mathcal{K} = \big\{(x_1, x_2, \dots, x_n) \in \mathcal{K}_0 \mid \alpha_i \|x_i\|_C \le \|x_i\|_{L^1} (1 \le i \le n-1); \\ \|x_i\|_{L^1} \le \|x_{i-1}\|_C (2 \le i \le n-1);\ \|x_n\|_{L^1} \le 2\|x_{n-1}\|_{L^1}\big\}. \tag{2.17}$$

Clearly, $\mathcal{K}$ is a cone of $\mathcal{E}$. By Lemmas 2.2 and 2.3, A maps $\mathcal{K}$ into $\mathcal{K}$ and it is a completely continuous operator. In what follows, denote $\mathcal{K}_R = \{X \in \mathcal{K} \mid \|x_1\|_C < R\}$, $R > 0$.

Remark 2.1. If $X(t) = (x_1(t), \dots, x_n(t)) \in \mathcal{K}$, then for $i = 1, 2, \dots, n-1$,

$$\begin{aligned} \|x_i\|_{L^1} &\le \|x_{i-1}\|_C \le \frac{1}{\alpha_{i-1}} \|x_{i-1}\|_{L^1} \le \frac{1}{\alpha_{i-1}} \|x_{i-2}\|_C \le \cdots \\ &\le \frac{1}{\alpha_2 \alpha_3 \cdots \alpha_{i-1}} \|x_1\|_C, \\ \|x_n\|_{L^1} &\le 2\|x_{n-1}\|_{L^1} \le \frac{2}{\alpha_2 \alpha_3 \cdots \alpha_{n-2}} \|x_1\|_C. \end{aligned}$$

Denote

$$\Delta = 2 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2 \alpha_3} + \cdots + \frac{1}{\alpha_2 \alpha_3 \cdots \alpha_{n-2}} + \frac{2}{\alpha_2 \alpha_3 \cdots \alpha_{n-2}}, \tag{2.18}$$

then

$$\sum_{i=1}^{n} \|x_i\|_{L^1} \le \Delta \|x_1\|_C. \tag{2.19}$$

Similarly, denote

$$\Delta_1 = 1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2 \alpha_3} + \cdots + \frac{1}{\alpha_2 \alpha_3 \cdots \alpha_{n-1}}, \tag{2.20}$$

then

$$\begin{aligned} \sum_{i=1}^{n-1} \|x_i\|_C &\le \left(1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2 \alpha_3} + \cdots + \frac{1}{\alpha_2 \alpha_3 \cdots \alpha_{n-1}}\right) \|x_1\|_C \\ &= \Delta_1 \|x_1\|_C. \end{aligned} \tag{2.21}$$

3. Main theorems. For any $u = (x_1, x_2, \dots, x_n) \in K = (-1)^{\sigma_0} R_+ \times (-1)^{\sigma_1} R_+ \times \cdots \times (-1)^{\sigma_{n-2}} R_+ \times R^1 \subset R^n$, denote $\|u\| = \sum_{i=1}^{n-1} (-1)^{\sigma_{i-1}} x_i + |x_n|$.

Theorem 3.1. *Suppose that, for* $u = (x_1, \dots, x_n) \in K$,

(H1) $\displaystyle\lim_{\|u\| \to +\infty} \frac{f(t, x_1, \dots, x_n)}{\|u\|} = 0$ *uniformly on* $t \in I$.

(H2) $\displaystyle\liminf_{x_1 \to 0^+} \frac{f(t, x_1, \dots, x_n)}{x_1} > \lambda_1$ *uniformly on* $t \in I$, $x_i \in (-1)^{\sigma_{i-1}} R_+$, $i \neq 1, n$, $x_n \in R^1$, *where* $\lambda_1 = 1/r_1$ *and* r_1 *is given by* (2.8).

Then (1.1)–(1.2) *has at least one positive solution.*

Proof. Set $g = \max_{t,s \in I} G(t,s) > 0$. Choose $\varepsilon > 0$ such that $\varepsilon < (1/g\Delta)$, where Δ is defined by (2.18).

By (H1) we can choose a constant $\gamma > 0$ such that for any $u \in K$ and $\|u\| \geq \gamma$,

$$f(t, x_1, x_2, \dots, x_n) \leq \varepsilon \|u\|. \tag{3.1}$$

Denote $f_0 = \sup_{\substack{u \in K \\ \|u\| \leq \gamma}} f(t, x_1, \dots, x_n)$ and choose $R > 0$ such that

$$R > \max\left\{\gamma, 1 + \frac{g f_0}{1 - \varepsilon g \Delta}\right\}. \tag{3.2}$$

Let $\mathcal{K}_R$ be defined as above, then by (2.18) and Remark 2.1, we know that $\mathcal{K}_R$ is a bounded open set in $\mathcal{K}$. For any $X = (x_1, \dots, x_n) \in \partial\mathcal{K}_R$, where $\partial\mathcal{K}_R$ denotes the boundary of $\mathcal{K}_R$ with respect to $\mathcal{K}$, set $J = \{t \in I \mid \sum_{i=1}^{n-1} (-1)^{\sigma_{i-1}} x_i(t) + |x_n(t)| \leq \gamma\}$, then by (3.1) and (2.19)

$$\begin{aligned} A_1(x_1, x_2, \dots, x_n)(t) &= \int_0^1 G(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \\ &= \int_J G(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \\ &\quad + \int_{I \setminus J} G(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&\le g\cdot f_0+\varepsilon g\int_0^1\left(\sum_{i=1}^{n-1}(-1)^{\sigma_{i-1}}x_i(t)+|x_n(t)|\right)dt\\
&\le g\cdot f_0+\varepsilon g\sum_{i=1}^{n}\|x_i\|_{L^1}\\
&= g\cdot f_0+\varepsilon g\Delta\|x_1\|_C.
\end{aligned}
$$

Since $X\in\partial\mathcal{K}_R$, $\|x_1\|_C=R$, we have by (3.2), $A_1(x_1,\dots,x_n)(t)<\|x_1\|_C$. Therefore

$$X\neq\mu AX\quad\text{for } X\in\partial\mathcal{K}_R,\quad \mu\in(0,1].$$

It follows from the homotopy invariance of the fixed point index that (see, e.g., [**3**], [**8**]),

$$i(A,\mathcal{K}_R,\mathcal{K})=1. \tag{3.3}$$

By (H2), $R_1<R$ exists such that

$$f(t,x_1,\dots,x_n)>\lambda_1x_1\quad\text{for } t\in I,\ u\in K,\ 0<x_1<R_1. \tag{3.4}$$

We now prove that

$$X-AX\neq\mu X_0\quad\text{for } X\in\partial\mathcal{K}_{R_2},\quad \mu\ge 0,$$

where $X_0=(1,(-1)^{\sigma_1},(-1)^{\sigma_2},\dots,(-1)^{\sigma_{n-2}},0)\in\mathcal{K}$, $0<R_2\le R_1$. Otherwise, if $X-AX=\mu X_0$ holds some $X\in\partial\mathcal{K}_{R_2}$ and some $\mu\ge 0$, then $x_1(t)\ge A_1(x_1,\dots,x_n)(t)$, hence by (3.4),

$$
\begin{aligned}
\int_0^1 p_1(t)x_1(t)\,dt &\ge \int_0^1 p_1(t)\,dt\int_0^1 G(t,s)f(s,x_1(s),\dots,x_n(s))\,ds\\
&= r_1\int_0^1 p_1(s)f(s,x_1(s),\dots,x_n(s))\,ds\\
&> r_1\lambda_1\int_0^1 p_1(s)x_1(s)\,ds\\
&= \int_0^1 p_1(s)x_1(s)\,ds,
\end{aligned}
$$

a contradiction. Therefore $X - AX \neq \mu X_0$ for $X \in \partial\mathcal{K}_{R_2}$ and $\mu \geq 0$, it follows from the property of the fixed point index (see e.g. [**8**], Corollary 2.3.1), that

$$i(A, \mathcal{K}_{R_2}, \mathcal{K}) = 0. \tag{3.5}$$

By (3.3), (3.5) and the additivity of the fixed point index we have

$$i(A, \mathcal{K}_R \setminus \overline{\mathcal{K}_{R_2}}, \mathcal{K}) = 1,$$

which implies that A has at least one fixed point in $\mathcal{K}_R \setminus \overline{\mathcal{K}_{R_2}}$, i.e., (1.1)–(1.2) has at least one positive solution. This completes the proof.

Theorem 3.2. *Suppose that, for any $u = (x_1, \dots, x_n) \in K$,*

(H3) $\displaystyle\lim_{\nu \to 0^+} \frac{f(t, x_1, \dots, x_n)}{\|u\|} = 0$ *uniformly on $t \in I$, $x_n \in R^1$, where $\nu = \sum_{i=1}^{n-1} (-1)^{\sigma_{i-1}} x_i$.*

(H4) $\displaystyle\liminf_{x_1 \to +\infty} \frac{f(t, x_1, \dots, x_n)}{x_1} > \lambda_1$ *uniformly on $t \in I$, $x_i \in (-1)^{\sigma_{i-1}} R_+$, $i \neq 1, n$, $x_n \in R^1$.*

Then (1.1)–(1.2) *has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $\varepsilon < (1/g\Delta)$ where g, Δ are given as above. By (H3), $R_3 > 0$ exists such that

$$f(t, x_1, x_2, \dots, x_n) < \varepsilon \cdot \|u\| \quad \text{for } u \in K, \sum_{i=1}^{n-1} (-1)^{\sigma_{i-1}} x_i < R_3. \tag{3.6}$$

Now choose $0 < R_4 \leq R_3/\Delta_1$, then for $X \in \partial\mathcal{K}_{R_4}$, $\sum_{i=1}^{n-1} |x_i(s)| \leq \sum_{i=1}^{n-1} \|x_i\|_C \leq \Delta_1 \|x_1\|_C \leq R_3$ on account of (2.21), and by (3.6) and (2.19) we have

$$\begin{aligned} A_1(x_1, x_2, \dots, x_n)(t) &= \int_0^1 G(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \\ &\leq \varepsilon \int_0^1 G(t,s) \cdot \sum_{i=1}^n |x_i(s)|\, ds \\ &\leq \varepsilon g \sum_{i=1}^n \|x_i\|_{L^1} \leq \varepsilon g \Delta \|x_1\|_C < \|x_1\|_C, \end{aligned}$$

therefore $X \neq \mu AX$ for any $X \in \partial\mathcal{K}_{R_4}$, $\mu \in (0,1]$, so

$$i(A, \mathcal{K}_{R_4}, \mathcal{K}) = 1. \tag{3.7}$$

Set

$$Q = \Big\{ h \in C(I) \,\Big|\, \int_0^1 p_1(t)h(t)\,dt \geq r_1 \cdot \delta_1 \|h\|_C \Big\}. \tag{3.8}$$

Clearly, Q is a cone of $C(I)$ and $h_1 + h_2 \in Q$ for any $h_1, h_2 \in Q$.

By (H4), $\sigma > 0$ and $R_0 > 0$ exist such that for $t \in I$, $x_i \in (-1)^{\sigma_{i-1}} R_+$, $2 \leq i \leq n-1$, and $x_n \in R^1$,

$$f(t, x_1, \dots, x_n) > (\lambda_1 + \sigma)x_1 \quad \text{for } x_1 > R_0.$$

Therefore, for any $t \in I$, $x_i \in (-1)^{\sigma_{i-1}} R_+$ $(2 \leq i \leq n-1)$ and $x_n \in R^1$,

$$f(t, x_1, \dots, x_n) > (\lambda_1 + \sigma)x_1 - a_1 \quad \text{for } x_1 \geq 0, \tag{3.9}$$

where $a_1 = (\lambda_1 + \sigma)R_0$.

For any fixed $R_5 > R_4$ such that $R_5 > a_1 \int_0^1 p_1(t)\,dt \cdot (r_1 \sigma \delta_1)^{-1}$, we now show that

$$X - AX \neq \mu X_0 \quad \text{for } X \in \partial\mathcal{K}_{R_5}, \quad \mu \geq 0, \tag{3.10}$$

where $X_0 = (1, (-1)^{\sigma_1}, (-1)^{\sigma_2}, \dots, (-1)^{\sigma_{n-2}}, 0) \in \mathcal{K}$. In fact, if $X - AX = \mu X_0$ for some $X \in \partial\mathcal{K}_{R_5}$ and some $\mu \geq 0$, then $x_1 - A_1 X = \mu$. We have by (2.8) and (2.9)

$$\begin{aligned}
\int_0^1 p_1(t) A_1 X(t)\,dt &= \int_0^1 p_1(t)\,dt \int_0^1 G(t,s) f(s, x_1(s), \dots, x_n(s))\,ds \\
&= r_1 \int_0^1 p_1(s) f(s, x_1(s), \dots, x_n(s))\,ds \\
&\geq r_1 \delta_1 \int_0^1 G(\tau, s) f(s, x_1(s), \dots, x_n(s))\,ds \\
&= r_1 \delta_1 A_1 X(\tau), \quad \tau \in I.
\end{aligned}$$

Hence $A_1X(t) \in Q$ and so $x_1 = \mu + A_1X \in Q$, i.e., $\int_0^1 p_1(t)x_1(t)\,dt \geq r_1 \cdot \delta_1 \|x_1\|_C = r_1\delta_1 R_5$. Consequently, by (2.8), (2.9) and (3.9),

$$\begin{aligned}
0 \geq -\mu \int_0^1 p_1(t)\,dt &= \int_0^1 p_1(t)A_1X(t)\,dt - \int_0^1 p_1(t)x_1(t)\,dt \\
&= \int_0^1 p_1(t)\,dt \int_0^1 G(t,s)f(s,x_1(s),\dots,x_n(s))\,ds - \int_0^1 p_1(t)x_1(t)\,dt \\
&= r_1 \int_0^1 p_1(s)(\lambda_1+\sigma)x_1(s)\,ds - \int_0^1 p_1(t)x_1(t)\,dt - a_1r_1 \int_0^1 p_1(s)\,ds \\
&= r_1\sigma \int_0^1 p_1(s)x_1(s)\,ds - a_1r_1 \int_0^1 p_1(s)\,ds \\
&\geq r_1^2\sigma\delta_1R_5 - a_1r_1 \int_0^1 p_1(s)\,ds
\end{aligned}$$

in contradiction with the choice of R_5. Hence, (3.10) holds and so

$$i(A, \mathcal{K}_{R_5}, \mathcal{K}) = 0. \tag{3.11}$$

By (3.7), (3.11) and the additivity of the fixed point index, we have

$$i(A, \mathcal{K}_{R_5} \setminus \overline{\mathcal{K}_{R_4}}, \mathcal{K}) = -1,$$

which implies that A has at least one fixed point in $\mathcal{K}_{R_5} \setminus \overline{\mathcal{K}_{R_4}}$, that is, (1.1)–(1.2) has at least one positive solution. This completes the proof. $\square$

In what follows, we investigate multiple positive solutions of (1.1)–(1.2). Let us list some conditions first:

(H5) There exists $l \geq 1$ such that for any $u = (x_1, x_2, \dots, x_n) \in K$, $0 \leq x_1 \leq l$,

$$f(t, x_1, \dots, x_n) \leq \frac{\sqrt{\alpha_1}}{g}\sqrt{x_1},$$

where α_1, g are given as above.

(H6) There exist $\varepsilon > 0$ and $l > \displaystyle\int_0^1 \frac{p_1(t)}{\delta_1}\,dt$ such that

$$f(t, x_1, \dots, x_n) \geq \frac{1+\varepsilon}{r_1}\,x_1 - \varepsilon \quad \text{for } 0 \leq x_1 \leq l.$$

Theorem 3.3. *If* (H2), (H4) *and* (H5) *are satisfied, then* (1.1)–(1.2) *has at least two positive solutions* x, y *such that* $0 < \|x\|_C < l < \|y\|_C$.

Proof. By (3.5), (3.11) and the additivity of the fixed point index, it suffices to prove that

$$i(A, \mathcal{K}_l, \mathcal{K}) = 1. \tag{3.12}$$

In fact, we have $X \neq \mu AX$ for any $X \in \partial\mathcal{K}_l$, $\mu \in (0,1]$. Otherwise, $X = \mu AX$ for some $X \in \partial\mathcal{K}_l$ and some $\mu \in (0,1]$ then by (H5) we have

$$\begin{aligned}
x_1(t) &= \mu A_1 X(t) \leq A_1 X(t) \\
&= \int_0^1 G(t,s) f(s, x_1(s), \dots, x_n(s))\, ds \\
&\leq \frac{\sqrt{\alpha_1}}{g} \int_0^1 G(t,s) \sqrt{x_1(s)}\ ds \\
&\leq \frac{\sqrt{\alpha_1}}{g} \left(\int_0^1 G^2(t,s)\, ds \right)^{1/2} \left(\int_0^1 x_1(s)\, ds \right)^{1/2} \\
&\leq\not\equiv \sqrt{\alpha_1} \|x_1\|_{L^1}^{1/2},
\end{aligned}$$

hence

$$\|x_1\|_{L^1} = \int_0^1 x_1(t)\, dt < \sqrt{\alpha_1}\, \|x_1\|_{L^1}^{1/2},$$

and so $\|x_1\|_{L^1} < \alpha_1$.

Consequently, it follows from $AX \in \mathcal{K}$ and $x_1 = \mu A_1 X$ that

$$\alpha_1 > \|x_1\|_{L^1} = \mu \|A_1 X\|_{L^1} \geq \mu\alpha_1 \|A_1 X\|_C = \alpha_1 \|x_1\|_C = \alpha_1 l,$$

in contradiction with $l \geq 1$. Therefore, $X \neq \mu AX$ for any $X \in \partial\mathcal{K}_l$, $\mu \in (0,1]$, and so (3.12) holds.

Thus A has at least two fixed points in $\mathcal{K}_{R_5} \setminus \overline{\mathcal{K}_l}$ and $\mathcal{K}_l \setminus \overline{\mathcal{K}_{R_2}}$, respectively, that is, (1.1)–(1.2) has a positive solution x satisfying $0 < \|x\|_C < l$, and has a positive solution y satisfying $\|y\|_C > l$. The proof is completed. □

Theorem 3.4. *If* (H1), (H3) *and* (H6) *are satisfied, then* (1.1)–(1.2) *has at least two positive solutions* x, y *such that* $0 < \|x\|_C < l < \|y\|_C$.

Proof. By (3.3), (3.7) and the additivity of the fixed point index, it suffices to prove that

$$i(A,\mathcal{K}_l,\mathcal{K})=0. \tag{3.13}$$

Hence, by [**8**], Corollary 2.3.1 it is sufficient to prove that

$$X-AX\neq \mu X_0 \quad \text{for any } X\in\partial\mathcal{K}_l,\quad \mu\geq 0, \tag{3.14}$$

where $X_0 = (1,(-1)^{\sigma_1},(-1)^{\sigma_2},\dots,(-1)^{\sigma_{n-2}},0)\in\mathcal{K}$. In fact, if $X-AX=\mu X_0$ for some $X\in\partial\mathcal{K}_l$, $\mu\geq 0$, then $x_1=A_1X+\mu\in Q$, i.e., $\int_0^1 p_1(t)x_1(t)\,dt\geq r_1\delta_1\|x_1\|_C=r_1\delta_1 l$. By (H6),

$$\begin{aligned}\int_0^1 p_1(t)x_1(t)\,dt &\geq \int_0^1 p_1(t)\,dt\int_0^1 G(t,s)f(s,x_1(s),\dots,x_n(s))\,ds\\ &\geq r_1\int_0^1 p_1(s)f(s,x_1(s),\dots,x_n(s))\,ds\\ &\geq r_1\int_0^1 p_1(s)\frac{1+\varepsilon}{r_1}x_1(s)\,ds - r_1\varepsilon\int_0^1 p_1(s)\,ds.\end{aligned}$$

Therefore,

$$r_1\int_0^1 p_1(s)\,ds\geq\int_0^1 p_1(s)x_1(s)\,ds\geq r_1\delta_1 l,$$

in contradiction with $l>\int_0^1 \frac{p_1(t)}{\delta_1}\,dt$. Thus (3.13) and (3.14) hold. The following proof is similar to that of Theorem 3.3. This completes the proof. $\square$

Remark 3.1. Theorems 3.3 and 3.4 improve Theorems 3 and 4 of [**10**], respectively.

In order to understand our results easily, we apply them to a special example and to see how our hypotheses, say (H5) and (H6), can be satisfied.

Example 3.1. Consider the following problem

$$\begin{cases} -x^{(4)} = f(t, x, x', x'', x'''), & 0 < t < 1, \\ x(0) = x''(0) = 0, x'(1) = x'''(1) = 0. \end{cases} \tag{3.15}$$

To study (3.15) we use all the notations as have been used for problem (1.1)–(1.2). So for (3.15), $\sigma_0 = \sigma_1 = 3$, $\sigma_2 = \sigma_3 = 2$. We assume hereinafter that

(i)′ $f \in C[I \times K, R_+]$ where $K = (-R_+) \times (-R_+) \times R_+ \times R^1$.

(iii)′ For $t \in I,\ x_1, x_2 \leq 0,\ x_3 \geq 0,\ |x_4| \geq c > 0$,

$$f(t, x_1, x_2, x_3, x_4) \leq a(t, x_1, x_2, x_3) + b(t, x_1, x_2, x_3)|x_4|$$

for some nonnegative continuous functions a, b.

Corollary 3.1. *Suppose that* (i)′, (iii)′ *are satisfied and*

(H2)′ $\liminf_{x_1 \to 0^-} \dfrac{f(t, x_1, x_2, x_3, x_4)}{-x_1} > \dfrac{\pi^4}{16}$ *uniformly on* $t \in I,\ x_2 \leq 0,\ x_3 \geq 0,\ x_4 \in R^1$.

(H4)′ $\liminf_{x_1 \to -\infty} \dfrac{f(t, x_1, x_2, x_3, x_4)}{-x_1} > \dfrac{\pi^4}{16}$ *uniformly on* $t \in I,\ x_2 \leq 0,\ x_3 \geq 0,\ x_4 \in R^1$.

(H5)′ *There exists* $l \geq 1$ *such that*

$$f(t, x_1, x_2, x_3, x_4) \leq \frac{2}{\sqrt{3}} \sqrt{-x_1} \quad \text{for } -l \leq x_1 \leq 0.$$

Then (3.15) *has at least two negative solutions* x, y *such that* $0 < \|x\|_C < l < \|y\|_C$.

Proof. By Theorem 3.3 and the other former results as well as their proofs, it is sufficient to show that (H5)′ implies (H5) for (3.15).

The Green's function corresponding to (3.15) is

$$G(t, s) = \begin{cases} -\dfrac{6ts - 3st^2 - s^3}{6}, & 0 \leq s \leq t \leq 1, \\ -\dfrac{6ts - 3ts^2 - t^3}{6}, & 0 \leq t \leq s \leq 1. \end{cases}$$

It is not difficult to show that

$$g \equiv -\max_{t,s\in I} G(t,s) = \frac{1}{2},$$

$$-\int_0^1 G(t,s)\,dt = \frac{8s-4s^3+s^4}{24} \geq -\frac{1}{3}\,G(\tau,s) \quad \text{for } t,s,\tau \in I.$$

Hence for $x_i \in (-1)^{\sigma_{i-1}}P$, $i = 1,2,3$, $x_4 \in L^1(I)$,

$$\begin{aligned}
&\|A_1(x_1,x_2,x_3,x_4)\|_{L^1} \\
&\quad = \int_0^1 \left| \int_0^1 G(t,s) f(s,x_1(s),x_2(s),x_3(s),x_4(s))\,ds \right| dt \\
&\quad = -\int_0^1 f(s,x_1(s),x_2(s),x_3(s),x_4(s))\,ds \int_0^1 G(t,s)\,dt \\
&\quad \geq -\frac{1}{3}\int_0^1 G(\tau,s) f(s,x_1(s),x_2(s),x_3(s),x_4(s))\,ds \\
&\quad = \frac{1}{3}|A_1(x_1,x_2,x_3,x_4)(\tau)|,
\end{aligned}$$

i.e., for problem (3.15), $\alpha_1 = (1/3)$ satisfies (2.13). Therefore

$$\frac{\sqrt{\alpha_1}}{g} = \frac{2}{\sqrt{3}}$$

and (H5)′ implies (H5). This completes the proof. □

Similarly, by Theorem 3.4, one can prove that

Corollary 3.2. *Suppose that* (i)′, (iii)′ *are satisfied and*

(H1)′ $\displaystyle\lim_{\|u\|\to+\infty} \frac{f(t,x_1,x_2,x_3,x_4)}{\|u\|} = 0$ *uniformly on* $t \in I$, *where* $\|u\| = \sum_{i=1}^4 |x_i|$.

(H3)′ $\displaystyle\lim_{x_1\to 0^-} \frac{f(t,x_1,x_2,x_3,x_4)}{\|u\|} = 0$ *uniformly on* $t \in I$, $x_2 \leq 0$, $x_3 \geq 0$, $x_4 \in R^1$.

(H6)′ *There exist* $\varepsilon > 0$ *and* $l \geq 1$ *such that*

$$f(t,x_1,x_2,x_3,x_4) \geq -\frac{\pi^4}{16}(1+\varepsilon)x_1 - \varepsilon \quad \text{for } -l \leq x_1 \leq 0.$$

Then (3.15) *has at least two negative solutions* x, y *such that* $0 < \|x\|_C < l < \|y\|_C$.

Proof. In fact one can show that $l \geq 1$ implies $l > \int_0^1 \frac{p(s)}{\delta_1}\, ds$ for problem (3.15).

Remark 3.2. The following constructed functions f_1 and f_2 satisfy all the hypotheses of Corollaries 3.1 and 3.2, respectively:

$$\begin{aligned} f_1(t, x, x', x'', x''') &= \frac{\pi^4 + 1}{48}(1 - (x+1)^3) - \frac{1}{\sqrt{3}} \frac{x|x'''|}{1 + (x''')^2}, \\ f_2(t, x, x', x'', x''') &= \frac{\pi^4}{4}(1 - (x+1)^{1/3}) - \frac{x\|u\|}{1 + \|u\|^2}, \end{aligned}$$

with $\|u\| = \sum_{i=0}^3 |x^{(i)}(t)|$.

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