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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS

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ABSTRACT. This work is a continuation of [**18,19,26–28**]. The concept of weak compatibility between a set-valued mapping and a single-valued mapping of Jungck and Rhoades [**19**] is used as a tool for proving some common fixed point theorems on metric spaces. Generalizations of known results, especially theorems by Fisher [**7**], are thereby obtained. As an application of this generalization, one example is given.

1. Introduction. In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [**11**], Jungck introduced more generalized commuting mappings, called *compatible mappings*, which are more general than commuting and weakly commuting mappings (Definition 1.4). This concept has been useful for obtaining more comprehensive fixed point theorems (see, e.g., [**1, 2, 4, 5, 9–18, 20–25, 29, 32, 34, 35**]).

Recently, Jungck and Rhoades [**18, 19**] defined the concepts of δ-compatible and weakly compatible mappings which extend the concept of compatible mappings in the single-valued setting to set-valued mappings. Several authors used these concepts to prove some common fixed point theorems (see, e.g., [**18, 19, 26–28**]).

Throughout this paper, let (X, d) be a complete metric space unless mentioned otherwise and $B(X)$ is the set of all nonempty bounded subsets of X. As in [**6, 8**], let $\delta(A, B)$ and $D(A, B)$ be the functions

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defined by

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$

for all A, B in $B(X)$.

If A consists of a single point a, we write $\delta(A,B) = \delta(a,B)$. If B also consists of a single point b, we write $\delta(A,B) = d(a,b)$.

It follows immediately from the definition that

$$\begin{aligned}
&\delta(A,B) = \delta(B,A) \geq 0,\\
&\delta(A,B) \leq \delta(A,C) + \delta(C,B),\\
&\delta(A,B) = 0 \quad \text{iff} \quad A = B = \{a\},\\
&\delta(A,A) = \operatorname{diam} A,
\end{aligned}$$

for all A, B, C in $B(X)$.

Definition 1.1 [**6**]. A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, \dots$, and $\{a_n\}$ converges to a.

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 1.1 [**6, 8**]. *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 1.2 [**8**]. *Let $\{A_n\}$ be a sequence in $B(X)$ and y a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Definition 1.2 [**8, 33**]. A set-valued mapping F of X into $B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point in X.

Lemma 1.3 [**8**]. *Let $\{A_n\}$ be a sequence of nonempty subsets of X and z in X such that*

$$\lim_{n\to\infty} a_n = z,$$

z independent of the particular choice of each $a_n \in A_n$. If a self-map I of X is continuous, then $\{Iz\}$ is the limit of the sequence $\{IA_n\}$.

Definition 1.3 [**33**]. The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be *weakly commuting on* X if $IFx \in B(X)$ and

$$\delta(FIx, IFx) \le \max\{\delta(Ix, Fx), \operatorname{diam} IFx\} \tag{1}$$

for all x in X.

Note that if F is a single-valued mapping, then the set IFx consists of a single point. Therefore, $\operatorname{diam} IFx = 0$ for all $x \in X$ and condition (1) reduces to the condition given by Sessa [**31**], that is,

$$d(FIx, IFx) \le d(Ix, Fx) \tag{2}$$

for all x in X.

Two *commuting mappings* F and I clearly *weakly commute* but two weakly commuting F and I do not necessarily commute as shown in [**33**].

In a recent paper, Jungck [**11**] generalized the concept of weakly commuting for single-valued mappings as follows:

Definition 1.4. Two single-valued mappings f and g of a metric space (X, d) into itself are *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X.

It can be seen that two *weakly commuting mappings* are *compatible* but the converse is false. Examples supporting this fact can be found in [**11**].

In [**18**], Jungck and Rhoades extended Definition 1.4 of compatibility to set-valued mappings setting as follows:

Definition 1.5. The mappings $I : X \to X$ and $F : X \to B(X)$ are *δ-compatible* if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$,

$$Fx_n \to \{t\} \quad \text{and} \quad Ix_n \to t$$

for some t in X.

The following definition is given by Jungck and Rhoades [**19**].

Definition 1.6. The mappings $I : X \to X$ and $F : X \to B(X)$ are *weakly compatible* if they commute at coincidence points, i.e., for each point u in X such that $Fu = \{Iu\}$, we have $FIu = IFu$. (Note that the equation $Fu = \{Iu\}$ implies that Fu is a singleton).

It can be seen that any *δ-compatible pair* $\{F, I\}$is *weakly compatible*. Examples of weakly compatible pairs which are not δ-compatible are given in [**19**].

In [**7**], Fisher proved the following theorem:

Theorem 1.1. *Let F, G be mappings of X into $B(X)$ and I, J be mappings of X into itself satisfying*

$$\delta(Fx, Gy) \le c \quad \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\},$$

for all $x, y \in X$, where $0 \le c < 1$. If F commutes with I and G commutes with J, $G(X) \subseteq I(X)$, $F(X) \subseteq J(X)$ and I or J is continuous, then F, G, I and J have a unique common fixed point u in X.

On the other hand, Fisher [**7**] proved the following fixed point theorem on compact metric spaces:

Theorem 1.2. *Let F, G be continuous mappings of a compact metric space (X, d) into $B(X)$ and I, J continuous mappings of X into itself* *satisfying the inequality*

$$\delta(Fx, Gy) < \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}, \tag{3}$$

for all $x, y \in X$ for which the righthand side of the inequality (3) is positive. If the mappings F and I commute and G and J commute and $G(X) \subset I(X)$, $F(X) \subset J(X)$, then there is a unique point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

The aim of the present paper is to prove a common fixed point theorem on complete metric spaces. Also, an example is given to satisfy our theorem. The result extends and generalizes Theorem 12 of Sastry and Naidu [**30**] and Theorem 1.1, respectively. At the end, a common fixed point theorem on compact metric spaces which generalizes Theorem 1.2 is verified.

2. Main results.

Theorem 2.1. *Let I, J be mappings of a metric space (X, d) into itself and $F, G : X \to B(X)$ set-valued mappings such that*

$$\cup F(X) \subseteq J(X), \quad \cup G(X) \subseteq I(X). \tag{4}$$

Also, the mappings I, J, F and G satisfy the following inequality:

(5)

$$\delta(Fx, Gy) \leq \alpha \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\} + (1-\alpha)[aD(Ix, Gy) + bD(Jy, Fx)],$$

for all $x, y \in X$, where

$$0 \leq \alpha < 1, \quad a + b < 1, \quad a \geq 0, \quad b \geq 0, \quad \alpha|a - b| < 1 - (a + b). \tag{6}$$

Suppose that one of $I(X)$ or $J(X)$ is complete. If both pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible, then there exists $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

Proof. Let x_0 be an arbitrary point in X. By (4), we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$. For this point x_1 there exists a point x_2 in X such that $Ix_2 \in Gx_1 = Z_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$Jx_{2n+1} \in Fx_{2n} = Z_{2n}, \quad Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \tag{7}$$

for $n = 0, 1, 2, \dots$. For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$, for $n = 0, 1, 2, 3, \dots$.

By (5) and (7), we have

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\le \alpha \max\{d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1})\} \\ &\quad + (1-\alpha)[aD(Ix_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, Fx_{2n})] \\ &\le \alpha \max\{\delta(Gx_{2n-1}, Fx_{2n}), \delta(Fx_{2n}, Gx_{2n+1})\} \\ &\quad + (1-\alpha)a\delta(Gx_{2n-1}, Gx_{2n+1}) \\ &\le \alpha \max\{V_{2n-1}, V_{2n}\} + (1-\alpha)a(V_{2n-1} + V_{2n}) \le \beta V_{2n-1}, \end{aligned}$$

for $n = 1, 2, 3, \dots$, where $\beta = \max\{(\alpha + (1-\alpha)a/1 - (1-\alpha)a), (a/1-a)\}$. The last inequality above, $\le \beta V_{2n-1}$, follows easily upon considering the cases: $V_{2n} \le V_{2n-1}$ and $V_{2n-1} \le V_{2n}$. Similarly, one can show that

$$V_{2n+1} \le \gamma V_{2n},$$

for $n = 0, 1, 2, \dots$, where $\gamma = \max\{(\alpha + (1-\alpha)b/1 - (1-\alpha)b), (b/1-b)\}$. Let $c = \beta\gamma$. If $a, b \in [0, (1/2))$, then $\beta < 1$ and $\gamma < 1$. So that $0 \le c < 1$.

If $\max\{a, b\} \ge 1/2$, then, since

$$\frac{\alpha + (1-\alpha)x}{1-(1-\alpha)x} \le \frac{x}{1-x} \Longleftrightarrow \frac{1}{2} \le x \quad \forall\, x \in [0, 1),$$

by hypotheses (6), it is easily seen that $0 \le c < 1$. Then we deduce that

(8)

$$V_{2n} = \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \le c^n \delta(Fx_0, Gx_1) = c^n V_0$$

and

(9)

$$V_{2n+1}=\delta(Z_{2n+1},Z_{2n+2})=\delta(Gx_{2n+1},Fx_{2n+2})\leq c^n\delta(Gx_1,Fx_2)=c^nV_1,$$

for $n=0,1,2,\dots$. We put

$$M=\max\{\delta(Fx_0,Gx_1),\delta(Gx_1,Fx_2)\}.$$

By inequalities (8) and (9), then if z_n is an arbitrary point in the set Z_n, for $n=0,1,2,3,\dots$, it follows that

$$d(z_{2n+1},z_{2n+2})\leq\delta(Z_{2n+1},Z_{2n+2})\leq c^n.M,$$
$$d(z_{2n+2},z_{2n+3})\leq\delta(Z_{2n+1},Z_{2n+2})\leq c^n.M.$$

Therefore the sequence $\{z_n\}$, and hence any subsequence thereof, is a Cauchy sequence in X.

Suppose that $J(X)$ is complete. Let $\{x_n\}$ be the sequence defined by (7). But $Jx_{2n+1}\in Fx_{2n}=Z_{2n}$, for $n=0,1,2,\dots$

$$d(Jx_{2m+1},Jx_{2n+1})\leq\delta(Z_{2m},Z_{2n})<\varepsilon,$$

for $m,n\geq n_0$, $n_0=1,2,3,\dots$. Therefore by the above, the sequence $\{Jx_{2n+1}\}$ is Cauchy and hence $Jx_{2n+1}\to p=Jv\in J(X)$, for some $v\in X$. But $Ix_{2n}\in Gx_{2n-1}=Z_{2n-1}$ by (7), so that we have

$$d(Ix_{2n},Jx_{2n+1})\leq\delta(Z_{2n-1},Z_{2n})=V_{2n-1}\to 0.$$

Consequently, $Ix_{2n}\to p$. Moreover, we have for $n=1,2,3,\dots$

$$\delta(Fx_{2n},p)\leq\delta(Fx_{2n},Ix_{2n})+\delta(Ix_{2n},p)=\delta(Z_{2n},Z_{2n-1})+d(Ix_{2n},p).$$

Therefore, $\delta(Fx_{2n},p)\to 0$. In like manner it follows that $\delta(Gx_{2n-1},p)\to 0$.

Since, for $n=1,2,3,\dots$,

$$\begin{aligned}\delta(Fx_{2n},Gv)&\leq\alpha\max\{d(Ix_{2n},Jv),\delta(Ix_{2n},Fx_{2n}),\delta(Jv,Gv)\}\\&\quad+(1-\alpha)[aD(Ix_{2n},Gv)+bD(Jv,Fx_{2n})]\\&\leq\alpha\quad\max\{d(Ix_{2n},Jv),\delta(Ix_{2n},Fx_{2n}),\delta(Jv,Gv)\}\\&\quad+(1-\alpha)[a\delta(Ix_{2n},Gv)+b\delta(Jv,Fx_{2n})]\end{aligned}$$

and since $\delta(Ix_{2n}, Gv) \to \delta(p, Gv)$ when $Ix_n \to p$, we get as $n \to \infty$

$$\delta(p, Gv) \le \alpha\delta(p, Gv) + (1-\alpha)a\delta(p, Gv) \Longrightarrow (1-\alpha)(1-a)\delta(p, Gv) \le 0.$$

Hence $Gv = \{p\} = \{Jv\}$, since $a < 1$. But $\cup G(X) \subseteq I(X)$, so $u \in X$ exists such that $\{Iu\} = Gv = \{Jv\}$. Now if $Fu \ne Gv$, $\delta(Fu, Gv) \ne 0$, so that we have

$$\begin{aligned}\delta(Fu, Gv) &\le \alpha \max\{d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)[aD(Iu, Gv) + bD(Jv, Fu)] \\ &\le \alpha \max\{d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)[a\delta(Iu, Gv) + b\delta(Jv, Fu)].\end{aligned}$$

So, we have

$$\delta(Fu, p) \le \alpha\delta(Fu, p) + (1-\alpha)b\delta(Fu, p) \Longrightarrow (1-\alpha)(1-b)\delta(Fu, p) \le 0$$

and $b < 1$; it follows that $Fu = \{p\} = Gv = \{Iu\} = \{Jv\}$.

Since $Fu = \{Iu\}$ and the pair $\{F, I\}$ is weakly compatible, we obtain $Fp = FIu = IFu = \{Ip\}$.

Using inequality (5), we have

$$\begin{aligned}\delta(Fp, p) &\le \delta(Fp, Gv) \\ &\le \alpha \max\{d(Ip, Jv), \delta(Ip, Fp), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)[aD(Ip, Gv) + bD(Jv, Fp)] \\ &\le \alpha\delta(Fp, p) + (1-\alpha)(a+b)\delta(Fp, p) \\ &\Longrightarrow (1-\alpha)[1-(a+b)]\delta(Fp, p) \le 0,\end{aligned}$$

and since $a + b < 1$, it follows that $\{p\} = Fp = \{Ip\}$. Similarly, $\{p\} = Gp = \{Jp\}$ if the pair $\{G, J\}$ is weakly compatible. Therefore, we obtain $\{p\} = \{Ip\} = \{Jp\} = Fp = Gp$.

To see the p is unique, suppose that $\{q\} = \{Iq\} = \{Jq\} = Fq = Gq$. If $p \ne q$, then

$$\begin{aligned}d(p, q) &\le \delta(Fp, Gq) \le \alpha d(p, q) + (1-\alpha)[ad(p, q) + bd(p, q)] \\ &\Longrightarrow (1-\alpha)[1-(a+b)]d(p, q) \le 0,\end{aligned}$$

and, since $a + b < 1$, it follows that $p = q$.

Remark 2.1. In Theorem 2.1, if F and G are single-valued mappings, then we obtain a generalization of Theorem 12 of Sastry and Naidu [**30**] for four single-valued mappings.

Remark 2.2. If we put $a = b = 0$ in Theorem 2.1, we obtain a generalization of Theorem 1.1.

Remark 2.3. As another generalization of Theorem 1.1, the authors [**26**, Theorem 2.1] proved a theorem by using the inequality

(10)

$$\delta(Fx, Gy) \leq \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)),$$

for all $x, y \in X$, where $\phi : [0, \infty)^5 \to [0, \infty)$ is a function which satisfies the following conditions.

(i) ϕ is upper semi-continuous from the right and nondecreasing in each coordinate variable,

(ii) for each $t > 0$

$$\Psi(t) = \max\{\phi(t, t, t, t, t), \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\} < t.$$

Condition (5) is not deducible from condition (10) since the function h of $[0, \infty)^5$ into $[0, \infty)$ defined as

$$h(t_1, t_2, t_3, t_4, t_5) = \alpha \max\{t_1, t_2, t_3\} + (1 - \alpha)[at_4 + bt_5],$$

for all t_1, t_2, t_3, t_4, t_5 in $[0, \infty)$, where a, b, α are as in condition (6), does not generally satisfy condition (ii). Indeed, we have that

$$\Psi(t) = t \max\{\alpha + (1 - \alpha)(a + b), \alpha + (1 - \alpha)(2a), \alpha + (1 - \alpha)(2b)\},$$

for all $t > 0$ and this does not imply $\Psi(t) < t$ for all $t > 0$.

It suffices to consider $\alpha = 1/4$, $a = 2/3$, $b = 1/6$ and then a, b, α satisfy

$$0 \leq \alpha < 1, \quad a \geq 0, \quad b \geq 0, \quad a + b < 1, \quad \alpha|a - b| < 1 - (a + b)$$

but $\Psi(t) = (5t/4) > t$, for all $t > 0$. Therefore Theorem 2.1 in [**26**] and Theorem 2.1 are two different generalizations of Theorem 1.1.

Now, we give an example to show that Theorem 2.1 is more general than Theorem 1.1.

Example. Let $X = [0, \infty)$ endowed with the Euclidean metric d. Define

$$Fx = [0, (x^6/6)], \quad Gx = [0, (x^3/6)],$$
$$Ix = x^6 + 6x^3, \quad Jx = \frac{x^{12}}{2} + x^6 + \frac{x^3}{2}$$

for all $x \in X$. We have

$$\bigcup F(X) = J(X) = \bigcup G(X) = I(X) = X.$$

For any sequence $\{x_n\}$ in X, we have

$$Ix_n \to 0 \quad \text{as} \quad x_n \to 0, \qquad Fx_n \to \{0\} \quad \text{as} \quad x_n \to 0,$$

and

$$\delta(FIx_n, IFx_n) = \max\left\{\frac{(x_n^6 + 6x_n^3)^6}{6}, \left(\frac{x_n^6}{6}\right)^6 + 6\left(\frac{x_n^6}{6}\right)^3\right\} \longrightarrow 0 \quad \text{as} \quad x_n \to 0,$$

$IFx_n \in B(X)$, thus F and I are δ-compatible and so they are weakly compatible. Similarly, G and J are δ-compatible and so they are weakly compatible.

For any $x, y \in X$, $x \neq y$

$$\begin{aligned}\delta(Fx, Gy) &= \max\left\{\frac{x^6}{6}, \frac{y^3}{6}\right\} \\ &= \max\left\{\frac{1}{3}\frac{x^6}{2}, \frac{1}{3}\frac{y^3}{2}\right\} \\ &\leq \max\left\{\frac{1}{3}(x^6 + 6x^3), \frac{1}{3}\left(\frac{y^{12}}{2} + y^6 + \frac{y^3}{2}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{3}\max\left\{\left|(x^6+6x^3)-\left(\frac{y^{12}}{2}+y^6+\frac{y^3}{2}\right)\right|,\right.\\
&\quad\left.(x^6+6x^3),\left(\frac{y^{12}}{2}+y^6+\frac{y^3}{2}\right)\right\}\\
&=\frac{1}{3}\max\{d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy)\}\\
&\leq \frac{1}{3}\max\{d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy)\}\\
&\quad+\left(1-\frac{1}{3}\right)\left[\frac{1}{4}D(Ix,Gy)+\frac{1}{5}D(Jy,Fx)\right].
\end{aligned}
$$

We see that the inequality (5) holds with $a = 1/4$, $b = 1/5$, $\alpha = 1/3$ and 0 is the unique common fixed point of I, J, F and G. Hence the hypotheses of Theorem 2.1 are satisfied. Theorem 1.1 is not applicable because F and G do not commute with I and J, respectively.

According to the technique of Chang [**3**], we prove the following theorem on compact metric spaces:

Theorem 2.2. *Let I, J be functions of a compact metric space (X, d) into itself and $F, G : X \to B(X)$ two set-valued functions with $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. Suppose that the inequality*

$$
\delta(Fx,Gy) < \alpha\max\{d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy)\} + (1-\alpha)[aD(Ix,Gy)+bD(Jy,Fx)], \tag{11}
$$

for all $x, y \in X$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$, $a \leq 1/2$, $b < 1/2$, $\alpha|a-b| < 1-(a+b)$, holds whenever the righthand side of (11) *is positive. If the pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible, and if the functions F and I are continuous, then there is a unique point u in X such that*

$$
Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.
$$

Proof. Let $\eta = \inf_{x\in X}\{\delta(Ix, Fx)\}$. Since X is a compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$
\delta(Ix_n, Fx_n) \to \eta \quad \text{as} \quad n \to \infty.
$$

Since

$$\delta(Ix_0, Fx_0) \leq d(Ix_0, Ix_n) + \delta(Ix_n, Fx_n) + \delta(Fx_n, Fx_0),$$

by the continuity of F and I and $\lim_{n\to\infty} x_n = x_0$ we get $\delta(Ix_0, Fx_0) \leq \eta$ and thus $\delta(Ix_0, Fx_0) = \eta$.

Since $\cup F(X) \subseteq J(X)$, there exists a point y_0 in X with $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \leq \eta$.

If $\eta > 0$, then

$$\begin{aligned}\delta(Jy_0, Gy_0) &\leq \delta(Fx_0, Gy_0)\\ &< \alpha \max\{d(Ix_0, Jy_0), \delta(Ix_0, Fx_0), \delta(Jy_0, Gy_0)\}\\ &\quad + (1-\alpha)[aD(Ix_0, Gy_0) + bD(Jy_0, Fx_0)]\\ &\leq \alpha \max\{\eta, \delta(Jy_0, Gy_0)\}\\ &\quad + (1-\alpha)a[d(Ix_0, Jy_0) + \delta(Jy_0, Gy_0)]\\ &\leq \alpha \max\{\eta, \delta(Jy_0, Gy_0)\} + (1-\alpha)a[\eta + \delta(Jy_0, Gy_0)].\end{aligned}$$

If $\delta(Jy_0, Gy_0) > \eta$ in the last inequality, then we obtain from $0 \leq \alpha < 1$ and $a \leq 1/2$ that

$$\delta(Jy_0, Gy_0) < [\alpha + 2(1-\alpha)a]\delta(Jy_0, Gy_0) \leq \delta(Jy_0, Gy_0).$$

This contradiction implies that $\delta(Jy_0, Gy_0) \leq \eta$.

Since $\cup G(X) \subseteq I(X)$, then there is a point z_0 in X such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) < \eta$. Hence we have from $0 \leq \alpha < 1$ and $b < 1/2$ that

$$\begin{aligned}\eta &\leq \delta(Iz_0, Fz_0) \leq \delta(Fz_0, Gy_0)\\ &< \alpha \max\{d(Iz_0, Jy_0), \delta(Iz_0, Fz_0), \delta(Jy_0, Gy_0)\}\\ &\quad + (1-\alpha)[aD(Iz_0, Gy_0) + bD(Jy_0, Fz_0)]\\ &\leq \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b\delta(Jy_0, Fz_0)\\ &\leq \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b[d(Jy_0, Iz_0) + \delta(Iz_0, Fz_0)]\\ &< \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b[\eta + \delta(Iz_0, Fz_0)]\\ &\leq [\alpha + 2(1-\alpha)b]\delta(Iz_0, Fz_0) < \delta(Iz_0, Fz_0).\end{aligned}$$

This contradiction demands that $\eta = 0$. Therefore, we have $Gy_0 = \{Jy_0\} = Fx_0 = \{Ix_0\} = \{Iz_0\}$.

Since F and I are weakly compatible and $Fx_0 = \{Ix_0\}$, we get $F^2x_0 = FIx_0 = IFx_0 = \{I^2x_0\}$.

If $I^2x_0 \neq Ix_0$, then we have

$$\begin{aligned} d(I^2x_0, Ix_0) &= \delta(F^2x_0, Gy_0) \\ &< \alpha \max\{d(IFx_0, Jy_0), \delta(IFx_0, F^2x_0), \delta(Jy_0, Gy_0)\} \\ &\quad + (1-\alpha)[aD(IFx_0, Gy_0) + bD(Jy_0, F^2x_0)] \\ &= \alpha d(I^2x_0, Ix_0) + (1-\alpha)(a+b)d(I^2x_0, Ix_0) \\ &= [\alpha + (1-\alpha)(a+b)]d(I^2x_0, Ix_0) \end{aligned}$$

and since $[\alpha + (1-\alpha)(a+b)] < 1$, then we have $I^2x_0 = Ix_0$. Hence $FIx_0 = \{Ix_0\} = \{I^2x_0\}$. Similarly, we have $GJy_0 = \{Jy_0\} = \{J^2y_0\}$. Let $u = Ix_0 = Jy_0$. Then $Fu = \{u\} = \{Iu\} = \{Ju\} = Gu$.

Suppose that the point y in X is a common fixed point of F, G, I and J with $y \neq u$. If either $\delta(y, Fy) \neq 0$ or $\delta(y, Gy) \neq 0$, then we have that

$$\begin{aligned} \delta(y, Fy) \leq \delta(Fy, Gy) &< \alpha \max\{d(y, y), \delta(y, Fy), \delta(y, Gy)\} \\ &\quad + (1-\alpha)[aD(y, Gy) + bD(y, Fy)] \\ &= \alpha \max\{\delta(y, Fy), \delta(y, Gy)\} \\ &\quad + (1-\alpha)[a\delta(y, Gy) + b\delta(y, Fy)] \\ &\leq \lambda\delta(y, Gy), \end{aligned}$$

where $\lambda = \max\left\{\dfrac{\alpha + (1-\alpha)a}{1-(1-\alpha)b}, \dfrac{a}{1-b}\right\} < 1$, it follows that $\delta(y, Fy) < \delta(y, Gy)$.

By symmetry, we have that $\delta(y, Gy) < \delta(y, Fy)$, which is impossible. So $\delta(y, Fy) = \delta(y, Gy) = 0$, that is, $Fy = Gy = \{y\}$.

Now

$$\begin{aligned} d(y, u) = \delta(Fy, Gu) &< \alpha \max\{d(y, u), \delta(y, Fy), \delta(u, Gu)\} \\ &\quad + (1-\alpha)[aD(y, Gu) + bD(u, Fy)] \\ &= \alpha d(y, u) + (1-\alpha)(a+b)d(y, u) \\ &= [\alpha + (1-\alpha)(a+b)]d(y, u) \end{aligned}$$

and since $\alpha + (1-\alpha)(a+b) < 1$, it follows that $u = y$, whence u is the unique common fixed point of F, G, I and J.

Remark 2.4. If we put $a = b = 0$ in Theorem 2.2, we obtain a generalization of Theorem 1.2.

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