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ON THE SPACE OF VECTOR-VALUED FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. If the space of all real-valued functions of bounded variation on a real closed interval is endowed with the topology of simple convergence, then every bounded subset which is bounded for the values of total variation is relatively sequentially compact by Helly's selection principle. In this paper, by treating vector-valued functions on a linearly ordered set, we consider an extension of this classical result.

1. Introduction. If the space $\mathrm{BV}[a,b]$ of all real-valued functions of bounded variation on the real closed interval $[a,b]$ is endowed with the topology of simple convergence, then, by Helly [**1**], every subset of $\mathrm{BV}[a,b]$, of the form $\{f \mid \max|f(x)| \leqq C, V_b^a[f] \leqq K\}$, is sequentially compact, where the positive numbers C, K are constants and $V_b^a[f]$ denotes the total variation of f.

In this paper, to give an interesting extension of this classical result, we consider a space of vector-valued functions on a linearly ordered interval of bounded variation, which take values in a locally convex space E. Then we examine conditions on E under which analogs of Helly's result are valid.

Let $E(\tau)$ be a sequentially complete Hausdorff locally convex space over the real or complex field. E' denotes the dual of $E(\tau)$. We write $\Gamma = \{p_\lambda \mid \lambda \in \Lambda\}$ for a system of saturated semi-norms on E generating the topology τ. A linearly ordered interval with a maximum and a minimum element is denoted by $[a,b]$ and its cardinal number is denoted by $\overline{\overline{[a,b]}}$. For simplicity we write $F(\tau_p)$ for the product space $\prod_{\alpha\in[a,b]} E_\alpha(\tau_\alpha)$, where $E_\alpha(\tau_\alpha) = E(\tau)$ for all $\alpha \in [a,b]$ and write F' for the direct sum $\oplus_{\alpha\in[a,b]} E'_\alpha$. The reader is referred to [**4**] for terminology used in this article.

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2. Definition.

DEFINITION. Let $[a,b]$ be a linearly ordered interval and $E(\tau)$ be a locally convex space. For an arbitrary E-valued function f defined on $[a,b]$, we set $V_\Delta(f,p_\lambda)=\sum_{i=0}^{n-1}p_\lambda(f(a_{i+1})-f(a_i))$ for every $p_\lambda\in\Gamma$ and for every finite subset $\Delta=\{a_i, 0\leqq i\leqq n, a=a_0<a_1<\cdots<a_n=b\}$ of $[a,b]$. Then the extended real number $V(f,p_\lambda)=\sup_\Delta V_\Delta(f,p_\lambda)$ is called the total variation of f on $[a,b]$ with p_λ.

The E-valued function $f(x)$ is said to be of bounded variation if $V(f,p_\lambda)<\infty$ for every $p_\lambda\in\Gamma$. The space of all E-valued functions on $[a,b]$ that are of bounded variation is denoted by $\mathrm{BV}_E[a,b]$, for simplicity denoted by BV_E.

3. On $\mathrm{BV}_{\mathbf{E}}[\mathbf{a},\mathbf{b}]$.

PROPOSITION 1. *BV_E is a dense subspace of $F(\tau_p)$.*

PROOF. BV_E contains $\oplus_{\alpha\in[a,b]}E_\alpha$ which is dense in $F(\tau_p)$. □

NOTATION 1. Using every semi-norm $p_\lambda\in\Gamma$, we define a semi-norm s_λ on BV_E by $s_\lambda(f)=p_\lambda(f(b))+V(f,p_\lambda)$ for all $f\in\mathrm{BV}_E$. The locally convex topology which is generated by the system of semi-norms $\{s_\lambda(\cdot)\}_{\lambda\in\Lambda}$ is denoted by τ_v.

If $\overline{\tau}_p$ is the induced topology on BV_E from $F(\tau_p)$, we obtain

PROPOSITION 2. *τ_v is finer than $\overline{\tau}_p$.*

PROOF. If, for each $\lambda\in\Lambda$, we take a τ_v-neighborhood $W=\{f|s_\lambda(f)\leqq 1\}$ of 0, then $p_\lambda(f(\gamma))\leqq p_\lambda(f(\gamma)-f(b))+p_\lambda(f(b))\leqq s_\lambda(f)\leqq 1$ for all $f\in W$ and $\gamma\in[a,b]$, which leads to the conclusion. □

Since it is assumed that $E(\tau)$ is sequentially complete, we can show

PROPOSITION 3. *$\mathrm{BV}_E(\tau_v)$ is sequentially complete.*

PROOF. Let $(f_n)_n$ be an arbitrary τ_v-Cauchy sequence. Since each sequence $(f_n(\alpha))_n, \alpha \in [a,b]$ is a Cauchy sequence in $E_\alpha(\tau_\alpha)$ by Proposition 2, we can define an E-valued function f on $[a,b]$ with $f(\alpha) = \lim_n f_n(\alpha)$ and easily see that f belongs to BV_E and f_n converges to f in $\mathrm{BV}_E(\tau_v)$. □

REMARK 1. (1) By Proposition 1, BV_E and F' form a dual pair.

(2) If the dual of $\mathrm{BV}_E(\tau_v)$ is denoted by $W_E[a,b]$ (We write W_E for short.), F' which is the dual of $\mathrm{BV}_E(\tau_p)$ is a subspace of W_E by Proposition 2.

Before giving the next proposition, we prepare some notations.

NOTATION 2. (1) For each $\lambda \in \Lambda$, let τ_λ be a locally convex topology on E generated by the semi-norm p_λ (τ_λ is not necessarily Hausdorff.). Then we denote by $E'(\lambda)$ the dual of $E(\tau_\lambda)$, which is a linear subspace of E'. We can give a well defined norm $||\cdot||_{\lambda'}$ on $E'(\lambda)$ such that $||u||_{\lambda'} = \min\{m \,|\, |\langle u, x\rangle| \leqq m \cdot p_\lambda(x),\ x \in E\}$.

(2) For each $\lambda \in \Lambda$, we can define a norm $||\cdot||_\infty^\lambda$ on $F'_\lambda = \oplus_{\alpha\in[a,b]} E'_\alpha(\lambda)$, where $E'_\alpha(\lambda) = E'(\lambda)$ for each $\alpha \in [a,b]$, such that $||v||_\infty^\lambda = \sup_{1\leqq j\leqq n} ||\sum_{i=1}^{j} v(a_i)||'_\lambda$ for all $v = (v(\alpha))_\alpha \in F'_\lambda$, provided each $v(a_i), 1 \leqq i \leqq n$, is a nonzero element of $E'_{a_i}(\lambda)$ and $a \leqq a_1 < a_2 < \cdots < a_n \leqq b$.

Now we obtain

PROPOSITION 4. *On the linear subspace* $F'_\lambda = \oplus_{\alpha\in[a,b]} E'_\alpha(\lambda)$ *of* $F', B_m^\lambda = \{u \,|\, u = (u(\alpha))_\alpha \in F'_\lambda, ||u||_\infty^\lambda \leqq m\}$ *for each* $m \in N$ *and, for each* $\lambda \in \Lambda$, *is* $\sigma(F', \mathrm{BV}_E)$*-bounded.*

PROOF. For each $f \in \mathrm{BV}_E$ and each $u \in B_1^\lambda$, we have

$$|\langle u, f\rangle| = \left|\sum_{j=1}^{n}\Big\langle \sum_{i=1}^{j} u(a_i), f(a_j) - f(a_{j+1})\Big\rangle + \Big\langle \sum_{i=1}^{n} u(a_i), f(b)\Big\rangle\right| \leqq s_\lambda(f),$$

where u vanishes except at $a_i, 1 \leqq i \leqq n$ and $a_{n+1} = b$. Hence this inequality shows the boundedness of B_1^λ. □

PROPOSITION 5. *The polar $B_m^{\lambda_0}$ in BV_E of B_m^λ in Proposition 4 is $\{f \mid s_\lambda \leqq 1/m, f \in \mathrm{BV}_E\}$ for each $m \in N$ and for each $\lambda \in \Lambda$. Hence each $s_\lambda, \lambda \in \Lambda$, is a $\beta(\mathrm{BV}_E, F')$-continuous semi-norm.*

PROOF. From the proof of Proposition 4, we have $s_\lambda(f) = V(f, p_\lambda) + p_\lambda(f(b)) = \sup\{|\langle u, f\rangle| \mid ||u||_\infty^\lambda \leqq 1, u \in F'_\lambda\}$. Hence we obtain $B_1^{\lambda_0} = \{f \mid s_\lambda(f) \leqq 1, f \in \mathrm{BV}_E\}$. Similarly, the polar $B_m^{\lambda_0}$ of B_m^λ in BV_E is shown by replacing 1 with $1/m$. □

PROPOSITION 6. *In BV_E, $\beta(\mathrm{BV}_E, F')$-boundedness is identical with τ_v-boundedness.*

PROOF. Since τ_v is finer than $\overline{\tau}_p$ on BV_E by Proposition 2, $\beta(\mathrm{BV}_E,\ W_E)$ is finer than $\beta(\mathrm{BV}_E, F')$. Further, $\beta(\mathrm{BV}_E, F')$ is finer than τ_v by Proposition 5. On the other hand, τ_v-bounded and $\beta(\mathrm{BV}_E, W_E)$-bounded subsets of BV_E are the same by Proposition 3 and the proposition in Köthe [**4** see Chapter 4, §20-11-(3)]. Hence $\beta(\mathrm{BV}_E, F')$-boundedness is identical with τ_v-boundedness. □

By Proposition 6, any $\beta(\mathrm{BV}_E, F')$-bounded subset is contained in a subset of the form $B = \{f \mid s_\lambda(f) \leqq M_\lambda, M_\lambda > 0$ for each $\lambda \in \Lambda, f \in \mathrm{BV}_E\}$. Finally we examine the closure in $F(\sigma(F, F'))$ of B.

PROPOSITION 7. *The closure in $F(\sigma(F, F'))$ of every $\beta(\mathrm{BV}_E, F')$-bounded subset is contained in BV_E.*

PROOF. As is mentioned above, it is sufficient to show that each $\beta(\mathrm{BV}_E, F')$-bounded subset $B = \{f \mid s_\lambda(f) \leqq M_\lambda\ (>0)$ for each $\lambda \in \Lambda, f \in \mathrm{BV}_E\}$ is closed in $F(\sigma(F, F'))$. By the proofs of Proposition 4 and 5, every subset $B_\lambda = \{f \mid s_\lambda(f) \leqq M_\lambda,\ M_\lambda > 0, f \in F\}, \lambda \in \Lambda$, is $\sigma(F, F')$-closed. Since $B = \cap_{\lambda\in\Lambda} B_\lambda \subset \mathrm{BV}_E$, we can easily verify the above fact.

4. Main theorem. In this section, we consider the following question: For what locally convex spaces $E(\tau)$ is every bounded subset of $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ such that, for each semi-norm $p_\lambda \in \Gamma$, the values of total variation are uniformly bounded (i.e., every τ_v-bounded subset) relatively compact? Since a subset B of $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is a bounded subset mentioned above if and only if B is strongly bounded by Proposition 6, the problem we consider is transformed as follows: under what locally convex space $E(\tau)$, is every strongly bounded subset of $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ relatively compact or β-semi-Montel (see Kitahara [**3**]). Now, using the propositions of §3, we can state

THEOREM. $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ *is β-semi-Montel if and only if* $\mathbf{E}(\tau)$ *is semi-reflexive. In particular,* $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ *is semi-Montel if and only if $E(\tau)$ is semi-reflexive and $\overline{\overline{[a,b]}}$ is finite.*

PROOF. If $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is β-semi-Montel, the closed subspace $E_o = \{f \mid f(a) = x,\ x \in E,\ f(y) = 0$ for all $y \in (a, b]\}$ of $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is β-semi-Montel by Proposition 5 in Kitahara [**3**]. Furthermore, $E_o(\sigma(\mathrm{BV}_E, F'))$ is linearly homeomorphic to $E(\sigma(E, E'))$ (see Chapter 4, §22-5-(3) in Köthe [**4**]). Since $E(\tau)$ is sequentially complete, $\sigma(E, E')$-boundedness is identical with $\beta(E, E')$-boundedness by the same argument as in the proof of Proposition 6. Thus an arbitrary closed $\sigma(E, E')$-bounded subset is $\sigma(E, E')$-compact. Conversely suppose that $E(\tau)$ is semi-reflexive and $F(\sigma(F, F'))$ is semi-Montel. Since an arbitrary closed $\beta(\mathrm{BV}_E, F')$-bounded subset in $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is $\sigma(F, F')$-bounded and closed in $F(\sigma(F, F'))$ by Proposition 7, it is $\sigma(\mathrm{BV}_E, F')$-compact. Hence $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is β-semi-Montel.

Suppose that $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is semi-Montel and $\overline{\overline{[a,b]}}$ is not finite. Then we can easily find a Cauchy sequence converging to a function of which the total variation is not finite, which leads to a contradiction. Finally, if $E(\tau)$ is semi-reflexive and $\overline{\overline{[a,b]}}$ is finite, $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is semi-Montel from $F = \mathrm{BV}_E$.

COROLLARY 1. *If $E(\tau)$ is a Banach space, $\mathrm{BV}_E(\sigma(\mathrm{BV}_E, F'))$ is β-semi-Montel if and only if $E(\tau)$ is reflexive.*

From sequence spaces, we set $bv = \{f \mid f \in K^N,\ K$ is R or C, $\lim_n \sum_{i=1}^n |f(i+1) - f(i)|$ exists.$\}$ (see Chapter 2, §2 in Kamthan and Gupta [**2**]).

COROLLARY 2. *The sequence space $bv(\tau_0)$ endowed with the topology of simple convergence is β-Montel.*

PROOF. As a linearly ordered interval, we consider the positive integers and infinity $\mathbf{Z}_\infty^+$ with the usual order. By Theorem, $\mathrm{BV}_K[\mathbf{Z}_\infty^+](\tau_s)$ is β-semi-Montel, where K is the real or complex field and τ_s is the topology of simple convergence. Then $\mathrm{BV}_K[\mathbf{Z}_\infty^+]_0(\overline{\tau}_s) = \{f \mid f \in \mathrm{BV}_K[\mathbf{Z}_\infty^+], f(\infty) = 0\}$ is linearly homeomorphic to $bv(\tau_0)$ and is a closed subspace of $\mathrm{BV}_K[\mathbf{Z}_\infty^+](\tau_s)$. Thus $bv(\tau_0)$ is β-semi-Montel by Proposition 5 in Kitahara [**3**]. Clearly, since each strongly bounded subset in the weak dual of $bv(\tau_0)$ is finite dimensional, $bv(\tau_0)$ is infrabarrelled. □

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