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COMPOUND MATRICES AND ORDINARY DIFFERENTIAL EQUATIONS

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This paper is dedicated to the memory of Geoffrey Butler

ABSTRACT. A survey is given of a connection between compound matrices and ordinary differential equations. Some typical linear results are presented. For nonlinear autonomous systems, a criterion for orbital asymptotic stability of a closed trajectory given by Poincaré in two dimensions is extended to systems of any finite dimension. A criterion of Bendixson for the nonexistence of periodic solutions of a two dimensional system is also extended to higher dimensional systems.

1. Introduction. Let X be any $n \times m$ matrix of real or complex numbers, and let $x^{j_1 \dots j_k}_{i_1 \dots i_k}$ denote the minor of X determined by the rows $(i_1, \dots, i_k)$ and the columns $(j_1, \dots, j_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, $1 \le j_1 < j_2 < \cdots < j_k \le m$. The k-th multiplicative compound $X^{(k)}$ of X is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries, written in lexicographic order, are $x^{j_1 \dots j_k}_{i_1 \dots i_k}$. In particular, when X is $n \times k$ with columns $x^1, \dots, x^k$, then $X^{(k)}$ is the exterior product $x^1 \wedge \cdots \wedge x^k$ represented as a column vector. The term "multiplicative" is used since the Binet-Cauchy Theorem [**13**, p. 17] states that

$$(AB)^{(k)} = A^{(k)} B^{(k)} \tag{1.1}$$

for any matrices A and B of dimension consistent with the multiplication. An immediate consequence of (1.1) is that, for any nonsingular $n \times n$ matrix X, $(X^{(k)})^{-1} = (X^{-1})^{(k)}$, since $I^{(k)}$ is clearly the $\binom{n}{k} \times \binom{n}{k}$ identity, if I is the $n \times n$ identity.

When $m = n$, the k-th additive compound $X^{[k]}$ of X is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined by

$$X^{[k]} = D(I + hX)^{(k)}|_{h=0}, \tag{1.2}$$

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where D denotes differentiation with respect to h. For any integer $i = 1, \dots, \binom{n}{k}$, let $(i) = (i_1, \dots, i_k)$ be the i-th member in the lexicographic ordering of all k-tuples of integers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$. Then, if $Y = X^{[k]}$, we have the following formula for y_i^j from (1.2):
(1.3)

$$y_i^j = \begin{cases} x_{i_1}^{i_1} + \cdots + x_{i_k}^{i_k}, & \text{if } (i) = (j), \\ (-1)^{r+s} x_{i_s}^{j_r}, & \text{if exactly one entry } i_s \text{ in } (i) \text{ does not occur} \\ & \text{in } (j) \text{ and } j_r \text{ does not occur in } (i), \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases}$$

In the special cases $k = 1, k = n$, we find

$$X^{[1]} = X, \quad X^{[n]} = \operatorname{Tr} X.$$

The term "additive" is used since

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}, \tag{1.4}$$

and, indeed, the map $X \to X^{[k]}$ is linear. This may be deduced directly from the Binet-Cauchy formula (1.1) and the definition (1.3), since

$$(I+hA)^{(k)}(I+hB)^{(k)} = ((I+hA)(I+hB))^{(k)} = (I+h(A+B)+h^2AB)^{(k)},$$

which implies (1.4). Alternatively, (1.3) implies (1.4).

The connection between compound matrices and differential equations is as follows. If $X(t)$ is any $n \times m$ matrix solution of the system

$$x' = A(t)x, \tag{1.5}$$

where $A(t)$ is a continuous real or complex matrix-valued function of the real variable t and $1 \le k \le \min\{m, n\}$, then $Y(t) = X^{(k)}(t)$ is a $\binom{n}{k} \times \binom{m}{k}$ matrix solution of the k-th compound system

$$y' = A^{[k]}(t)y. \tag{1.6}$$

To see this, we may suppose that $X(t)$ is $n \times n$ and nonsingular. Then (1.5) implies $X(t+h) = (I + hA(t))X(t) + o(h)$, near $h = 0$, and hence, from (1.1),

$$X^{(k)}(t + h) = (I + hA(t))^{(k)} X^{(k)}(t) + o(h).$$

Thus $Y(t) = X^{(k)}(t)$ is a fundamental matrix for (1.6), by (1.2). From the preceding discussion, we find that, for any $n \times n$ matrix A,

$$(\exp(A))^{(k)} = \exp(A^{[k]}). \tag{1.7}$$

In the case $k = 1$, (1.6) is the original equation (1.5) and, when $k = n$, $A^{[k]}(t) = \operatorname{Tr} A(t)$ and (1.6) is the well-known Abel-Jacobi-Liouville formula for the determinant of an $n \times n$ matrix solution of (1.5)

Although cases of the relationship between (1.5) and (1.6) are considered for special equations by Mikusínski [**15**], Nehari [**19**] and less directly by Hartman [**9**; Corollary 3.1, Chapter IV], the first treatment in full generality is due to Schwarz [**21**] who considers the question of when a fundamental matrix $X(t)$ satisfying $X(t_0) = I$ is totally positive or strictly totally positive for all $t \geq t_0$ and for each choice of t_0. Consideration of the compound equations (1.6) arises in a natural way in this study, and a complete answer is obtained to the question raised in the form of concrete necessary and sufficient conditions on the entries of $A(t)$ for total positivity and strict total positivity. In addition, comprehensive results are obtained which, broadly speaking, describe the properties of solutions of such systems with respect to oscillation.

In this paper, we wish to describe the connection between (1.6) and the dimension of certain sets of solutions to differential equations as demonstrated in [**17, 18**] and discussed in §3. We also show how such considerations give new information about questions of orbital stability and nonexistence of periodic trajectories for nonlinear systems.

Most of the developments on compound matrices have occurred in the context of linear and multilinear algebra. There is an extensive classical body of work dealing with the algebraic aspects of multiplicative compounds; for example, see [**1, 6, 10, 13, 14** and **24**]. Besides the Binet-Cauchy Theorem, the best-known results are due to Jacobi, Franke and Sylvester. Good historical accounts may be found in [**16**] and [**20**].

In contrast, the literature on additive compounds is quite sparse. In the final chapter of the lecture notes [**25**], Wielandt discusses algebraic and spectral properties of both multiplicative and additive compounds. The same approach is taken in the book of Marshall and Olkin [**14**]. Major applications are the work of Schwarz [**21**] on differential equations and of Fiedler [**5**] on stochastic matrices. London [**12**] derives a large number of interesting properties of additive compounds based on the relationship between (1.5) and (1.6) and shows how properties of compounds may be used to greatly simplify many classical spectral inequalities.

The elegant paper of Fiedler [**5**] presents algebraic aspects of additive and multiplicative compounds in a coordinate-free setting. Fiedler also identifies $k-1$ other generalized compounds associated with the k-th additive and multiplicative compounds. However, their pertinence to differential equations is not so obvious, so we do not discuss them here.

London [**12**] also shows how a relationship analogous to that between (1.5) and (1.6) may be developed from a Binet-Cauchy-type formula when the k-th multiplicative compound of X is replaced by the k-th induced matrix of X, essentially substituting permanents for determinants. A similar development is possible based on Kronecker powers and Kronecker sums of matrices. Some remarks on this are contained in the book of Bellman [**2**, Chapter 12]. It is interesting to note, however, that, while the spectral properties associated with these developments are quite similar to those of the analogous multiplicative and additive compounds, they do not seem to be as effective in extracting information about the solution space of (1.5) as the concepts considered here.

2. Spectral and metric properties of compounds. Spectral properties of compound matrices are readily deduced from the Binet-Cauchy formula (1.1). If A, J and T are $n \times n$ matrices such that $AT = TJ$, then (1.1) implies $A^{(k)}T^{(k)} = T^{(k)}J^{(k)}$. If J is triangular, then so also is $J^{(k)}$, and the diagonal entries of $J^{(k)}$ are products of diagonal entries of J taken k at a time. In particular, if $x^1, x^2, \ldots, x^k$ are independent eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, then the exterior product $x^1 \wedge x^2 \wedge \cdots \wedge x^k$ is an eigenvector of $A^{(k)}$ with corresponding eigenvalue $\lambda_1\lambda_2\cdots\lambda_k$. Similarly, $AT = TJ$ implies $(I + hA)T = T(I + hJ)$ and hence, by (1.1), $(I+hA)^{(k)}T^{(k)} = T^{(k)}(I+hJ)^{(k)}$. Thus, from (1.2), $A^{[k]}T^{(k)} = T^{(k)}J^{[k]}$ and $J^{[k]}$ is triangular if J is. The diagonal of $J^{[k]}$ is composed of sums of diagonal entries of J taken k at a time. Thus, if $x^1, x^2, \ldots, x^k$ are as before, then $x^1 \wedge x^2 \wedge \cdots \wedge x^k$ is an eigenvector of $A^{[k]}$ with eigenvalue $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

Let $|\cdot|$ denote any norm in $\mathbf{R}^n$ and the matrix norm which it induces. The Lozinskiĭ logarithmic norm of an $n \times n$ matrix A is then defined [**3**, p. 41] to be the right-hand derivative

$$\mu(A) = D_+|I + hA|_{h=0}. \tag{2.1}$$

It has the property that

$$|x(t)| \exp\left(-\int_{t_0}^{t} \mu(A)\right), \quad |x(t)| \exp\left(\int_{t_0}^{t} \mu(-A)\right)$$

are nonincreasing and nondecreasing, respectively, when x is a solution of (1.5). It follows, therefore, that a sufficient condition for the system (1.5) to be, respectively, stable; asymptotically stable; uniformly stable is that $\int_{t_0}^{t} \mu(A) \le K, \quad t_0 \le t < \infty$ (K independent of t); $\lim_{t\to\infty} \int_{t_0}^{t} \mu(A) = -\infty$; $\int_{s}^{t} \mu(A) \le M, \quad t_0 \le s \le t < \infty$ (M independent of s and t). If we replace $\mu(A)$ by $-\mu(-A)$ in these expressions, then we obtain a necessary condition for (1.5) to have the corresponding stability property.

The value of $\mu(A)$ depends on $|\cdot|$. In the cases $|x| = \sup_i |x_i|, \sum_i |x_i|$, $(\sum_i |x_i|^2)^{1/2}$, the Lozinskiĭ norm is given by

$$\mu(A) = \sup_i \left(\operatorname{Re} a_i^i + \sum_{j\neq i} |a_i^j|\right), \quad \sup_j \left(\operatorname{Re} a_j^j + \sum_{i\neq j} |a_i^j|\right), \quad \lambda_1,$$

respectively, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of

$(A^* + A)/2$. More generally, we have the following expressions, for $k = 1, \ldots, n$,

$$\mu(A^{[k]}) = \begin{cases} \sup_{(i)} \left[\operatorname{Re}(a_{i_1}^{i_1} + \cdots + a_{i_k}^{i_k}) + \sum_{j\notin(i)} (|a_{i_1}^{j}| + \cdots + |a_{i_k}^{j}|)\right] \\ \sup_{(j)} \left[\operatorname{Re}(a_{j_1}^{j_1} + \cdots + a_{j_k}^{j_k}) + \sum_{i\notin(j)} (|a_{i}^{j_1}| + \cdots + |a_{i}^{j_k}|)\right] \\ \lambda_1 + \cdots + \lambda_k, \end{cases} \tag{2.2}$$

respectively. Here $(i) = (i_1, \dots, i_k)$ is as described in §1.

It is noteworthy that the first two expressions given for $\mu(A)$ in the preceding paragraph are the upper bounds one obtains from Geršgorin's Theorem [**11**, §10.6] for the real parts of the eigenvalues of A. It is well-known that, in contrast to the autonomous case, the real parts of the eigenvalues of $A(t)$ being negative does not imply the stability of the system (1.5). However, the Geršgorin upper bounds on the real parts of the eigenvalues of $A(t)$ being negative does imply that the system is stable.

Geršgorin's Theorem also illustrates how the consideration of compounds leads to interesting new information: Every sum of k eigenvalues of A lies in at least one of the $\binom{n}{k}$ disks

$$\left\{z : |z - a^{i_1}_{i_1} - \cdots - a^{i_k}_{i_k}| \le \rho(i)\right\},$$

where

$$\rho(i) = \min\left\{\sum_{j \notin (i)} (|a^j_{i_1}| + \cdots + |a^j_{i_k}|), \quad \sum_{j \notin (i)} (|a^{i_1}_j| + \cdots + |a^{i_k}_j|\right\}.$$

Moreover, a set of m of these disks having no points in common with the remaining $\binom{n}{k} - m$ disks contains exactly m sums of k eigenvalues of A. This statement follows directly by applying Geršgorin's Theorem to $A^{[k]}$, whose eigenvalues are sums of k eigenvalues of A.

3. Linear differential equations. Let $\mathcal{X}$ be a subspace of $C([0,\infty) \to \mathbf{R}^n)$ and let $\mathcal{X}^{(k)}$ denote its k-th exterior power, $1 \le k \le n$:

$$\mathcal{X}^{(k)} = \mathrm{sp}\,\{x^1 \wedge x^2 \wedge \cdots \wedge x^k : x_i \in \mathcal{X}\}.$$

It will be assumed that $\mathcal{X}$ satisfies the conditions that, if $x \in \mathcal{X}$,

$$\limsup_{t\to\infty} |x(t)| < \infty, \tag{3.1}$$

$$\liminf_{t\to\infty} |x(t)| = 0 \Rightarrow \lim_{t\to\infty} x(t) = 0. \tag{3.2}$$

Further, we consider subspaces $\mathcal{X}_0, \mathcal{X}_0^{(k)}$ of $\mathcal{X}$, $\mathcal{X}^{(k)}$, respectively, defined by

$$\mathcal{X}_0 = \{x \in \mathcal{X} : \lim_{t\to\infty} x(t) = 0\}$$
$$\mathcal{X}_0^{(k)} = \{y \in \mathcal{X}^{(k)} : \lim_{t\to\infty} y(t) = 0\}.$$

THEOREM 3.1. *Let* $\mathcal{X}$ *satisfy* (3.1), (3.2). *Then*

$$\operatorname{codim} \mathcal{X}_0 < k \iff \mathcal{X}_0^{(k)} = \mathcal{X}^{(k)}.$$

This theorem includes some results of [**17**] and [**18**]. The reader is referred to [**17**] for a bibliography on results of this type, of which the following is typical.

COROLLARY 3.2. *Suppose the system* (1.5) *is uniformly stable. Then a necessary and sufficient condition that* (1.5) *have an* $(n-k+1)$-*dimensional set of solutions satisfying* $\lim_{t\to\infty} x(t) = 0$ *is that the system* (1.6) *be asymptotically stable.*

This follows by choosing $\mathcal{X}$ to be the solution space of (1.5) so that $\mathcal{X}^{(k)}$ is the solution space of (1.6). Uniform stability of (1.5) implies that $\mathcal{X}$ satisfies (3.1) and (3.2), and the condition $\mathcal{X}_0^{(k)} = \mathcal{X}^{(k)}$ is the asymptotic stability of (1.6). In more concrete terms, we have

COROLLARY 3.3. *Suppose there exists a constant* M *such that*

$$\int_s^t \mu(A) \le M, \quad 0 \le s \le t < \infty,$$

where M *is independent of* s, t. *Then* (1.5) *has an* $(n-k+1)$-*dimensional set of solutions* x *such that* $\lim_{t\to\infty} x(t) = 0$ *if*

$$\liminf_{t\to\infty} \int_0^t \mu(A^{[k]}) = -\infty$$

and only if, for $1 \le l \le n$,

$$\lim_{t\to\infty} \int_0^t \mu(-A^{[l]}) = \infty.$$

This is an extension of a result in [**17**]. Results of this type may also be proved, as in §5 of [**18**], when $\mathcal{X}$ is a stable subspace of the solution set of (1.5), as in the case of dichotomies.

PROOF OF THEOREM 3.1. Suppose $\operatorname{codim} \mathcal{X}_0 < k$ and let $x^1, x^2, \dots, x^k$ be linearly independent elements of $\mathcal{X}$. Then, there is a nontrivial $x \in \operatorname{sp}\{x^1, x^2, \dots, x^k\}$ such that $\lim_{t\to\infty} x(t) = 0$, which implies that $y = x^1 \wedge x^2 \wedge \cdots \wedge x^k$ satisfies $\lim_{t\to\infty} y(t) = 0$, by (3.1). Thus $\mathcal{X}_0^{(k)} = \mathcal{X}^{(k)}$. Conversely, suppose $\mathcal{X}_0^{(k)} = \mathcal{X}^{(k)}$ and let $x^1, x^2, \dots, x^k$ be any elements of $\mathcal{X}$. Then $y = x^1 \wedge x^2 \wedge \cdots \wedge x^k$ satisfies $\lim_{t\to\infty} y(t) = 0$. Now (3.1) implies that there exists a sequence $t_i \to \infty$ such that $\lim_{i\to\infty} X(t_i) = C$ exists where X is the $n \times k$ matrix whose columns are $x^1, x^2, \dots, x^k$. We have $C^{(k)} = \lim_{i\to\infty} y(t_i) = 0$ so that $\operatorname{rk} C < k$, and there exists a nonzero vector $\xi \in \mathbf{R}^n$ such that $C\xi = 0$. Therefore, $x = X\xi$ satisfies $\lim_{i\to\infty} x(t_i) = \lim_{i\to\infty} X(t_i)\xi = C\xi = 0$, which implies $\lim_{t\to\infty} x(t) = 0$, by (3.2). Thus $\operatorname{codim} \mathcal{X}_0 < k$. □

4. Nonlinear differential equations. Consider the nonlinear autonomous system

$$x' = f(x), \tag{4.1}$$

where $f \in C^1(\mathbf{R}^n \to \mathbf{R}^n)$. If $x_0 \in \mathbf{R}^n$, let $x = x(t, x_0)$ be the solution of (4.1) which satisfies $x(0, x_0) = x_0$. We consider the system of linear equations

$$y' = \frac{\partial f^{[k]}}{\partial x}(x(t, x_0))y, \quad k = 1, \dots, n, \tag{4.2}$$

where $\partial f/\partial x$ is the Jacobian matrix of f and $\partial f^{[k]}/\partial x$ denotes its k-th additive compound. The case $k = 1$ of (4.2) is the equation of first variation of (4.1) (cf. [**9**, Chapter V]), and the other cases of (4.2) are the various compound equations corresponding to the variational equation. In particular, when $k = n$, (4.2) is the scalar linear equation

$$y' = \operatorname{div} f(x(t, x_0))y. \tag{4.3}$$

The matrix $Y(t) = \partial x(t, x_0)/\partial x_0$ is a fundamental matrix for the variational equation and satisfies $Y(0) = I$. Thus $Y^{(k)}(t)$, the matrix of

Jacobian determinants $\partial(x_i, \dots, x_{i_k})(t, x_0)/\partial(x_{0j_1}, \dots, x_{0j_k})$, $1 \le i_1 < \cdots < i_k \le n$, $1 \le j_1 < \cdots < j_k \le n$ is the fundamental matrix for (4.2) satisfying $Y^{(k)}(0) = I$. From (4.3), we have the familiar formula [**9**; Chapter V, Corollary 3.1]

$$\frac{\partial(x_1, \dots, x_n)}{\partial(x_{01}, \dots, x_{0n})}(t, x_0) = \exp \int_0^t \operatorname{div} f(x(s, x_0))\, ds.$$

The equations (4.2) may be used to describe the local evolution in $\mathbf{R}^n$ of measures of k-dimensional surface content under the dynamics of (4.1). Let $\{dx_{01}, \dots, dx_{0n}\}$ be a basis for the vector space of differential 1-forms on $\mathbf{R}^n$. By the map $x_0 \to x(t, x_0)$, this basis corresponds to the basis $\{dx_1, \dots, dx_n\}$ given by

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial x_{0j}}(t, x_0) dx_{0j}, \quad i = 1, \dots, n.$$

Thus the basis evolves in time t as a solution of the variational equation of (4.1). The corresponding lexicographically ordered basis $\{dx_{x_{i_1}} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ for the differential k-forms in $\mathbf{R}^n$ therefore satisfies

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{(j)} \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(x_{0j_1}, \dots, x_{0j_k})}(t, x_0) dx_{0j_1} \wedge \cdots \wedge dx_{0j_k}, \tag{4.4}$$

and evolves in time as a solution of (4.2).

Let $h_0 \in C^1(\mathbf{R}^k \to \mathbf{R}^n)$, $h_0 = h_0(r)$, and consider

$$\sigma_k(h_0) = \int_D \left| \frac{\partial h_0}{\partial r_1} \wedge \cdots \wedge \frac{\partial h_0}{\partial r_k} \right| = \int_D \left| \frac{\partial h_0^{(k)}}{\partial r} \right|, \tag{4.5}$$

where $D \subset \mathbf{R}^k$ is the domain of h_0 and is assumed to be such that the integral exists. Now $\sigma_k(h_0)$ is a measure of the k-content of the k-surface h_0 defined by the norm $|\cdot|$. Different norms give different measures σ_k, but any two such measures are comparable in the sense that the corresponding norms are. Suppose $h_t(r) = x(t, h_0(r))$ is defined for all t in a neighborhood of 0 and $r \in D$. Then h_t is a k-surface with content σ_k given by

$$\sigma_k(h_t) = \int_D \left| \frac{\partial h_t^{(k)}}{\partial r} \right|. \tag{4.6}$$

Now the $n \times k$ matrix $\partial h_t/\partial r = (\partial x(t, y_0)/\partial x_0)(\partial h_0/\partial r)$ satisfies the variational equation (4.2), $k = 1$, with $x_0 = h_0$. Therefore $\partial h_t^{(k)}/\partial r$ satisfies (4.2) and the discussion in §2 implies

$$\tag{4.7} \begin{aligned} &\left|\frac{\partial h_s^{(k)}}{\partial r}\right| \exp\left[-\int_s^t \mu\left(-\frac{\partial f^{[k]}}{\partial x}(x(u,h_0))\right) du\right] \\ &\qquad \leq \left|\frac{\partial h_t^{(k)}}{\partial r}\right| \leq \left|\frac{\partial h_s^{(k)}}{\partial r}\right| \exp\left[\int_s^t \mu\left(\frac{\partial f^{[k]}}{\partial x}(x(u,h_0))\right) du\right] \end{aligned}$$

if $t \geq s$. From (4.6), (4.7) we find that $\sigma_k(h_t)$ decreases as t increases (respectively decreases) if the trace of the k-surface h_t is in a region where $\mu(\partial f^{[k]}/\partial x) < 0$ (respectively, $\mu(-\partial f^{[k]}/\partial x) < 0$). For each of the specific vector norms considered in §2, the corresponding Lozinskiĭ norm $\mu(\partial f^{[k]}/\partial x)$ is given by the expression (2.2) with $a_i^j = \partial f_i/\partial x_j$. In the case $k = n$ we find from (4.3) that the map $x_0 \to x(t, x_0)$ decreases Lebesgue measure in $\mathbf{R}^n$ when $t > 0$ (respectively, $t < 0$) if $\operatorname{div} f < 0$ (respectively $\operatorname{div} f > 0$).

The main results of this section are extensions to higher dimensions of two results for the system (4.1) when $n = 2$: Bendixson's negative criterion and Poincaré's stability criterion. These results and their generalizations are as follows.

Bendixson's Negative Criterion. *When $n = 2$, a sufficient condition for the nonexistence of nonconstant periodic solutions of* (4.1) *is that, for each $x \in \mathbf{R}^2$,*

$$\operatorname{div} f(x) \neq 0.$$

Theorem 4.1. *Suppose that one of the inequalities*

$$\mu\left(\frac{\partial f^{[2]}}{\partial x}\right) < 0, \quad \mu\left(-\frac{\partial f^{[2]}}{\partial x}\right) < 0$$

holds for all $x \in \mathbf{R}^n$. Then the system (4.1) *has no nonconstant periodic solutions.*

POINCARÉ'S STABILITY CRITERION. *When* $n = 2$, *a periodic orbit* $\gamma = \{p(t) : 0 \le t \le \omega\}$ *of* (4.1) *is orbitally asymptotically stable if*

$$\int_0^\omega \operatorname{div} f(p(t))\, dt < 0.$$

THEOREM 4.2. *A sufficient condition for a periodic trajectory* $\gamma = \{p(t) : 0 \le t \le \omega\}$ *of* (4.1) *to be orbitally asymptotically stable is that the linear system*

$$y' = \frac{\partial f^{[2]}}{\partial x}(p(t))y$$

be asymptotically stable.

COROLLARY 4.3. *Suppose that, for some Lozinskiĭ norm* μ,

$$\int_0^\omega \mu\left(\frac{\partial f^{[2]}}{\partial x}(p(t))\right) dt < 0.$$

Then γ *is orbitally asymptotically stable.*

When $n = 2, \partial f^{[2]}/\partial x = \operatorname{Tr} \partial f/\partial x = \operatorname{div} f$, so that Theorems 4.1 and 4.2 give the results of Bendixson and Poincaré, respectively, in that case. By (2.2), any one of the following three expressions may be used as $\mu(\partial f^{[2]}/\partial x)$ in Theorem 4.1 and Corollary 4.3:

$$\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \ne r,s} \left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right| : r, s = 1, \ldots, n, r \ne s\right\},$$

$$\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \ne r,s} \left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right| : r, s = 1, \ldots, n, r \ne s\right\},$$

$$\lambda_1 + \lambda_2,$$

where λ_1, λ_2 are the two largest eigenvalues of $(\partial f^*/\partial x + \partial f/\partial x)/2$.

The usual proof of Bendixson's criterion based on Green's theorem in the plane does not lend itself readily to higher dimensional generalization. However, the criterion may also be established by recalling from the preceding discussion that, if $n = 2$ and $\operatorname{div} f < 0$ (respectively $\operatorname{div} f > 0$) on $\mathbf{R}^2$, then the flow defined by (4.1) is area diminishing (respectively enhancing). But the area bounded by a nontrivial periodic trajectory is a positive constant under the flow, thus contradicting $\operatorname{div} f \neq 0$. When $n \geq 2$ and $|x| = (\sum_i x_i^2)^{1/2}$, the result of Douglas [**4**] implies that any nontrivial periodic trajectory γ is the boundary of a surface for which the corresponding area σ_2 is a minimum and positive. The condition of Theorem 4.1 implies that γ is invariant and σ_2 is decreased by the flow as t increases (decreases), contradicting the minimality of the surface area so that no such periodic trajectory exists. The idea of this proof is extended to general norms.

PROOF OF THEOREM 4.1. Suppose $\gamma = \{p(t) : 0 \leq t \leq \omega\}$ is the trajectory of a nontrivial ω-periodic solution p of (4.1). Consider the family $\mathcal{H}$ of simply connected surfaces $h \in C^1(\mathbf{R}^2 \to \mathbf{R}^n), h = h(r,s), (r,s) \in [0,\omega]\times[0,1]$ whose boundary is γ. For example, $h(r,s) = (1-s)p(0) + sp(r)$ is such a surface. Let $\delta = \inf\{\sigma_s(h) : h \in \mathcal{H}\}$. The result of Douglas [**4**] implies $\delta > 0$ in the case of the norm $|x| = (\sum_i x_i^2)^{1/2}$, and this implies $\delta > 0$ for the area σ_2 corresponding to any norm. Let $\mathcal{H}_0 \subset \mathcal{H}$ be a sequence of surfaces h_0 whose traces lie in a ball B. Then, for some $\varepsilon > 0, h_t(r,s) = x(t, h_0(r,s))$ exists for all $(r,s) \in [0,\omega] \times [0,1], t \in [-\varepsilon, \varepsilon]$ and for all $h_0 \in \mathcal{H}_0$. Then $h_t \in \mathcal{H}$ and, if $\mu(\partial f^{[2]}/\partial x) < 0$, we find from (4.5), (4.6), (4.7) that there is a number $\eta \in (0,1)$ such that

$$\sigma_2(h_\varepsilon) \leq \eta\sigma_2(h_0)$$

if $h_0 \in \mathcal{H}_0$. By choosing $\mathcal{H}_0$ to be a minimizing sequence for σ_2, we find

$$0 < \delta \leq \eta\delta, \quad 0 < \eta < 1,$$

since we may choose h_0 so that $\sigma_2(h_0)$ is arbitrarily close to δ and $\delta \leq \sigma_2(h_\varepsilon)$ by definition. This contradiction shows that no such trajectory γ exists in the case $\mu(\partial f^{[2]}/\partial x) < 0$. When $\mu(-\partial f^{[2]}/\partial x) < 0$, we consider the evolution of the surfaces $\mathcal{H}_0$ backward in time rather than forward. $\square$

REMARKS.

(a) When $n = 2$, it is sometimes convenient to assume that the Bendixson criterion is satisfied only on some open subset E of $\mathbf{R}^2$. If E is simply connected, no nontrivial periodic orbit of (4.1) lies in E. If E is multiply connected, one easily finds a bound on the number of closed orbits based on the connectivity of E. When $n > 2$, it seems to be necessary to impose additional restrictions on the structure of E or the dynamics of (4.1) as in Theorem 8 of [**23**].

(b) For example, if E is open and convex such that $\mu(\partial f^{[2]}/\partial x) < 0$ (or $\mu(-\partial f^{[2]}/\partial x) < 0$) in E, the proof given above shows that E cannot contain a simply connected 2-surface whose boundary is invariant under the flow of (4.1). In particular, E contains no nontrivial periodic trajectory.

(c) Suppose E is open with compact closure, is positively (or negatively) invariant with respect to (4.1) and $\mu(\partial f^{[2]}/\partial x) < 0$ (or $\mu(-\partial f^{[2]}/\partial x) < 0$) in E. Suppose also that $\gamma \subset E$ is such that all smooth 2-surfaces h with boundary γ and trace in E satisfy $0 < \delta \leq \sigma_2(h)$, and that one such surface h_0 satisfies $\sigma_2(h_0) < \infty$. Then γ cannot be invariant. An argument similar to that used in the proof of Theorem 4.1 shows $0 < \delta \leq \sigma_2(h_t) \to 0, \quad t \to \infty$ (or $t \to -\infty$), a contradiction.

(d) If $0 < r \in C^1(\mathbf{R}^n \to \mathbf{R})$, then the orbits of $x' = r(x)f(x)$ are the same as those of (4.1). This can be seen by making the transformation $y(s) = x(t), s = \int_0^t r(x(u))\,du$, in this system. Thus we may replace f by rf in the statement of Theorem 4.1 without altering the conclusion of the theorem. The corresponding modification of Bendixson's criterion is due to Dulac (cf. [**22**]).

Poincaré's stability criterion pertains to the orbital stability of a periodic trajectory associated with (4.1). Suppose the system has a periodic solution $x = p(t)$ with least period $\omega > 0$ and trajectory $\gamma = \{p(t) : 0 \leq t \leq \omega\}$. This trajectory is *orbitally stable* if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less than δ, remains at a distance less than ε from γ for all $t \geq 0$. It is *orbitally asymptotically stable* if the distance of $x(t)$ from γ also tends to zero as $t \to \infty$.

PROOF OF THEOREM 4.2. Let $x = p(t)$ be a nontrivial ω-periodic solution of (4.1). Then the variational equation (4.2), $k = 1, x(t, x_0) = p(t)$, is a linear system with ω-periodic coefficient matrix $\partial f(p(t))/\partial x$. By Floquet's theorem [**3**, p. 47], a fundamental matrix $Y(t)$ of (4.2), $k = 1$, may be written in the form

$$Y(t) = P(t)\exp(Lt), \tag{4.8}$$

where the $n \times n$ matrices $P(t), L$ are ω-periodic and constant, respectively. The stability character of (4.2), $k = 1$, is, therefore, determined by the eigenvalues of L which are called the *characteristic exponents*. Since $y = p'(t)$ is a nontrivial periodic solution of (4.2), $k = 1$, it follows that one of the characteristic exponents is equal to zero (mod $2\pi i/\omega$). A fundamental result in stability theory is that γ is orbitally asymptotically stable if the remaining $n - 1$ characteristic exponents have negative real part. There are several essentially different proofs of this fact in the literature; for example, see [**3, 7, 8, 9**]. Now, the equation (4.2), $k = 2, x(t, x_0) = p(t)$, has fundamental matrix

$$Y^{(2)}(t) = P^{(2)}(t)\exp(L^{[2]}t)$$

by (1.1), (1.7) and (4.8). The characteristic exponents of (4.2), $k = 2$, are thus the eigenvalues of $L^{[2]}$ which are sums of pairs of eigenvalues of L. Since L has at least one eigenvalue zero, it follows that all the remaining $(n - 1)$ eigenvalues of L are also eigenvalues of $L^{[2]}$. These eigenvalues must, therefore, all have negative real part since $Y^{(2)}(t) \to 0, t \to \infty$. Hence, γ is orbitally asymptotically stable.

The condition of Corollary 4.3 may be shown to be a sufficient condition for the system (4.2) with $k = 2$, $x(t, x_0) = p(t)$, to be asymptotically stable. Thus the assertion about orbital asymptotic stability follows from Theorem 4.2.

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