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SPACES ON WHICH UNCONDITIONALLY CONVERGING OPERATORS ARE WEAKLY COMPLETELY CONTINUOUS

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ABSTRACT. Let Ω be a compact Hausdorff space, and let E be a Banach space with unconditional reflexive decomposition, then every unconditionally converging operator T on $C(\Omega, E)$, the space of E-valued continuous functions on Ω, is weakly completely continuous, i.e., T sends weakly Cauchy sequences into sequences that converge weakly.

Introduction. Let $T : X \to Y$ be a bounded linear operator from a Banach space X into a Banach space Y. We say that T is *weakly compact* (w.c.) if for every bounded sequence (x_n) in X, there is a subsequence (x_{n_k}) such that (Tx_{n_k}) converges weakly in Y. We say that T is *weakly completely continuous* (w.c.c.) (also called Dieudonné operator) if for every weakly Cauchy sequence (x_n) in X, the sequence (Tx_n) converges weakly in Y, and we say that T is *unconditionally converging* (u.c.) if for every weakly unconditionally Cauchy series (w.u.c.) $\sum_n x_n$ in X, the series $\sum_n Tx_n$ converges unconditionally in Y. Here recall that a series $\sum_n x_n$ is weakly unconditionally Cauchy if for each x^* in X^* the series $\sum_n |x^*(x_n)|$ is convergent. It is clear that T weakly compact implies T weakly completely continuous which in turn implies T unconditionally converging. In his fundamental paper [**9**] A. Pelczynski looked at spaces on which every unconditionally converging operator is weakly compact. Such spaces are said to have Pelczynski's property (V). In [**9**] Pelczynski showed that among classical Banach spaces, the spaces $C(\Omega)$ of scalar-valued continuous functions on a compact Hausdorff space Ω have property (V), and in [**7**] W. Johnson and M. Zippin showed that more generally any Banach space whose dual is isometric to an L^1 space have property (V). Also in [**9**] spaces with property (u) were introduced; for this recall that a Banach space E has *property* (u) if for any weakly Cauchy sequence (e_n) in E there

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exists a weakly unconditionally Cauchy series (w.u.c) $\sum_n x_n$ in E such that the sequence $(e_n - \sum_{i=1}^n x_i)$ converges weakly to zero in E. Any Banach space E with unconditional basis or more generally any space with unconditional reflexive decomposition has property (u) and so is the case of any weakly sequentially complete Banach space and any order continuous Banach lattice [**8**]. It is clear that if a Banach space E has property (u), then every unconditionally converging operator on E is weakly completely continuous. The main object of this paper is to see what other Banach spaces share the above property with spaces that have property (u). For this, let us fix some notations and terminology. If Ω is a compact Hausdorff space and if E is a Banach space, we let $C(\Omega, E)$ stand for the Banach space of all E-valued continuous functions on Ω under supnorm. In this paper we shall show that if E is a Banach space with unconditional reflexive decomposition and if Ω is a compact Hausdorff space, then every unconditionally converging operator on $C(\Omega, E)$ is weakly completely continuous. For all undefined notions, we refer the reader to [**5**] or [**8**].

The main result. Throughout, Ω is a compact Hausdorff space, E is a Banach space and $C(\Omega, E)$ stands for the Banach space of continuous E-valued functions on Ω under supnorm. We shall denote by $M(\Omega, E^*)$ the space of all regular E^*-valued vector measures μ defined on the σ-field of Borel subsets of Ω that are of bounded variation. It is well known [**5**, p. 182] that $M(\Omega, E^*)$ is a Banach space under the variation norm $||\mu|| = |\mu|(\Omega)$ and that $M(\Omega, E^*)$ is isometrically isometric to the dual $C(\Omega, E)^*$ of $C(\Omega, E)$. Recall also that if F is a Banach space, then any bounded linear operator $T : C(\Omega, E) \to F$ has a finitely additive representing measure G that is defined on the σ-field of Borel subsets of Ω and that takes its values in $L(E, F^{**})$, the space of all bounded linear operators from E into the second dual F^{**} of F. Among many properties of G we should point out that G is said to be of bounded semi-variation. If for each $y^* \in F^*$ we denote by G_{y^*} the element of $M(\Omega, E^*)$ defined by

$$\langle G_{y^*}(B), e\rangle = y^*(\langle G(B), e\rangle)$$

for each Borel subset B of Ω and each e in E, the *semi-variation* of G is the extended nonnegative function $||G||$ whose value on a Borel subset B of Ω is given by

$$||G||(B) = \sup\{|G_{y^*}|(B) : y^* \in F^*, ||y^*|| \leq 1\}.$$

For a series $\sum_n x_n$ in a Banach space X we say that $\sum_n x_n$ is a *weakly unconditionally Cauchy series* in X if it satisfies one of the following equivalent statements

a) $\sum_n |x^*(x_n)| < \infty$, for every $x^* \in X^*$;

b) $\sup \{|| \sum_{n\in\sigma} x_n|| : \sigma \text{ finite subset of } \mathbf{N}\} < \infty$

c) $\sup_n \sup_{\sigma_i=\pm 1} || \sum_{i=1}^n \sigma_i x_i|| < \infty$.

Our first lemma is well known; its proof is an easy consequence of the Lebesgue bounded convergence theorem and of the fact that for any compact Hausdorff space Ω and any Banach space E, $C(\Omega, E)$ embeds isometrically in the space $C(\Omega \times B(E^*))$ of the scalar-valued continuous functions on the product of Ω and the dual unit ball $B(E^*)$ with its weak*-topology, where the embedding associates to each element ϑ in $C(\Omega, E)$ the element $\bar{\vartheta}$ in $C(\Omega \times B(E^*))$ defined by

$$\bar{\vartheta}(\omega, e^*) = e^*(\vartheta(\omega)) \qquad \omega \in \Omega, e^* \in E^*.$$

Lemma 1. *Let Ω be a compact Hausdorff space and let E be a Banach space. A bounded sequence (f_n) is weakly Cauchy in $C(\Omega, E)$ if and only if for each $\omega \in \Omega$, the sequence $(f_n(\omega))$ is weakly Cauchy in E.*

Finally, we say that a Banach space has an *unconditional Schauder decomposition into subspaces* $\{E_n\}$, if each $x \in E$ can be uniquely written as $x = \sum_n x_n$ with each $x_n \in E_n$ and such that the series $\sum_n x_n$ converges unconditionally in E. If a Banach space has such an unconditional decomposition $\{E_n\}$, then for each $n \geq 1$, we will denote by Q_n the bounded linear operator on E defined by $Q_n\ (\sum_{i=1}^\infty x_i) = \sum_{i=1}^n x_i$. We will also denote by $P_1 = Q_1$ and $P_n = Q_n - Q_{n-1}$ for $n \geq 2$. Here what is important to note is that when the decomposition is unconditional, then $\sup\{|| \sum_{n\in\sigma} P_n|| : \sigma \text{ finite subset of } \mathbf{N}\} < \infty$. If the decomposition is unconditional and E_n is reflexive for each $n \geq 1$, then we say that E has an *unconditional reflexive decomposition.* With these notations, we can now state our next Lemma which basically tells us how well some spaces with unconditional decomposition enjoy property (u).

Lemma 2. *If E is a Banach space that has an unconditional reflexive* *decomposition* $\{E_n\}$ *and if* (y_k) *is a weakly Cauchy sequence in* E *with* y^{**} *its weak** *limit in* E^{**}, *then*

$$y^{**} = \text{weak}^* \sum_n P_n^{**}(y^{**}).$$

Proof. Let $x^{**} \in E^{**}$. Let $P_n : E \to E_n$ be the projection associated to the decomposition with range E_n, then $P_n^{**} : E^{**} \to E_n$, since each space E_n is reflexive. Notice now that the series $\sum_n P_n^{**}(x^{**})$ is a weakly unconditionally Cauchy series since for each finite subset σ of $\mathbf{N}$

$$\begin{aligned}\left\|\sum_{n\in\sigma} P_n^{**}(x^{**})\right\| &\le \|x^{**}\| \left\|\sum_{n\in\sigma} P_n^{**}\right\| \\ &= \|x^{**}\| \left\|\sum_{n\in\sigma} P_n\right\| \\ &\le C||x^{**}||\end{aligned}$$

where $C = \sup\{||\sum_{n\in\sigma} P_n|| : \sigma$ finite subset of $\mathbf{N}\}$ is finite because the decomposition $\{E_n\}$ is unconditional. So what the lemma is asserting is that any element y^{**} in E^{**} that is the weak* limit of a weakly Cauchy sequence in E can be realized as the weak* sum of a very special weakly unconditionally Cauchy series in E. Let y^{**} in E^{**} such that $y^{**} = \text{weak}^* \lim_k y_k$ with (y_k) a weakly Cauchy sequence in E. We need to show that for each $z^* \in E^*$

$$y^{**}(z^*) = \sum_n \langle P_n^{**}(y^{**}), z^*\rangle.$$

For this, let $z^* \in E^*$ be fixed and note that since the decomposition $\{E_n\}$ is unconditional, one can easily show that the series $\sum_n P_n^*(z^*)$ is a weakly unconditionally Cauchy series in E^*. Since (y_k) is a weakly Cauchy sequence in E, it follows from [**9**] that

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} |\langle P_n^*(z^*), y_k\rangle| = 0$$

uniformly for all $k \ge 1$. Fix $\varepsilon > 0$, then there exists $N_0 > 0$ such that

$$\sum_{n=N_0+1}^{\infty} |\langle P_n^*(z^*), y_k\rangle| < \varepsilon/3$$

for all $k \geq 1$. Since y^{**} is the weak* limit of the sequence (y_k) we can find $K > 0$ such that for all $k \geq K$, we have

$$|z^*(y^{**}) - z^*(y_k)| < \varepsilon/3$$

and

$$\sum_{n=1}^{N_0} |P_n^*(z^*)(y_k - y^{**})| < \varepsilon/3.$$

Hence, for $k \geq K$, we have

$$\begin{aligned}
&\left| y^{**}(z^*) - \sum_{n=1}^{N_0} \langle P_n^*(z^*), y^{**} \rangle \right| \\
&\qquad \leq |y^{**}(z^*) - z^*(y_k)| + \left| z^*(y_k) - \sum_{n=1}^{N_0} \langle P_n^*(z^*), y^{**} \rangle \right| \\
&\qquad \leq \varepsilon/3 + \left| \sum_{n=1}^{\infty} \langle P_n(y_k), z^* \rangle - \sum_{n=1}^{N_0} \langle P_n^*(z^*), y^{**} \rangle \right| \\
&\qquad \leq \varepsilon/3 + \left| \sum_{n=1}^{N_0} \langle P_n^*(z^*)(y_k - y^{**}) \rangle \right| + \left| \sum_{n=N_0+1}^{\infty} \langle P_n^*(z^*), y_k \rangle \right| \\
&\qquad \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3.
\end{aligned}$$

This completes the proof. □

We are now ready to state and prove the main result of this paper. The proof refines ideas found in [**2**] and [**3**].

Theorem 3. *Let Ω be a compact Hausdorff space, and let E be a Banach space with unconditional reflexive decomposition $\{E_n\}$. Let F be a Banach space. If $T : C(\Omega, E) \to F$ is an unconditionally converging operator, then T is weakly completely continuous.*

Proof. Let $T : C(\Omega, E) \to F$ be unconditionally converging with G its representing vector measure. It follows from [**6**] that there exists a nonnegative scalar measure λ on Ω such that

$$(\dagger) \qquad \lim_{\lambda(B) \to 0} ||G||(B) = 0.$$

Without loss of generality, we may assume that Ω is metrizable, for one can proceed as in [**1**] (see also [**3**]) to reduce to the case where Ω is metrizable.

Let $(f_n)_{n\geq 1}$ be a weakly Cauchy sequence in $C(\Omega, E)$. Then for each $\omega \in \Omega$, there is $f(\omega) \in E^{**}$ such that

$$f(\omega) = \text{weak}^* \lim_n f_n(\omega).$$

Without loss of generality, we may assume that $||f_n(\omega)|| \leq 1$ for all $n \geq 1$; hence, for all $\omega \in \Omega$, $||f(\omega)|| \leq 1$. By Lemma 2, we have

$$f(\omega) = \text{weak}^* \sum_{n=1}^{\infty} P_n^{**}(f(\omega)).$$

For each $n \geq 1$, let $\Psi_n(\omega) = P_n^{**}(f(\omega))$ for each $\omega \in \Omega$. Since $P_n^{**} : E^{**} \to E_n$, then Ψ_n takes its values in E_n. We claim that $\Psi_n : \Omega \to E_n$ is λ-measurable. For this, note that for each $x^* \in E_n^*$, $P_n^* x^* \in E^*$, hence for each $k \geq 1$, the mapping $\omega \to \langle f_k(\omega), P_n^* x^* \rangle$ is continuous, therefore the mapping

$$\begin{aligned} \omega \to \langle \Psi_n(\omega), x^* \rangle &= \langle f(\omega), P_n^* x^* \rangle \\ &= \lim_k \langle f_k(\omega), P_n^* x^* \rangle \end{aligned}$$

is a scalarly λ-measurable function because it is the pointwise limit of a sequence of continuous functions. It follows from the Pettis Measurability Theorem [**5**, p. 42] that $\Psi_n : \Omega \to E_n$ is λ-measurable.

Since for every $\omega \in \Omega$ we have that $\sum_n \Psi_n(\omega)$ is a weakly unconditionally Cauchy series in E, then

$$\sup_n \sup_{\sigma_i = \pm 1} \left\| \sum_{i=1}^{n} \sigma_i \Psi_i(\omega) \right\| = M(\omega) < \infty$$

and the mapping $\omega \to M(\omega)$ is λ-measurable.

Let $\varepsilon > 0$ be given. It follows from (†) that one can find $\delta > 0$ such that $\lambda(B) < \delta$ implies

$$||G||(B) < \frac{\varepsilon}{1 + C},$$

where $C = \sup\{||\sum_{n\in\sigma} P_n|| : \sigma \text{ finite subset of } \mathbf{N}\} < \infty$ and where for each $n \geq 1$, P_n is the projection associated to the decomposition $\{E_n\}$. Let Ω_ε be a closed subset of Ω such that $\lambda(\Omega\backslash\Omega_\varepsilon) < \delta$ and each Ψ_n is continuous on Ω_ε and

$$\sup_{\omega\in\Omega_\varepsilon} M(\omega) = M_\varepsilon < \infty.$$

By the Borsuk-Dugundji Theorem [**10**, p. 365], there exists a linear extension operator $S : C(\Omega_\varepsilon, E) \to C(\Omega, E)$ with $||S|| = 1$ and $S(g)(\omega) = g(\omega)$ for all $g \in C(\Omega_\varepsilon, E)$ and $\omega \in \Omega_\varepsilon$. For each $k \geq 1$, let

$$g_{k,\varepsilon} = S(\Psi_k|\Omega_\varepsilon)$$

then

$$\left\|\sum_{j=1}^{n} \sigma_j g_{j,\varepsilon}\right\| \leq M_\varepsilon$$

for all $n \geq 1$ and $\sigma_j = \pm 1$. Therefore, the series $\sum_j g_{j,\varepsilon}$ is a weakly unconditionally Cauchy series in $C(\Omega, E)$. Since T is unconditionally converging the series $\sum_j Tg_{j,\varepsilon}$ converges unconditionally in F.

For $n \geq 1$, consider the quantity

$$Tf_n - \sum_{j=1}^{n} Tg_{j,\varepsilon} = \int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon}\right] dG + \int_{\Omega\backslash\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon}\right] dG,$$

and note that for each $n \geq 1$,

$$\begin{aligned}\left\|\sum_{j=1}^{n} g_{j,\varepsilon}\right\| &= \left\|\sum_{j=1}^{n} S(\Psi_j|\Omega_\varepsilon)\right\| \\ &= \sup_{\omega\in\Omega_\varepsilon} \left\|\sum_{j=1}^{n} P_j^{**}(f(\omega))\right\| \\ &\leq C,\end{aligned}$$

and, therefore, for each $n \geq 1$,

$$\sup_{\omega\in\Omega} \left\|f_n(\omega) - \sum_{j=1}^{n} g_{j,\varepsilon}(\omega)\right\| \leq 1 + C.$$

This of course implies that

$$\left\| \int_{\Omega\setminus\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{i,\varepsilon} \right] dG \right\| \\ \leq \sup_{\omega\in\Omega\setminus\Omega_\varepsilon} \left\| f_n(\omega) - \sum_{j=1}^{n} g_{j,\varepsilon}(\omega) \right\| \, ||G||(\Omega\setminus\Omega_\varepsilon) < \varepsilon$$

since $\lambda(\Omega\setminus\Omega_\varepsilon) < \delta$ implies $||G||(\Omega\setminus\Omega_\varepsilon) < (\varepsilon/(1+C))$. We claim that

$$\int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] dG$$

converges to zero weakly in F as $n \to \infty$. For this, let $x^* \in F^*$ be such that $||x^*|| \leq 1$,

$$x^*\left(\int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] dG \right) = \int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] dG_{x^*}.$$

For each $\omega \in \Omega_\varepsilon$, the sequence $(f_n(\omega) - \sum_{j=1}^{n} g_{j,\varepsilon}(\omega))$ converges weakly to zero in E. This, of course, follows from the fact that

$$f(\omega) = \text{weak}^* \sum_n \Psi_n(\omega)$$

and the fact that $g_{n,\varepsilon}(\omega) = \Psi_n(\omega)$ for $\omega \in \Omega_\varepsilon$. Since the sequence $(f_n - \sum_{j=1}^{n} g_{j,\varepsilon})$ is bounded in $C(\Omega_\varepsilon, E)$, it converges weakly to zero in $C(\Omega_\varepsilon, E)$ by Lemma 1. We also know that the vector measure G_{x^*} defines an element of $M(\Omega, E^*)$ the dual of $C(\Omega, E)$, hence the measure G_{x^*} restricted to Ω_ε defines an element of $M(\Omega_\varepsilon, E^*)$ the dual of $C(\Omega_\varepsilon, E)$, hence

$$x^*\left(\int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] dG \right) = \int_{\Omega_\varepsilon} \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] dG_{x^*} \\ = \langle G_{x^*}|\Omega_\varepsilon, \left[f_n - \sum_{j=1}^{n} g_{j,\varepsilon} \right] \rangle \to 0$$

as $n \to \infty$.

So far we have shown that for $\varepsilon > 0$, we can write

$$Tf_n = \sum_{j=1}^{n} Tg_{j,\varepsilon} + \alpha_{n,\varepsilon} + \gamma_{n,\varepsilon}$$

where $\alpha_{n,\varepsilon} \to 0$ weakly as $n \to \infty$ and $||\gamma_{n,\varepsilon}|| \leq \varepsilon$ for all $n \geq 1$. If we let $K_\varepsilon = \{\sum_{j=1}^{n} Tg_{j,\varepsilon} + \alpha_{n,\varepsilon} : n \geq 1\}$, then K_ε is relatively weakly compact since the sequence $(\sum_{j=1}^{n} Tg_{j,\varepsilon})$ converges in norm and $(\alpha_{n,\varepsilon})$ converges to zero weakly. Hence,

$$\{Tf_n\} \subset K_\varepsilon + B(0, \varepsilon)$$

where K_ε is relatively weakly compact and $B(0, \varepsilon)$ is the ball of radius ε. By a result of Grothendieck [**4**, p. 227], the sequence (Tf_n) is relatively weakly compact in F, since (Tf_n) is weakly Cauchy in F; it follows that (Tf_n) converges weakly in F. This completes the proof. □

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