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A NOTE ON DEDEKIND NON-D-RINGS

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1. Introduction. All rings considered will be commutative integral domains with unity. The term *prime ideal* will refer to a nonzero, proper, prime ideal. Following [**3, 5**], we define non-D-ring as follows:

Definition 1. A ring R is a *non-D-ring* provided there is a nonconstant polynomial $f(x) \in R[x]$ such that $f(a) \in U(R)$ (the unit group of R) for every $a \in R$. The polynomial $f(x)$ will be called a uv (*unit valued*) polynomial.

Roughly speaking, a non-D-ring is a ring in which the unit group is large and maximal ideals are sparse. It is reasonable to assume then that one should be able to draw some strong conclusions about the ideal structure of non-D-rings. In this direction, the following result was proven in [**5**].

Theorem 1. *If R is a Dedekind non-*D*-ring with $f(x) \in R[x]$ being a monic* uv*-polynomial with degree $n \geq 2$, then $\mathrm{Cl}\,(R)$, the ideal class group of R, is a torsion group with exponent d where d is a positive integer which divides n.*

Theorem 1 is interesting in that it places strong restrictions on the structure of the ideal class group of a Dedekind non-D-ring, but it provides no mechanism for constructing examples of Dedekind non-D-rings. In this note we will prove a theorem which will provide us with the means to construct a large class of Dedekind non-D-rings. We will then construct some specific examples and analyze the ideal class group structures.

We conclude this section with more terminology and results from [**5**] concerning non-D-rings.

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Definition 2. Suppose that R is a ring, $P \subseteq R$ is a prime ideal, and $f(x) \in R[x]$ is a nonconstant polynomial. We say that P is an f-non-D-ideal of R provided $f(x)$ is a uv-polynomial for the local ring R_P.

Proposition 1. *Suppose R is a ring, a and b are nonzero elements of R and $f(x) \in R[x]$ is a monic polynomial of degree $n \geq 2$. If $P \subseteq R$ is an f-non-*D*-ideal of R, then $b^n f(a/b) \in P$ if and only if $a, b \in P$.*

Corollary 1. *Suppose R is a Noetherian ring, $f(x) \in R[x]$ is a monic polynomial of degree $n \geq 2$, and $P \subseteq R$ is an f-non-*D*-ideal of R. Then there exists an element $a \in P$ such that if $P_1 \subseteq R$ is an f-non-*D*-ideal of R with $a \in P_1$, then $P \subseteq P_1$.*

2. Construction. In this section we will consider locally finite intersections of discrete valuation domains (i.e., Krull domains). We will show that the non-D property can be preserved under intersection and can be used to insure that the intersections are sparse enough to make the Krull domains Dedekind.

Proposition 2. *Let F be a field and let $f(x) \in F[x]$ be a nonconstant polynomial. Suppose that $\{T_i | i \in S\}$ is a collection of subrings of F such that for each $i \in S$, $f(x) \in T_i[x]$ and T_i is a non-D-ring with $f(x)$ serving as* uv*-polynomial. Then $T = \cap_{i \in S} T_i$ is a non-*D*-ring with $f(x)$ serving as a* uv*-polynomial.*

Proof. Let $a \in T$ and $i \in S$. Then $a \in T_i$ and so $1/f(a) \in T_i$. Since this is true for each $i \in S$, then $1/f(a) \in T$. Thus, $f(x)$ is a uv-polynomial for T. $\square$

Proposition 3. *Let R be a Noetherian ring with field of fractions F, let $f(x) \in R[x]$ be a nonconstant polynomial, let W be a discrete valuation domain such that $R \subseteq W \subseteq F$, and let Q_1 be the maximal ideal of W. Suppose that Q_1 is an f-non-*D*-ideal of W. Then $P_1 = R \cap Q_1$ is an f-non-*D*-ideal of R.*

Proof. Suppose that P_1 is not an f-non-D-ideal of R. Then $f(x)$ is not a uv-polynomial for R_{P_1}. Hence, there exist $a, b \in R$ with $b \notin P_1$ such that $f(a/b) \in P_1 R_{P_1}$. However, this gives $a/b \in W$ with $f(a/b) \in Q_1$ which implies that Q_1 is not an f-non-D-ideal of W. □

The following assumptions will hold for Propositions 4–7, Theorem 2 and Corollary 2.

Assumptions. 1. R is a Noetherian ring with field of fractions F.

2. $f(x) \in R[x]$ is a monic polynomial of degree $n \geq 2$.

3. $C = \{W_i | i \in S\}$ is a collection of discrete valuation domains such that for each $i \in S$, $R \subseteq W_i \subseteq F$ and $f(x)$ is a uv-polynomial for W_i.

4. $W = \cap_{i \in S} W_i$.

5. For each $i \in S$, Q_i is the maximal ideal of W_i, $P_i = Q_i \cap R$ and $J_i = Q_i \cap W$.

6. $D = \{V_i | i \in S\}$ is the collection of valuations corresponding to the W_i's.

7. If $a \in W$, then $V_i(a) > 0$ for at most finitely many valuations V_i of D.

Proposition 4. *If a and b are nonzero elements of W and $V_i \in D$, then $V_i(b^n f(a/b)) > 0$ if and only if $V_i(a) > 0$ and $V_i(b) > 0$.*

Proof. Choose $V_i \in D$ and a, b nonzero elements of W. Then $a, b \in W_i$ and the result follows immediately from Proposition 1 applied to W_i and Q_i. □

Proposition 5. *For each $V_i \in D$ there exists $d_i \in W$ such that $V_i(d_i) > 0$ and $V_j(d_i) = 0$ if $V_j \in D$ and $V_j \neq V_i$.*

Proof. Choose $V_1 \in D$. Consider all of the valuations $V_i \in D$ such that $Q_i \cap R \supseteq P_1$. It follows from assumption 7 that there are at most finitely many such valuations. Let $\{V_1, \dots, V_t\}$ be the collection of all such valuations, and let $\{P_1, P_2, \dots, P_t\}$ be the corresponding prime ideals of R (note that the P_i's may not all be distinct). Let $T = W_1 \cap W_2 \cap \cdots \cap W_t$. Then T is a PID and the maximal ideals of T are exactly the ideals $Q_i \cap T$ for $1 \leq i \leq t$. Hence, we can find $r_1 \in T$ such that $V_1(r_1) > 0$ and $V_i(r_1) = 0$ for $2 \leq i \leq t$. Then we can write $r_1 = a_1/b_1$ where $a_1, b_1 \in R$, $V_1(a_1) > V_1(b_1)$ and $V_i(a_1) = V_i(b_1)$ for $2 \leq i \leq t$. Now apply Corollary 1 to choose an element $z_1 \in P_1$ such that if P' is an f-non-D-ideal of R and $z_1 \in P'$, then $P_1 \subseteq P'$. Clearly, $V_i(z_1) > 0$ for $1 \leq i \leq t$. Also, by our method of choosing $P_1, \dots, P_t$, we know that for all $i \in S$, $P_i \supseteq P$ implies $P_i \in \{P_1, P_2, \dots, P_t\}$. Thus, it follows easily from Proposition 3 that $V_i(z_1) = 0$ if $V_i \in D$ and $V_i \notin \{V_1, \dots, V_t\}$. Without loss of generality, $V_i(z_1) > V_i(a_1)$ for $1 \leq i \leq t$ (if not, simply replace z_1 by z_1^m with a sufficiently large m). Let $q_1 = z_1^n f(a_1/z_1)$ and let $y_1 = z_1^n f(b_1/z_1)$. Let $d_1 = q_1/y_1$. Then for $1 \leq i \leq t$, we have $V_i(q_1) = V_i(a_1)$ and $V_i(y_1) = V_i(b_1)$ since $V_i(z_1) > V_i(a_1) \geq V_i(b_1)$. Thus, $V_1(d_1) > 0$ and $V_i(d_1) = 0$ for $2 \leq i \leq t$. Also, if $V_i \in D$ and $V_i \notin \{V_1, \dots, V_t\}$, then since $V_i(z_1) = 0$ we have $V_i(y_1) = V_i(q_1) = 0$ by Proposition 4 and so $V_i(d_1) = 0$. $\square$

Proposition 6. *The ideals $\{J_i | i \in S\}$ are exactly the minimal prime ideals of W.*

Proof. By Theorem 110 of [**4**] we know that each prime ideal of W contains J_i for some $i \in S$. Hence, each minimal prime ideal of W is one of the J_i's. Then Proposition 5 implies that the ideals $\{J_i | i \in S\}$ are all distinct and that if $J_i \subseteq J_j$, then $J_i = J_j$. Hence, J_i is a minimal prime ideal of W for each $i \in S$. $\square$

Proposition 7. *Every prime ideal of W is maximal.*

Proof. Theorem 110 of [**4**] implies that every prime ideal of W contains a minimal prime ideal of W. Working from Propostion 6 we need only show that J_i is maximal in W for each $i \in S$. Choose $i \in S$ and choose $r_i \in W$ such that $r_i \notin J_i$. We need to find an element $z_i \in W$ such that $r_i z_i \equiv 1 \pmod{J_i}$. Write $r_i = a_i/b_i$ with $a_i, b_i \in R$. Since $V_i(r_i) = 0$, then $V_i(a_i) = V_i(b_i)$. Use Proposition 5 to find an element $d_i \in W$ such that $V_i(d_i) > 0$ and $V_j(d_i) = 0$ for all $V_j \in D$ with $V_j \neq V_i$. Without loss of generality, $V_i(d_i) > V_i(a_i)$ (if not, replace d_i by d_i^m

for some sufficiently large integer m). Let $z_i = (b_i a_i^{n-1})/[d_i^n f(a_i/d_i)]$. Since $V_i(d_i) > V_i(a_i)$ we know that $V_i(z_i) = 0$. Also, Proposition 4 implies that if $V_j \in D$ and $V_j \neq V_i$, then $V_j(d_i^n f(a_i/d_i)) = 0$. Hence, $V_j(z_i) \geq 0$ for all $V_j \in D$ and so $z_i \in W$. Next, observe that $r_i z_i - 1 = (a_i/b_i)((b_i a_i^{n-1})/[d_i^n f(a_i/d_i)]) - 1 = [a_i^n - d_i^n f(a_i/d_i)]/[d_i^n f(a_i/d_i)]$. Since $V_i(d_i) > V_i(a_i)$ it is easy to see that $V_i(d_i^n f(a_i/d_i)) = V_i(a_i^n)$ and that $V_i(a_i^n - d_i^n f(a_i/d_i)) \geq V_i(d_i a_i^{n-1}) > V_i(a_i^n)$. Hence, $V_i(r_i z_i - 1) > 0$ and so $r_i z_i \equiv 1 \pmod{J_i}$. □

Theorem 2. *W is a Dedekind non-*D*-ring with $f(x)$ as a* uv-*polynomial.*

Proof. It follows immediately from Proposition 2 that W is a non-D-ring with $f(x)$ as a uv-polynomial. A ring which can be expressed as a locally finite (assumption 7) intersection of discrete valuation domains within its field of fractions is known as a Krull domain (see [**4**, p. 82]). Hence, by definition, W is a Krull domain. It is known that a Krull domain in which every prime ideal is maximal is a Dedekind domain (see [**4**, #2 p. 83]). □

Corollary 2. *The ideal class group of W is a torsion group with exponent d where d is a positive integer which divides n.*

Proof. This follows immediately from Theorems 1 and 2. □

3. Examples. In this section we will define a class of discrete valuations on $Q(x)$, the field of rational functions over the rational numbers and use the results of Section 2 to construct Dedekind domains.

Definition 3. Let $P \subseteq Z[x]$ be a prime ideal. We define $V_p : Z[x]^* \to Z$ $(Z[x]^* = Z[x]\backslash\{0\})$ by

$$V_p(f(x)) = \begin{cases} 0 & \text{if } f(x) \notin P; \\ t & \text{if } f(x) \in P^t \backslash P^{t+1}. \end{cases}$$

Then if $f(x), g(x) \in Z[x]^*$ we define $V_P(f(x)/g(x)) = V_P(f(x)) - V_P(g(x))$. In this way, we extend V_P to a discrete valuation on

$Q(x)$ with Z as the value group. We will designate by W_P and Q_P, respectively, the valuation ring and its maximal ideals which are associated with V_P.

It is easy to see that $V_P : Q(x) \to Z$ as defined above is well defined if P is a principal prime ideal of $Z[x]$. Before we proceed with the construction, we prove the following proposition which guarantees that V_P is well defined if P is a maximal ideal of $Z[x]$.

Proposition 8. *Let $M \subseteq Z[x]$ be a maximal ideal and let $a, b \in M$ with $V_M(a) = t$ and $V_M(b) = r$. Then $V_M(ab) = t + r$.*

Proof. If $t = r = 0$ the result is obvious. If $t > r = 0$ or $r > t = 0$, the result follows easily from the fact that M^n is primary for any $n > 0$. Suppose then that $r, t > 0$. Suppose also that M is generated by a prime $p \in Z$ and a monic polynomial $f(x) \in Z[x]$. Since $V_M(a) = t$ and $V_M(b) = r$, we can write

$$a = \sum_{i=0}^{t} g_i(x) p^i (f(x))^{t-i}$$

with $g_i(x) \notin M$ for some i and

$$b = \sum_{j=0}^{r} h_j(x) p^j (f(x))^{r-j}$$

with $h_j(x) \notin M$ for some j. Then by multiplying the above expressions for a and b, we obtain

$$ab = \sum_{k=0}^{t+r} l_k(x) p^k (f(x))^{t+r-k} \tag{1}$$

with $l_k(x) \notin M$ for some k. Clearly, $V_M(ab) \geq t + r$. Suppose that $V_M(ab) \geq t + r + 1$. Then we can write

$$ab = \sum_{i=0}^{t+r+1} L(x) p^i (f(x))^{t+r+1-i}. \tag{2}$$

If we combine equations (1) and (2) and simplify we can obtain an equation of the form

$$\sum_{i=0}^{t+r} F_i(x)p^i(f(x))^{t+r-i} = \sum_{j=0}^{t+r+1} G_j(x)p^j(f(x))^{t+r+1-j} \tag{3}$$

with $F_i(x) = 0$ or $F_i \notin M$ for each i and $F_i(x) \neq 0$ for some i.

To see that equation (3) leads to a contradiction, we note that equation (3) implies the existence of at least one positive integer m for which we can find polynomials $H_0(x), \ldots, H_m(x)$, $K_0(x)$, $K_1(x), \ldots, K_{m+1}(x) \in Z[x]$ which satisfy the equation

$$\sum_{i=0}^{m} H_i(x)p^i(f(x))^{m-i} = \sum_{j=0}^{m+1} K_j(x)p^j(f(x))^{m+1-j} \tag{4}$$

with $H_i(x) = 0$ or $H_i(x) \notin M$ for each i and $H_i(x) \neq 0$ for some i.

Let m be the minimal such positive integer. Suppose first that $H_m(x) \neq 0$. Then since M is maximal we can assume without loss of generality that $H_m(x) = 1$. Let d be a root of $f(x)$. Then if we substitute $x = d$ in equation (4) we obtain $p^m = K_{m+1}(d)p^{m+1}$ and so $K_{m+1}(d) = 1/p$. Since $K_{m+1}(x) \in Z[x]$ and d is an algebraic integer, this is a contradiction. Now suppose that $H_m(x) = 0$. If $m > 1$, this yields a contradiction of the minimality of m. If $m = 1$, this yields $H_0(x) \neq 0$ and $H_0(x) \in M$ which contradicts the defining conditions of equation (4). In either case we have reached a contradiction which tells us that equation (4) is impossible for all positive integers m and, in particular, equation (3) ($m = t + r$) is impossible. Thus, $V_M(ab) \geq t + r + 1$ is impossible and so $V_M(ab) = t + r$. $\square$

Note. The referee has informed me that the valuation V_M described above with M a maximal ideal of $Z[x]$ is well known and is referred to in a more general context as the *ord* valuation of a regular local ring (for order of an element in a power series ring). Further, the referee tells me that Proposition 8, which guarantees that V_M is well defined, can be proven in a more general context by using the fact that the completion of a regular local ring is a ring of formal power series over either a field or complete discrete valuation ring (see [**1**]).

The following seven assumptions describe the Dedekind domains that we wish to consider.

1. $f(t) \in Z[x][t]$ is a monic polynomial in t of degree $n \geq 2$.

2. $C_1 = \{M_i | i \in S\}$ is a collection of maximal ideals of $Z[x]$ such that each $M_i \in C_1$ is an f-non-D-ideal of $Z[x]$ and if $g(x) \in Z[x]$ then $g(x)$ is contained in at most finitely many M_i's.

3. For each $i \in S$, M_i is generated by p_i and $f_i(x)$ where $p_i \in Z$ is prime and $f_i(x) \in Z[x]$ is a monic polynomial which is irreducible $\text{mod} p_i$.

4. $C_2 = \{P | P$ is an f-non-D-ideal of R and $P \subseteq M_i$ for some $M_i \in C_1\}$. Note that $C_1 \subseteq C_2$.

5. For each $P \in C_2$, let V_P, W_P and Q_P be as in Definition 3.

6. $W = \cap_{P \in C_2} W_P$

7. For each $P \in C_2$, let $J_P = Q_P \cap W$.

We know by Theorem 2 that W, as defined above, is a Dedekind non-D-ring with $f(x)$ serving as a uv-polynomial. We will now give a list of further properties of W.

Proposition 9. a) *Suppose $g(x), h(x) \in Z[x]^*$ and that $g(x)/h(x) \in W$. Suppose also that $g(x)/h(x)$ is in lowest terms and that $k(x) \in Z[x]^*$ is an irreducible polynomial such that $k(x)|h(x)$. Then $V_P(k(x)) = 0$ for every $P \in C_2 \backslash C_1$. In particular, $k(x)$ is either a unit in W or else generates a prime ideal in $Z[x]$ which is not an f-non-D-ideal.*

b) *Suppose $k(x)$ is as in* a) *above. Then either $k(x) \notin M_i$ for all $M_i \in C_1$ or else $k(x)|(k_2(x))^n f(k_1(x)/k_2(x))$ for some $k_1(x), k_2(x) \in Z[x]^*$ with $k_1(x), k_2(x)$ having no common factors.*

c) *If $I \subseteq W$ is an ideal, then I^n is principal.*

d) *The ideals J_P for $P \in C_2$ are exactly the prime ideals of W.*

e) *If $M_i \in C_1$, then J_{M_i} is generated by p_i and $f_i(x)$.*

f) *$CL(W)$, the ideal class group of W, is generated by the set $\{J_{M_i} | M_i \in C_1\}$.*

Proof. a) Choose $P \in C_2 \backslash C_1$ and suppose that $V_P(k(x)) > 0$. Since

$g(x)/h(x) \in W$ then $V_P(g(x)) \geq V_P(h(x)) \geq V_P(k(x)) > 0$. Since P is a principal prime ideal and $Z[x]$ is a UFD, we must have $k(x)|g(x)$ and $k(x)|h(x)$ which is a contradiction.

b) Suppose $k(x) \in M_i$ for some $M_i \in C_1$. Let P be the ideal generated by $k(x)$ in $Z[x]$. Then P is not an f-non-D-ideal by (a). Hence, $f(x)$ is not a uv-polynomial for $Z[x]_P$. This implies that there exist $k_1(x), k_2(x) \in Z[x]^*$ such that $k_1(x)/k_2(x)$ is in lowest terms, $k(x) \nmid k_2(x)$ in $Z[x]$ and $k(x)|f(k_1(x)/k_2(x))$ in $Z[x]_P$. This implies that $k(x)|(k_2(x))^n f(k_1(x)/k_2(x))$.

c) This is the content of Corollary 2.

d) Immediate from Propositions 6 and 7.

e) Choose $M_i \in C_1$. Since M_i is the only prime ideal of $Z[x]$ which contains both p_i and $f_i(x)$, it follows that J_{M_i} is the only prime ideal of W which contains both p_i and $f_i(x)$. Thus, p_i and $f_i(x)$ generate some power of J_{M_i}. Since $V_{M_i}(p_i) = V_{M_i}(f_i(x)) = 1$, then p_i and $f_i(x)$ generate J_{M_i}.

f) For each $M_i \in C_1$ choose $\alpha_i \in W$ such that $M_i^n = \alpha_i W$. Let $W' = W[\{1/\alpha_i | i \in S\}]$. Then $W' = \cap_{P \in C_2 \setminus C_1} W_P$ (see [**3**, Theorem III]). However if $P \in C_2 \setminus C_1$, then W_P is simply the local ring $Z[x]_P$ (i.e., $Z[x]$ localized at P). Hence, W' is seen to be a localization of $Z[x]$ and so is a UFD. Since W' is also a localization of W, it is a Dedekind domain. Hence, W' is a PID. $\square$

We now examine the way that the above construction works out in a particular case.

Example 1. Let W be as above with $f(t) = t^2 + 1$ and with C_1 consisting of a single maximal ideal M_1 of $Z[x]$ which is generated by p_1 and $f_1(x)$. Then the following statements are true of W.

a) The prime ideals which are contained in M_1 and are not f-non-D-ideals of $Z[x]$ are exactly those which are generated by irreducbile polynomials of the form $(k_1(x))^2 + (k_2(x))^2$ with $k_1(x), k_2(x) \in M_1^*$.

b) The nonzero elements of W are exactly the fractions $h(x)/g(x)$ with $h(x), g(x) \in Z[x]^*$ and $h(x), g(x)$ having no common factors such that if $k(x) \in M_1$ is irreducible and $k(x)|g(x)$ then $k(x) = (k_1(x))^2 + (k_2(x))^2$ for some $k_1(x), k_2(x) \in M_1^*$.

c) $\mathrm{Cl}\,(W)$, the ideal class group of W, has order 2 and is generated by J_{M_1}.

d) J_{M_1} is generated by p_1 and $f_1(x)$ and $J^2_{M_1}$ is generated by $p_1^2 + f_1(x)^2$.

e) Let $P \subseteq M_1$ be a nonmaximal f-non-D-ideal of $Z[x]$ generated by an irreducible polynomial $k(x)$. If $V_{M_1}(k(x)) = 2t$ for some $t > 0$, then J_P is principal and is generated by $k(x)/(f_1(x)^2 + p_1^2)^t$. If $V_{M_1}(k(x)) = 2t - 1$ for some $t > 0$, then J_P is nonprincipal and is generated by $p_1 k(x)/(f_1(x)^2 + p_1^2)^t$ and $f_1(x)k(x)/(f_1(x)^2 + p_1^2)^t$.

We observe now that b, d and e follow easily from (a) and (c) above and Proposition 9. Hence, we will give arguments for (a) and (c). First we note that if $k_1(x), k_2(x) \in M_1^*$ and $k_1(x)$ and $k_2(x)$ have no common factors, then Proposition 1 tells us that any f-non-D-ideal of $Z[x]$ which contains $(k_1(x))^2 + (k_2(x))^2$ must contain both $k_1(x)$ and $k_2(x)$. Hence, if $k(x) = (k_1(x))^2 + (k_2(x))^2$ is irreducible for some $k_1(x), k_2(x) \in M_1^*$, then $k(x)$ generates a prime ideal which is not an f-non-D-ideal. Now suppose that P is a prime ideal of $Z[x]$ such that $P \subseteq M_1$ and P is not an f-non-D-ideal of $Z[x]$. We can infer from the proof of Proposition 9b that P can be generated by an irreducible polynomial $k(x)$ such that $k(x)|(k_1(x))^2 + (k_2(x))^2$ for some $k_1(x), k_2(x) \in Z[x]^*$ which have no common factors. However, the fact that $Z[i][x]$ is a UFD forces $k(x) = (k_3(x))^2 + (k_4(x))^2$ for some $k_3(x), k_3(x) \in Z[x]^*$. Then another application of Proposition 1 implies that $k_3(x), k_4(x) \in M_1^*$. This proves (a). To prove (c) we need only observe that if $k(x) = (k_3(x))^2 + (k_4(x))^2$ as above, then it follows from the factorization $k(x) = (k_3(x) + (ik_4(x))(k_3(x) - (ik_4(x))$ that $V_{M_1}(k(x))$ is even. (Note that since M is an f-non-D-ideal of $Z[x]$, then M extends to a maximal ideal in $Z[i][x]$).

Example 1 demonstrated that if W is given by assumptions 1–7 preceding Proposition 9 then $\mathrm{Cl}\,(W)$ is nontrivial for some particular cases. It seems reasonable to conjecture that in the general case $\mathrm{Cl}\,(W)$ is a direct sum of cyclic groups of order n with each factor of the direct sum generated by J_{M_i} for some $M_i \in C_1$. Unfortunately, this is false as the next example indicates.

Example 2. Let W be defined by assumptions 1–7 preceding

Proposition 9 with $f(t) = t^4 + 1$ and C_1 consisting of the single maximal ideal M_1 generated by x and 3. Consider the polynomial $g(x) = x^4 - 2x^2 + 9$. $g(x)$ is the minimal polynomial for $i - \sqrt{2}$ and so it is irreducible. Let $h(x) = (1/12)(x^3 - 3x^2 - 5x + 3)$. Then $h(i - \sqrt{2}) = (i+1)/\sqrt{2}$. Now let $k(x) = (x^3 - 3x^2 - 5x + 3)^4 + (12)^4$. Then $k(i - \sqrt{2}) = 0$ and so $g(x)|k(x)$. It follows that $g(x)$ generates a prime ideal of $Z[x]$ which is not an f-non-D-ideal of $Z[x]$. However, $V_{M_1}(g(x)) = 2$. Hence, $J_{M_1}^2$ is a principal ideal which is generated by $g(x)$ and so $|\mathrm{Cl}\,(W)| \leq 2$ although $n = 4$.

4. Conclusions. In [**2**] Eakin and Heinzer used a similar method to ours to construct Dedekind domains with preassigned finitely generated abelian group as ideal class groups. Although we have not achieved the same amount of control over the ideal class group as they did, we feel that our constructions allow the use of more simplistic valuations and hence lead to nontrivial Dedekind domains whose ideal structure can, potentially, be known in much more minute detail.

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