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SOME INTERESTING BANACH SPACES

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ABSTRACT. If a Banach space X has an unconditionally basic skipped-blocking finite-dimensional decomposition (UB-SBFDD), then each of the properties RNP, KMP, PCP, and CPCP is equivalent to X not having a subspace isomorphic with c_0. If X fails PCP and is a subspace of a space with a UB-SBFDD, then X has a subspace isomorphic with c_0. Many examples are described that have no c_0-subspaces and fail RNP. Some of these have PCP. Some fail PCP and have CPCP. Some are not contained in a space with an unconditional basis, but this remains an open question for others. Sufficient conditions are given for a boundedly complete skipped-blocking decomposition to imply CPCP.

Introduction. A bounded closed convex subset K of a Banach space X has the *Radon-Nikodým property* (RNP) if, for any finite-measure space (S, Σ, μ) and any μ-continuous measure $\lambda : \Sigma \to X$ with $\lambda(E)/\mu(E) \in K$ for each $E \in \Sigma$, there is a Bochner-integrable function $f : S \to X$ such that $\lambda(E) = \int_E f\, d\mu$ for each $E \in \Sigma$. For X to have RNP means that the unit ball has RNP.

A bounded closed convex subset K of a Banach space X has the *Krein-Milman property* (KMP) if each closed convex subset of K is the closure of the convex span of its extreme points. For X to have KMP means that the unit ball has KMP. A Banach space has KMP if it has RNP [**18**], but whether the converse is true remains an important unsolved problem.

A bounded closed convex subset K of a Banach space X has the *point-of-continuity property* (PCP) if, for each nonempty closed subset C of K, there is a point x of C such that the weak and norm topologies (restricted to C) coincide at x; K has the *convex-point-of-continuity property* (CPCP) if this condition is satisfied for all nonempty closed convex subsets of K. For X to have PCP (CPCP) means that the unit ball has PCP (CPCP). The CPCP was introduced in [**3**] to aid in

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proving that a space which fails RNP has a subspace which fails RNP and has a basis of finite-dimensional subspaces. This property became particularly important when it was proved that KMP$\Rightarrow$RNP for spaces with CPCP [**22**, Theorem 2.1]. The PCP was introduced in [**4**], where it was shown that PCP is strictly weaker than RNP. Both PCP and CPCP have been studied extensively.

A Banach space X (or a bounded closed convex subset of X) has RNP if and only if it does not contain a bush (for an easy proof that is easily adapted to the case of bounded closed convex subsets, see [**13**, Theorem 7, p. 354]). A *bush* in a Banach space is a bounded partially ordered subset B for which each member has finitely many successors, B has a first member b_{11}, each member of B can be joined to b_{11} by a linearly ordered chain of successive members of B, each member of B has at least two successors and is the average of its successors, and there is a positive *separation constant* δ such that $||y-x|| > \delta$ if y is a successor of x. The successors $\{b_{2i} : 1 \le i \le d(2)\}$ of b_{11} are said to be of *order* 2; the successors $\{b_{ni} : 1 \le i \le d(n)\}$ of members of B of order $n-1$ are said to be of *order* n. A difference $\Delta_{nj} = b_{n+1,j} - b_{ni}$ between b_{ni} and a successor $b_{n+1,j}$ is a *difference of order* n. A *branch* of a bush is an infinite linearly ordered subset whose first member is b_{11}; a *segment* is a linearly ordered subset; an *initial segment* is a segment whose first member is b_{11}.

A *decomposition* of a Banach space X is a sequence $\{G_n\}$ of closed subspaces for which $X = \overline{\text{lin}}\,\{G_n\}$ and $G_k \cap \overline{\text{lin}}\,\{G_i : i \ne k\} = \{0\}$ for each k. A decomposition $\{G_n\}$ is *basic* if there is a positive number ρ such that

$$\left\| \sum_{i=1}^{n} x_i \right\| \le \rho \left\| \sum_{i=1}^{n+p} x_i \right\| \tag{1}$$

if n and p are positive integers and $x_i \in G_i$ for each i; it is *monotone* if $\rho = 1$. A decomposition $\{G_n\}$ of X is basic if and only if each member x of X has a unique representation as $x = \sum_1^\infty x_n$, where $x_n \in G_n$ for each n and convergence is convergence in norm. It is *unconditionally basic* (a UBD) if this convergence is unconditional for each x; if each G_n is finite-dimensional, it is an *unconditionally basic finite-dimensional decomposition* (UBFDD). If a Banach space X is contained in a space with a UBFDD, then X is contained in a space with an unconditional basis [**16**, p. 51].

Given a decomposition $\{G_n\}$, a *skipping sequence* is a sequence $\{H_n\}$ for which there is a sequence of subsets $\{S_n\}$ of $\{G_n\}$ such that, for each n, $H_n = \overline{\text{lin}}\{G_i : i \in S_n\}$ and there is an r such that $\alpha < r < \beta$ if $G_\alpha \in S_n$ and $G_\beta \in S_{n+1}$. A decomposition $\{G_n\}$ is an *unconditionally basic skipped-blocking decomposition* (UBSBD) if each skipping sequence is a UBD. A UBSBD $\{G_n\}$ is an *unconditionally basic skipped-blocking finite-dimensional decomposition* (UBSBFDD) if each G_n is finite-dimensional. As noted in [**21**, p. 161], the James space J has a UBSBFDD, but J does not embed in a space with an unconditional basis.

A decomposition $\{G_n\}$ is a *boundedly complete skipped-blocking decomposition* if each skipping sequence $\{H_n\}$ is boundedly complete. The sequence $\{H_n\}$ being *boundedly complete* means that a series Σy_n converges if the partial sums are bounded and $y_n \in H_n$ for each n. A BCSBD $\{G_n\}$ is a *boundedly complete skipped-blocking finite-dimensional decomposition* (BCSBFDD) if each G_n is finite-dimensional.

1. Unconditional bases and c_0-subspaces. Although satisfied by many classical spaces, the assumption that X can be embedded in a Banach space with an unconditional basis is very severe. For example, it implies that X is reflexive unless X contains c_0 or l_1 [**2**]. The next two "almost-known" theorems also illustrate this. The proof of Theorem 1.1 involves only a natural modification of the proofs of Theorems 4.5 and 4.7 in [**14**] and the proof of Theorem 1.2 is a natural modification of the proof of Theorem 4.8 in [**14**].

Theorem 1.1. *Suppose X is a subspace of a Banach space Z which has a* UBSBFDD. *If no subspace of Z is isomorphic with c_0, then X has* RNP, KMP, PCP, *and* CPCP. *If X fails* PCP, *then X has a subspace isomorphic with c_0 (and X fails* RNP, KMP, *and* CPCP*).*

It is known that RNP and KMP are equivalent for X if X has CPCP or if X is a subspace of a space with an unconditional basis [**22**]. However, there is a space X [**15**] which is a subspace of a Banach space Z with an unconditional basis, X fails both RNP and KMP, X has PCP, and X has no subspace isomorphic with c_0. Then Z must have a subspace isomorphic with c_0 and, because of the next theorem, X cannot have a UBSBFDD.

Theorem 1.2. *If X has a* UBSBFDD*, then each of the properties* RNP, KMP, PCP*, and* CPCP *is equivalent to X not having a subspace isomorphic with c_0.*

These theorems give a great deal of information about the role c_0 plays with respect to RNP, KMP, PCP, and CPCP for spaces contained in spaces with a UBSBFDD. To obtain similar results for bounded closed convex subsets of a space with a UBSBFDD, we need two lemmas. The first lemma uses the following easy result which was first stated on page 138 of [**3**] for CPCP.

"If a bounded closed nonempty set $K \subset X$ fails PCP, then there is a closed nonempty set $A \subset K$ and a positive number $\varepsilon(A)$ such that

(2) $\quad \operatorname{diam}(U \cap A) > \varepsilon(A) \quad$ if $U \cap A \neq \varnothing \quad$ and U is w-open."

Lemma 1.3 is similar to [**3**, Lemma 9, p. 139], [**14**, Lemma 4.1, p. 307], and [**19**, Lemma 4.6, p. 312], but is much easier to prove. It suffices for Lemma 1.4, which is a stronger version of [**21**, Lemma 2.7, p. 176] and closely related to [**3**, Lemma 10, p. 140] and [**14**, Lemma 4.2, p. 308].

Lemma 1.3. *Suppose $A \subset K \subset X$ and $\varepsilon(A)$ are as in* (2)*. Then for any $\varepsilon < (1/2)\varepsilon(A)$, any $\mu > 0$, any $x \in A$, and any subspace E of X with finite codimension, there exists $\Delta \in X$ such that*

$$x + \Delta \in A, \qquad ||\Delta|| > \varepsilon, \qquad \operatorname{dist}(\Delta, E) < \mu.$$

Proof. Assume the hypotheses are satisfied. Let $E = \cap_{i=1}^{n} \operatorname{Ker}(f_i)$. Choose a positive number c such that $\operatorname{dist}(z, E) < \mu$ if $|f_i(z)| < c$ for each i. Now let

$$U = \{u : |f_i(u - x)| < c, 1 \leq i \leq n\}.$$

It follows from (2) that there are members y_1 and y_2 of $U \cap A$ such that $||y_1 - y_2|| > \varepsilon(A)$. Choose α as 1 or 2, so that $||x - y_\alpha|| > \varepsilon$. Let

$\Delta = y_\alpha - x$. Then $x + \Delta = y_\alpha \in U \cap A$ and $||\Delta|| > \varepsilon$. Since $y_\alpha \in U \cap A$, we have

$$|f_i(y_\alpha - x)| = |f_i(\Delta)| < c \qquad \text{for } 1 \leq i \leq n.$$

Therefore dist $(\Delta, E) < \mu$. □

Lemma 1.4. *Suppose $A \subset K \subset X$ and $\varepsilon(A)$ are as in* (2). *Then for any $\varepsilon < (1/2)\varepsilon(A)$, any $x \in A$, and any subspace E of X with finite codimension,*

$$x \in \overline{\text{conv}}\,\{x + \Delta : x + \Delta \in A, \Delta \in E, ||\Delta|| > \varepsilon\}.$$

Proof. Assume the hypotheses are satisfied and

$$x \notin \overline{\text{conv}}\,\{x + \Delta : x + \Delta \in A, \Delta \in E, ||\Delta|| > \varepsilon\} = D.$$

Then there is an f in X^* and a positive number μ such that $||f|| = 1$ and $f(x) - \mu > \sup\{f(z) : z \in D\}$. Let $F = E \cap \text{Ker}\,(f)$. It follows from Lemma 1.3 that there exists Δ_0 such that $x + \Delta_0 \in A$, $||\Delta_0|| > \varepsilon$, and dist $(\Delta_0, F) < (1/2)\mu$. Since dist $(\Delta_0, F) < (1/2)\mu$, we have $|f(\Delta_0)| < (1/2)\mu$,

$$f(x + \Delta_0) > f(x) - (1/2)\mu, \qquad \text{dist}\,(\Delta_0, E) < (1/2)\mu,$$
$$\text{dist}\,(x + \Delta_0, D) < (1/2)\mu.$$

The last inequality and $f(x) - \mu > \sup\{f(z) : z \in D\}$ imply $f(x + \Delta_0) < (1/2)\mu + [f(x) - \mu]$, or $f(x + \Delta_0) < f(x) - (1/2)\mu$, which contradicts $f(x + \Delta_0) > f(x) - (1/2)\mu$ and completes the proof. □

A *basic bush* is a bush for which $\{D_{ni}\}$ is basic, where D_{ni} is the linear span of the differences between b_{ni} and the successors of b_{ni}, and $\{D_{ni}\}$ is arranged as a sequence using lexicographic ordering. We will say that K contains "nearly monotone" basic bushes if K contains bushes with ρ in (1) arbitrarily near 1. A bush is *complemented* if there is a positive number θ such that, for each n, $||u - v|| \geq \theta ||u||$ if u belongs to the linear span of all followers of some member b_{ni} of order n and v belongs to the linear span of all followers of other members b_{nj} of order n with $j \neq i$. It follows from [**10**, Theorem A, p. 354] that a bounded closed convex set K fails KMP if K contains a complemented bush.

The next theorem is a significant strengthening of [**21**, Theorem 1.1], which states that if K is a closed bounded convex subset of a Banach space with a UBSBFDD and K fails PCP, then K contains a "well separated uniformly bounded bush $\mathcal{B}$ such that $\mathcal{B}$ is a strong martingale representation for its closed convex hull."

Theorem 1.5. *Let K be a bounded closed convex subset of a Banach space X. If K fails* PCP*, then K contains "nearly monotone" basic bushes. If X has a* UBSBFDD*, then K fails* KMP *because this bush can be complemented. Also, each branch can have the property that the sequence of differences along that branch is a natural basis for a subspace isomorphic with c_0.*

Proof. Let $A \subset K$ and ε be as in (2), and let $\{\delta_n\}$ be a sequence of positive numbers for which $\Sigma\delta_n < \varepsilon/3$. We will construct an *approximate bush* $\{x_{ni}\}$ in A with errors of approximation $\{\delta_n\}$ [**10**, p. 348]. For an arbitrary $x_{11} \in A$, choose a finite set $\{\Delta_{1i}\}$ in A for which there is a convex combination Δ of $\{\Delta_{1i}\}$ with $||\Delta|| < \delta_1$, each $x_{11} + \Delta_{1i} \in A$, and $||\Delta_{1i}|| > \varepsilon$ for each i. Without loss of generality, we can assume that Δ is the arithmetic average of $\{\Delta_{1i}\}$. Now we use δ_2 and repeat this for each $x_{11} + \Delta_{1i} = x_{2i}$, to obtain $x_{2i} + \Delta_{2j} = x_{3j}$. This process can be continued indefinitely, to define an approximate bush. Now we can obtain a bush $B = \{b_{ni}\}$ in K for which b_{ni} is the limit as $p \to \infty$ of the obvious natural weighted average of the followers of x_{ni} of order p [**10**, p. 352]. This bush B will be "nearly monotone" if, for each x_{ni}, the corresponding Δ_{nj}'s are chosen to be in the intersection of the null spaces of an appropriate finite set F_{ni} of members of the unit ball of X^* that are "nearly norming" for the linear span of all previously chosen Δ_{ni}'s. Now suppose that X has a UBSBFDD $\{G_p\}$. If $\{\mu_r\}$ is a set of positive numbers, then each F_{ni} can be chosen as before—but enlarged so that $\{G_p\}$ has a skipped-blocking $\{H_{ni}\}$, ordered lexicographically, such that, if D_{ni} is the set of $\Delta_{n+1,j}$'s of order n that follow x_{ni}, then each member of D_{ni} belongs to the closed linear span of those G_k's that follow the first member of $\{G_p\}$ that follows all members of $\{G_p\}$ that were used with previously determined H's, and

$$\operatorname{dist}(x, H_{ni}) \leq \lambda_{ni}||x||,$$

if $x \in \operatorname{lin}(D_{ni})$ and λ_{ni} is the first member of $\{\mu_r\}$ not used previously. Since $\{H_{ni}\}$ is unconditionally basic, it follows that, if the positive numbers $\{\mu_r\}$ are sufficiently small, then the bush B is complemented and the differences along a branch span a subspace isomorphic with c_0. □

It follows from Theorem 1.5 that, for bounded closed convex subsets of a Banach space with a UBSBFDD, KMP implies PCP. Since RNP and KMP are equivalent for spaces with CPCP [**22**], this implies that KMP implies RNP for bounded closed convex subsets of a space with a UBSBFDD. This was proved by Schachermayer [**22**] for spaces with an unconditional basis. It also follows from the result of Rosenthal and Wessel quoted above [**21**, Theorem 1.1].

For a bounded closed convex subset K of a space with a UBSBFDD, Theorem 1.5 establishes the existence in K of a nicer and more restricted bush than previously known to exist. Each branch generates a c_0-subspace, although the closure of the convex span of the branch need not contain a bush. This bush resulted from the failure of PCP. That such a bush need not exist because of the failure of RNP is shown by the example discussed following Theorem 1.1. Theorems giving the existence of basic bushes for Banach spaces that fail RNP are known (see [**3**] and [**20**, Theorem 3.9]).

2. Boundedly complete decompositions. The purpose of this section is to introduce some results that will be used in the next section to aid in proving that several spaces that fail RNP have PCP and that several spaces that fail PCP have CPCP. The next theorem is known. Other results concerning skipped-blocking decompositions are given in [**4**], [**6**], and [**19**] for finite-dimensional decompositions and in [**8**] for infinite-dimensional decompositions.

Theorem 2.1. *A separable Banach space has* PCP *if and only if it has a* BCSBFDD [**6**, Theorem 2.1].

Let $\{G_n\}$ be a basic decomposition of a Banach space X. If T is a subset of X, we will use the symbols $P_{(<s)}T$, $P_{(=s)}T$, and $P_{(>s)}T$ for the set of all projections of members of T onto $\operatorname{lin}(G_n : n < s\}$, G_s, and $\overline{\operatorname{lin}}\{G_n : n > s\}$, respectively.

Lemma 2.2. *A decomposition $\{G_n\}$ of a Banach space X is a* BCSBD *if there is an $\varepsilon > 0$ such that $||x + y|| \geq (1 + \varepsilon)||x||$ whenever $||x|| = ||y||$ and (x, y) is a* skipping pair *in the sense that there is an integer p for which one of x or y belongs to $\overline{\operatorname{lin}}\{G_i : i < p\}$ and the other belongs to $\overline{\operatorname{lin}}\{G_i : i > p\}$.*

Proof. Suppose first that (x, y) is a skipping pair. If $||x|| = \theta||y||$, $0 < \theta \leq 1$, and the hypothesis of the lemma is satisfied, then $||x + \theta y|| \geq (1 + \varepsilon)||x||$ and

$$\theta||x + y|| \geq ||x + \theta y|| - (1 - \theta)||x|| \geq (\varepsilon + \theta)||x||,$$

so that $||x + y|| \geq (1 + \varepsilon/\theta)||x|| \geq (1 + \varepsilon)||x||$. Thus for any skipping pair (x, y),

$$||x + y|| \geq (1 + \varepsilon) \min\{||x||, ||y||\}. \tag{3}$$

Now suppose a decomposition $\{G_n\}$ has the property described in the theorem but is not a BCSBD. Then there is a skipping sequence $\{u_i\}$ whose partial sums are bounded, but Σu_i is not convergent. This implies that there is a positive number Δ for which there is a sequence $\{\sigma_i : i \geq 1\}$ of sums of consecutive blocks of $\{u_i\}$ with $||\sigma_i|| > \Delta$ for each i. Let $\sigma_i^1 = \sigma_{2i-1} + \sigma_{2i}$ for each i. Then it follows from (3) that each $||\sigma_i^1|| \geq (1 + \varepsilon)\Delta$. Now let

$$\sigma_i^2 = \sigma_{2i-1}^1 + \sigma_{2i}^1 \qquad \text{for each } i.$$

Then each $||\sigma_i^2|| \geq (1 + \varepsilon)^2\Delta$. Since this can be continued indefinitely, we have the contradiction that the partial sums of $\{u_i\}$ are not bounded. □

Lemma 2.3. *Let X be a Banach space that has a monotone decomposition $\{G_n\}$ for which each G_n is an l_2-space. Suppose each bounded convex nonempty subset K of X has the property that, for* *any* $\Delta > 0$ *and any weakly open convex set* W *with* $W \cap K \neq \varnothing$, *there is a weakly open convex subset* V *which has the properties that* $V \cap W \cap K \neq \varnothing$ *and there are integers* r *and* N *such that, if* $z \in V \cap W \cap K$ *and* $t > r$, *then there exists* $\xi \in X$ *for which*

$$||z - \xi|| < \Delta \quad \textit{and} \quad \sum_1^\infty |a_i^t| \leq N,$$

where

$P_{(=t)}\xi = \sum_1^\infty a_i^t e_i^t$ *with* $\{e_i^t\}$ *the orthonormal basis of* G_t. *Then each bounded convex nonempty subset of* X *has the property that, for any positive numbers* ε *and* r, *there exist a weakly open convex set* Ω *and integers* s *and* t *such that* $r < s \leq t$, $\Omega \cap K \neq \varnothing$ *and, if* $z \in \Omega \cap K$, *then*:

(a) $\operatorname{diam}[P_{(<s)}(\Omega \cap K)] < \varepsilon$, $||P_{(<s)}z|| > \sup\{||x|| : x \in K\} - \varepsilon$,

(b) $||P_{(=t)}z|| < \varepsilon$.

Proof. For each t, let $\{e_i^t : i \geq 1\}$ be the orthonormal basis for G_t. Let X and K be as described in the lemma, but assume that K is contained in the unit ball of X. Let ε be an arbitrary positive number. Choose an integer $s > r$ for which there is a linear functional f such that $K(s, f) \neq \varnothing$ if

$$K(s, f) = (P_{(<s)}K) \cap \{x : f(x) > 1\},$$

and also $||z|| > \sup\{||x|| : x \in K\} - \varepsilon$ if $z \in K(s, f)$. Since $P_{(<s)}X$ is reflexive, it has PCP. Therefore there exists $u \in \overline{K}(s, f)$ which has a convex w-neighborhood U with $\operatorname{diam}[U \cap \overline{K}(s, f)] < \varepsilon$. Then $W \cap K \neq \varnothing$ if

$$W = U \cap \{x : f(x) > 1\}.$$

This s and $\Omega = W$ satisfy (a). For some t, (b) may be satisfied and the proof complete. But we do not know if such a t exists, so we proceed as follows.

Let K^* be $W \cap K$ translated a distance less than $(1/2)\varepsilon$ so that there is an integer τ for which there exists $\zeta \in K^*$ such that $P_{(\geq\tau)}\zeta = 0$. If we can find $t \geq s$ and a weakly open convex set U^* for which

$$||P_{(=t)}z|| < (1/2)\varepsilon \qquad \text{if } z \in U^* \cap K^*,$$

then (b) will be true for $\Omega = U_0 \cap W$, where U_0 is U^* translated back. Let t be any integer greater than both τ and s. To find U^*, we first choose a positive number $\Delta < (1/8)\varepsilon$. Let V, r, and N be as described in the lemma for $W \cap K$ replaced by K^*. For each $\xi \in X$, let

$$L(t,\xi) = \sum_1^\infty |a_i^t| \qquad \text{if } P_{(=t)}\xi = \sum_1^\infty a_i^t e_i^t.$$

For each weakly open convex set $U \subset V$ for which $U \cap K^* \neq \varnothing$, let $N(t,U)$ be the least number such that, if $x \in U \cap K^*$, then there exists $\xi \in X$ for which $||x-\xi|| < \Delta$ and $L(t,\xi) \leq N(t,U)$. Let

$$M = \inf\{N(t,U) : U \cap K^* \neq \varnothing\},$$

where U is a weakly open convex subset of V. Choose U' for which $N(t,U') < M + (1/2)\Delta$. Let λ be an integer for which $\lambda^{-1/2} < (1/8)\varepsilon$, and let $\{\alpha_1, \dots, \alpha_\lambda\}$ be a set of positive numbers to be restricted later. Because of the definitions of $N(t,U)$ and M, there exists $u_1^t \in U' \cap K^*$ such that

$$L(t,\xi) > M - (1/4)\Delta \qquad \text{if } \xi \in X \quad \text{and} \quad ||u_1^t - \xi|| < \Delta.$$

Let $\{f_i^t : 1 \leq i \leq n_1\}$ be coefficient functionals for basis vectors of G_t whose linear span "almost" contains u_1^t [$f_i^t(x) = 0$ if $x \in \operatorname{lin}\{G_n : n \neq t\}$]. Let

$$W_1^t = \{x \in X : |f_i^t(x)| < \alpha_1 \text{ for each } i\}.$$

Since $\zeta \in W_1^t$, $W_1^t \cap U' \cap K^* \neq \varnothing$. Next, choose u_2^t so that $u_2^t \in W_1^t \cap U' \cap K^*$ and

$$L(t,\xi) > M - (1/4)\Delta \qquad \text{if } \xi \in X \quad \text{and} \quad ||u_2^t - \xi|| < \Delta.$$

Define W_2^t for α_2 and $\{f_i^t : 1 \leq i \leq n_2\}$ similarly, and continue until we have sets $\{W_1^t, \dots, W_\lambda^t\}$ and $\{u_1^t, \dots, u_\lambda^t\}$. Then $\zeta \in \cap_1^\lambda W_i^t$, so $(\cap_1^\lambda W_i^t) \cap U' \cap K^* \neq \varnothing$. If "almost" was sufficiently restrictive and if each α_i is small enough, then

$$\text{(4)} \quad L(t,\xi) > M - (1/2)\Delta \qquad \text{if } \xi \in X \quad \text{and} \quad \left\| \sum_1^\lambda (u_i^t/\lambda) - \xi \right\| < \Delta/\lambda.$$

Now choose β and coefficient functionals $\{g_i^t : 1 \le i \le m\}$ of $\{e_i^t\}$ to be restricted later. Let

$$U^* = \left\{x \in X : \left|g_i^t(x) - g_i^t\left(\sum_1^\lambda u_i^t/\lambda\right)\right| < \beta \text{ if } 1 \le i \le m\right\}.$$

Then $\sum_1^\lambda u_i^t/\lambda \in U^* \cap K^*$, so $U^* \cap K^* \ne \varnothing$. Let E be the set of basis vectors corresponding to these coefficient functionals. For any $x \in X$, let x_E and $x_{\sim E}$ be the projections of $P_{(=t)}x$ onto $\operatorname{lin}(E)$ and onto the closure of the linear span of the complement of E in $\{e_i^t : i \ge 1\}$, respectively. If m, $\{g_i : 1 \le i \le m\}$, and β are chosen appropriately, if $z \in U^* \cap K^*$, and if ξ is chosen so that $||z - \xi|| < (1/2)\Delta/\lambda$ and

$$L(t,\xi) < M + (1/2)\Delta, \tag{5}$$

then it follows from $\lambda^{-1/2} < (1/8)\varepsilon$, K being in the unit ball, and $||z_E||$ being approximately $[\sum_1^\lambda ||u_i^t/\lambda||^2]^{1/2}$, that $||z_E|| < (1/4)\varepsilon$; and it follows from (4) that $L(t,\xi_E) > M - (1/2)\Delta$. It follows from this and (5) that

$$L(t,\xi_{\sim E}) < (M + (1/2)\Delta) - (M - (1/2)\Delta) = \Delta.$$

Therefore $||\xi_{\sim E}|| < \Delta$, and it follows from $||z - \xi|| < (1/2)\Delta/\lambda$ that $||z_{\sim E}|| < 2\Delta < (1/4)\varepsilon$, so we have

$$||P_{(=t)}z|| \le ||z_E|| + ||z_{\sim E}|| < (1/2)\varepsilon. \qquad \square$$

Theorem 2.4. *Let X be a Banach space that has a monotone decomposition $\{G_n\}$. Then X has* CPCP *and $\{G_n\}$ is a* BCSBD *if:*

(i) *For any $\alpha > 0$, there is a $\beta > 0$ such that $||x+y|| \ge ||x|| + \beta||y||$ if $||y|| > \alpha||x||$ and (x,y) is a skipping pair with respect to $\{G_n\}$.*

(ii) *Each bounded convex nonempty subset K has the property that, for any positive numbers ε and r, there exists a weakly open convex set Ω and integers s and t such that $r < s \le t$, $\Omega \cap K \ne \varnothing$, and, if $z \in \Omega \cap K$, then:*

(a) $\operatorname{diam}[P_{(<s)}(\Omega \cap K)] < \varepsilon$, $||P_{(<s)}z|| > \sup\{||x|| : x \in K\} - \varepsilon$,

(b) $||P_{(=t)}z|| < \varepsilon$.

Proof. It follows from (i) and Lemma 2.2 that $\{G_n\}$ is a BCSBD. If X fails CPCP, then there is a bounded convex subset K of X and $\Delta > 0$ such that

$$\text{diam}\,(U \cap K) > \Delta \tag{6}$$

if U is a weakly open subset of X for which $U \cap K \neq \varnothing$ [**3**, Proposition 1, p. 138]. Choose $\beta < 1$ so that, if (x, y) is a skipping pair with respect to $\{G_n\}$, then

$$||x + y|| \geq ||x|| + \beta||y|| \qquad \text{if } ||y|| > (1/4)\Delta||x||.$$

For this K, let Ω be a weakly open convex set and $s \leq t$ be numbers for which $\Omega \cap K \neq \varnothing$ and (a) and (b) of (ii) are satisfied for $\varepsilon = (1/8)\beta\Delta$. Without loss of generality, we assume that $\sup\{||x|| : x \in K\} = 1$. Since $\{G_n\}$ is monotone, it follows from (a) that $||P_{(<t)}z|| > 1 - \varepsilon$ if $z \in (\Omega \cap K)$. If $||P_{(>t)}z|| > (1/4)\Delta$, then

$$||P_{(<t)}z + P_{(>t)}z|| > (1 - \varepsilon) + (1/4)\beta\Delta.$$

Since $||P_{(=t)}z|| < \varepsilon$, this implies $||z|| > (1 - \varepsilon) + (1/4)\beta\Delta - \varepsilon = 1$. This contradiction implies $||P_{(>t)}z|| \leq (1/4)\Delta$ and, therefore,

$$||P_{(\geq t)}z|| < (1/4)\Delta + \varepsilon.$$

Now choose a weakly open convex set V such that $V \cap \Omega \cap K \neq \varnothing$ and

$$\text{diam}\,[P_{(<t)}V \cap \Omega \cap K] < \varepsilon.$$

Then if z_1 and z_2 are members of $V \cap \Omega \cap K$, we have

$$\begin{aligned} ||z_1 - z_2|| &\leq ||P_{(<t)}(z_1 - z_2)|| + ||P_{(\geq t)}z_1|| + ||P_{(\geq t)}z_2|| \\ &< \varepsilon + (1/2)\Delta + 2\varepsilon < (7/8)\Delta. \end{aligned}$$

Thus $U = V \cap \Omega$ contradicts (6), and the proof is complete. $\square$

3. Some examples. It was proved in [**4**] that JT^* and JH^* contain subspaces that fail RNP and have PCP, where JT is the James-Tree space [**12**] and JH is a tree space constructed by Hagler [**9**]. Now it is known that JT^* and JH^* themselves have PCP ([**5** and **17**]). Four new examples will be given of spaces that fail RNP and have PCP.

Four examples will be given of spaces that fail PCP (and RNP), but have CPCP. It was proved in [**7**] and [**8**, Theorem IV.8] that the space B_∞ (or $J_*T_{\infty,1}$) fails PCP and has CPCP. The space Λ_∞ to follow is B_∞, defined explicitly rather than as a subspace of a dual. The space Φ_∞ is similar, but is not isomorphic with Λ_∞.

We know that, if X has a UBSBFDD, then each of the properties RNP, KMP, PCP, and CPCP is equivalent to X not having a subspace isomorphic with c_0. Therefore, none of the spaces discussed in this section has a UBSBFDD and none has a subspace isomorphic with c_0. However, there does exist a bounded closed convex subset of c_0 that has CPCP and fails PCP [**1**].

The definition of each space will be very explicit, so they may provide some understanding of how such spaces may be constructed. Perhaps they will provide enough understanding so that one might construct a space without RNP or a c_0-subspace, for which all subspaces that fail RNP also fail CPCP. Because of [**22**], this is a necessary condition for a counterexample to the conjecture that KMP $\Rightarrow$ RNP.

To describe these spaces, we will use partially ordered sets E for which each member has at least two successors, E has a first member e_0, and for each member e of E there is a linearly ordered chain of successive members of E for which e_0 and e are the first and last members. If the chain that joins e_0 to e has n members, then e is said to be of *order* $n-1$. In particular, the successors of e_0 are of order 1. The meanings of *branches* and *segments* are the same as for bushes. Let L be the natural linear space of all formal linear combinations with real coefficients of such a set E, with e_0 the zero of L.

Various norms will be defined on L and completed to give Banach spaces. A symbol to denote one of these spaces will be used without a subscript if each member of E has finitely many successors; it will be used with the subscript ∞ if each member of E has infinitely many successors. For each space, the norm will be defined by using certain functions denoted by φ and then letting

$$||x|| = \inf\{\Sigma_k \varphi[w(k)] : x = \Sigma_k w(k)\},$$

where φ can vary with $w(k)$. For each of the resulting spaces, the sum of an initial segment will have norm 1. So that the union of sums of initial segments in a space of type Y will be a bush, we impose the condition on the E used to define Y that, for each $e \in E$, the sum of the successors of e is 0 (without these conditions, Y would be isomorphic with l_1); also (to simplify a later proof) let all members of E with the same order have the same number of successors. For each space of type Z, Λ, Φ, Ψ, and Ω, we assume that there is a sequence $\{\lambda_n\}$ such that each member of the corresponding E with order n has at least λ_n successors and λ_n increases sufficiently rapidly so that the set of sums of initial segments is an approximate bush [**10**, p. 348]. In particular, for Z we assume that there is a number $r > 1$ such that $\lambda_{n+1} > r\lambda_n$ for each n. A *bump with altitude* h is a function on E whose support is a segment on which the function is identically h. For each of the following spaces, $||x|| = |h|$ if x is a bump with altitude h whose support contains e_0. A set of bumps is *disjoint* if their supports are disjoint.

X: The space whose properties in the following table were established in [**15**].

Y: Let $\varphi(w) = |h|$, if w is a bump with altitude h.

Z & Z_∞: Let $\varphi_1(w) = |h|$ if w is a bump with altitude h. Let $\varphi_2(w) = (\Sigma w_i^2)^{1/2}$, if $\{w_i\}$ is the set of coefficients of w as a member of L.

Λ & Λ_∞: Let $\varphi(w) = (\Sigma h_k^2)^{1/2}$, if $w = \Sigma w_k$ where $\{w_k\}$ is a set of disjoint bumps with corresponding altitudes $\{h_k\}$.

Φ & Φ_∞: Let $\varphi(w) = (\Sigma h_k^2)^{1/2}$, if $w = \Sigma w_k$ where $\{w_k\}$ is a set of bumps with corresponding altitudes $\{h_k\}$ and the property that no segment in E contains points of the supports of two different bumps in $\{w_k\}$.

Ψ & Ψ_∞: Let $\varphi(w) = [\Sigma_\nu(\Sigma_{i=1}^\infty \mid u(\nu,i) - u(\nu,i+1)|)^2]^{1/2}$, if there are finitely many segments $\{S(\nu)\}$ whose first members are distinct members of E with the same order and $w = \Sigma u(\nu)$, where each $u(\nu) = \Sigma_{i=1}^\infty u(\nu,i)e(\nu,i)$ and $S(\nu)$ is the segment $\{e(\nu,i) : 1 \le i < \infty\}$.

Ω & Ω_∞: Let $\varphi(w) = [\Sigma_\nu(\sup\{\Sigma_{i=1}^\infty[u(\nu,p_i) - u(\nu,p_{i+1})]^2\})]^{1/2}$, where the sup is for all increasing sequences $\{p_i\}$ of positive integers and there are finitely many segments $\{S(\nu)\}$ whose first members are distinct members of E with the same order and $w = \Sigma u(\nu)$, where each

$u(\nu) = \Sigma_{i=1}^{\infty} u(\nu, i)e(\nu, i)$ and $S(\nu)$ is the segment $\{e(\nu, i) : 1 \leq i < \infty\}$.

The above definitions can be modified in many ways, not necessarily preserving the space within isomorphism. For example, for Z and Z_∞, one could require that $\varphi_2(w)$ be defined only if, for some n, w belongs to the span of members of E with order n; for Λ and Λ_∞, one could require that the bumps of $\{w_k\}$ have supports that are subsets of segments whose first members are distinct members of E with the same order; and for Ψ and Ψ_∞ or Ω and Ω_∞, one could require only that the segments $\{S(\nu)\}$ be disjoint.

Each space described above contains a bush. Therefore, each space fails RNP. It is easy to see that the obvious bushes are complemented, so each space fails KMP [**10**, p. 354]. This completes the first two columns of the following table, where X is the space whose properties were established in [**15**], except for noting that X would be reflexive if it did not have an l_1-subspace, since it has no c_0-subspaces and is contained in a space with an unconditional basis. We will now establish most of the remaining entries. The question marks indicate unresolved problems.

	RNP	KMP	PCP	CPCP	c_0 subspace	UBFDD	l_1 subspace	Subspace of UB-space
X	NO	NO	YES	YES	NO	NO	YES	YES
Y	NO	NO	NO	NO	NO	NO	YES	NO
Z	NO	NO	NO	NO	NO	NO	YES	NO
Λ	NO	NO	YES	YES	NO	NO	NO	NO
Φ	NO	NO	YES	YES	NO	NO	YES	?
Ψ	NO	NO	YES	YES	NO	NO	YES	?
Ω	NO	NO	YES	YES	NO	NO	YES	NO
Z_∞	NO	NO	NO	NO	NO	NO	YES	NO
Λ_∞	NO	NO	NO	YES	NO	NO	NO	NO
Φ_∞	NO	NO	NO	YES	NO	NO	YES	NO
Ψ_∞	NO	NO	NO	YES	NO	NO	YES	NO
Ω_∞	NO	NO	NO	YES	NO	NO	YES	NO

For each of the spaces other than X, let G_n be the closure of the linear span of all members of E with order n.

The space Y is one of the simplest spaces to fail RNP. To show that Y fails CPCP (and therefore PCP), let K be the closure of the convex span of all sums of initial segments of E. Suppose K has a point of continuity ω. Then there is a ω-neighborhood of ω,

$$U = \{x : |f_i(x-\omega)| < 2\varepsilon \text{ for } 1 \leq i \leq k\},$$

such that $U \cap K \subset B(\omega, 1/3)$ and $||f_i|| \leq 1$ for each i. Let $\omega_0 \in U \cap K$, $|f_i(\omega_0 - \omega)| < \varepsilon$ for each i, and $\omega_0 = \Sigma_{r=1}^n \lambda_r \sigma_r$, where each σ_r is the sum of an initial segment s_r of E. Consider a particular s_r with sum σ_r. Choose a length for segments that follow s_r so that there are more than p such segments, where p will be specified later. Note that the sum of these segments is 0. Let η_p^r be the number of successors of any one of these segments. For segment i, let the set A^i be $\varnothing$ if η_p^r is even, and A^i contain exactly one successor of segment i if η_p^r is odd. Separate the remaining successors of segment i into equinumerable disjoint sets S_1^i and S_2^i. Let Δ_i (or δ_i) be the weighted average of the members of $A^i \cup S_1^i$ (or $A^i \cup S_2^i$) with the weight of the member of A^i (if any) half that of the others. Then $\Delta_i = -\delta_i$ and $2/3 \leq ||\Delta_i|| \leq 1$, where the $2/3$ is attained only if $\eta_p^r = 3$. Consider f_1 and $D^r = \Sigma_1^p d_i/p$, where d_i is either Δ_i or δ_i for each i. For any set of more than $1/\varepsilon$ of the d_i's, there are values for some d_i and d_j such that $|f_1(d_i + d_j)| < \varepsilon$. With such choices of d_i's, $pD^r = \Sigma_1^p d_i$ can be regarded as the sum of at most $p/2$ pairs on each of which $|f_1(d_i + d_j)| < \varepsilon$ and at most $1/\varepsilon$ terms with $|f_1(d_i)| \leq 1$. Thus $|f_1(D^r)| < 1/(\varepsilon p) + \varepsilon/2$. Similarly, for any set of more than $1/\varepsilon$ such pairs, there is a sum of pairs $x = \pm(d_h + d_i) \pm (d_j + d_k)$ for which $|f_2(x)| < 2\varepsilon$. There are at most p/4 such x's. After such changes in the value of D^r, we have $|f_2(D^r)| < 1/(\varepsilon p) + 2/(\varepsilon p) + \varepsilon/2$. Since this can be continued up to f_k, it is possible to choose p great enough that $|f_i(D^r)| < \varepsilon$ for each i. Let $D = \Sigma_{r=1}^n \lambda_r D^r$. Then $\Sigma_{r=1}^n \lambda_r(\sigma_r + D^r) = \omega_0 + D$ and $||D|| > 2/3$. Since

$$|f_i(\omega_0 + D) - \omega| \leq |f_i(\omega_0 - \omega)| + |f_i(D)| < 2\varepsilon \quad \text{ for each } i,$$

we have $\omega_0 + D \in U$. Now we have the contradiction that $||\omega_0 - \omega|| < 1/3$, $||\omega_0 + D) - \omega|| < 1/3$, and $||D|| > 2/3$.

We will now show that $Z_\infty, \Lambda_\infty, \Phi_\infty, \Psi_\infty$, and Ω_∞ fail PCP. In each case, let S be the set of all sums of initial segments of E. Observe that the norm of the sum of n distinct members of E with the same order is $n^{1/2}$. Suppose S has a point of continuity σ. Let $B(\sigma, 1/2)$ be the 1/2-neighborhood of σ. Then there is a w-neighborhood of σ,

$$U = \{x : |f_k(x - \sigma)| < \varepsilon \text{ for } 1 \le k \le n\},$$

such that $U \cap S \subset B(\sigma, 1/2)$. Let e be the last member of the segment s whose sum is σ. Since e has infinitely many successors, there is a successor e' of e for which $|f_k(e')| < \varepsilon$ for each k. Then $\sigma + e' \in U \cap S$, but $||e'|| = 1$ implies $\sigma + e'$ does not belong to $B(\sigma, 1/2)$. This contradiction implies S has no points of continuity.

To show that Z_∞ fails CPCP, let K be the closure of the convex span of all sums of initial segments of E. Suppose K has a point of continuity ω. Then there is a w-neighborhood of ω,

$$U = \{x : |f_k(x - \omega)| < \varepsilon \text{ for } 1 \le k \le n\},$$

such that $U \cap K \subset B(\omega, 1/4)$. Let $\omega_0 \in U \cap K$ be a convex span, $\sum_{i=1}^n \lambda_i \sigma_i$, where each σ_i is the sum of an initial segment s_i of E. For an arbitrary positive number δ, let $\{\delta_j : j \ge 1\}$ be a sequence of positive numbers for which $\sum \delta_j < \delta$. Consider a particular s_i with sum σ_i. Since the last member e_i of s_i has infinitely many successors, there is a successor e_i^1 of e_i for which $|f_k(e_i^1)| < \delta_1$ for each k. Then there is a successor e_i^2 of e_i^1 for which $|f_k(e_i^2)| < \delta_2$ for each k. For any positive integer p, we can extend s_i in p steps to be a segment s_i^p with sum σ_i^p for which

$$|f_k(\sigma_i^p - \sigma_i)| < \sum_{j=1}^p \delta_j < \delta.$$

When this has been done for each i, we have $|f_k(\sum_{i=1}^n \lambda_i \sigma_i^p - \omega_0)| < \delta$. This yields a contradiction, since δ can be small enough that $\sum_{i=1}^n \lambda_i \sigma_i^p$ belongs to $U \cap K$ and p can be great enough that

$$\left\| \sum_{i=1}^n \lambda_i \sigma_i^p - \omega_0 \right\| > 3/4,$$

which implies that $|| \sum_{i=1}^n \lambda_i \sigma_i^p - \omega|| > 1/2$.

The sequence $\{G_n\}$ is not a BCSBFDD for Z. In fact, Z fails CPCP (and therefore fails PCP). Although the proof can have similarity with the preceding proofs for Y and Z_∞, it is much more difficult and does not seem sufficiently interesting to include.

That Λ, Φ, Ψ, and Ω have PCP (and therefore CPCP) follows from Theorem 2.1, Lemma 2.2, and the next three lemmas. These lemmas also provide needed information for proving that $\Lambda_\infty, \Phi_\infty, \Psi_\infty$, and Ω_∞ have CPCP.

Lemma 3.1. *The spaces $\Lambda, \Lambda_\infty, \Phi$, and Φ_∞ have the property that, for any $\alpha > 0$ and $\beta = (1/4)\alpha^3(4+\alpha^4)^{-1/2}$,*

$$||x+y|| \geq ||x|| + \beta||y|| \qquad \text{if } \ ||y|| > \alpha||x|| \tag{7}$$

and (x, y) is a skipping pair with respect to $\{G_n\}$.

Proof. The proof is the same for all four spaces. Suppose that (x, y) is a skipping pair. Let p be an integer for which $x \in \overline{\text{lin}}\,\{G_i : i < p\}$ and $y \in \overline{\text{lin}}\,\{G_i : i > p\}$. Without loss of generality, we assume that x and y belong to the corresponding linear space L, $||x|| = 1$, and $||y|| = a > \alpha$. Suppose (7) is not satisfied for some β. Then $||x+y|| < 1 + \beta a$ and there exists a finite set $\{w_k\}$ for which $\Sigma w_k = x + y$ and

$$1 + \beta a > \Sigma\varphi(w_k), \qquad \varphi(w_k) = [\Sigma_i h(k,i)^2]^{1/2}, \tag{8}$$

where each w_k is the sum of bumps with altitudes $\{h(k,i)\}$. If there is an $e \in E$ with order p for which some bump used in (8) is not zero at e, let $h(r,i)$ have the least absolute value of altitudes of such bumps. Then some w_s with $s \neq r$ has a bump with altitude $h(s,j)$ of opposite sign. Replace w_s by θw_s and $(1-\theta)w_s$, where $0 \leq \theta \leq 1$ and the bump in θw_s whose support contains e has altitude $-h(r,i)$. Relabel these bumps in w_r and θw_s so that the one whose support has an element of least order uses λ instead of h and the other uses μ instead of h (if the least order for elements in the support is the same for both bumps, truncate both bumps so their supports contain only members of E with orders greater than p). Repeat this until no w_k or replacement uses any bump that is nonzero at e, unless it has been relabeled. Then do this successively for other e's of order p at which some bump is nonzero. Finally, for the new $\{\bar{w}_k\}$ we have $\Sigma\bar{w}_k = x + y$ and the inequality in (8) can be replaced by

$$1 + \beta a > \Sigma\varphi(\bar{w}_k), \tag{9}$$

where

$$\varphi(\bar{w}_k) = [\Sigma_i \lambda(k,i)^2 + \Sigma_i \xi(k,i)^2 + \Sigma_i \mu(k,i)^2 + \Sigma_i \eta(k,i)^2]^{1/2}. \tag{10}$$

Here $\lambda(k,i)$ and $\mu(k,i)$ are relabeled altitudes as described above, $\xi(k,i)$ is the altitude of a bump whose support has members only of order less than p, and $\eta(k,i)$ is the altitude of a bump whose support has members only of order greater than p. It follows from (9), (10), and Minkowski's inequality that

$$\begin{aligned} 1+\beta a > [\{\Sigma_k[\Sigma_i \xi(k,i)^2]^{1/2}\}^2 + \{\Sigma_k[\Sigma_i \lambda(k,i)^2 \\ + \Sigma_i \mu(k,i)^2 + \Sigma_i \eta(k,i)^2]^{1/2}\}^2]^{1/2}. \end{aligned}$$

If we truncate all bumps used in the second of the two expressions in braces so that all bumps have support in the set of all members of E with orders greater than p, then we see that this expression is at least as great as $||y||$. Therefore,

$$1+\beta a > [\{\Sigma_k[\Sigma_i \xi(k,i)^2]^{1/2}\}^2 + a^2]^{1/2},$$

which implies that

$$\Sigma_k[\Sigma_i \xi(k,i)^2]^{1/2} < [(1+\beta a)^2 - a^2]^{1/2}. \tag{11}$$

Now observe that it follows from (9) and (10) that

$$\begin{aligned} 1+\beta a > [\{\Sigma_k[\Sigma_i \lambda(k,i)^2 + \Sigma_i \xi(k,i)^2]^{1/2}\}^2 \\ + \{\Sigma_k[\Sigma_i \mu(k,i)^2]^{1/2}\}^2]^{1/2}. \end{aligned} \tag{12}$$

The first of the two expressions in braces is at least as great as $||x|| = 1$, since the sum of the bumps with altitudes $\lambda(k,i)$ or $\xi(k,i)$ is x if each λ-bump is truncated so that its support contains only members of E with orders less than the order of members of the support of the corresponding μ-bump. Also,

$$\begin{aligned} \Sigma_k[\Sigma_i \mu(k,i)^2]^{1/2} \geq \Sigma_k[\Sigma_i \mu(k,i)^2 + \Sigma_i \xi(k,i)^2]^{1/2} \\ - \Sigma_k[\Sigma_i \xi(k,i)^2]^{1/2}, \end{aligned} \tag{13}$$

which is obtained by applying for each k the inequality $|x| \geq (x^2 + y^2)^{1/2} - |y|$. The first summation over k in the right member of (13) is at least as great as 1, since the sum of the μ-bumps and the ξ-bumps becomes x if each μ-bump with altitude $\mu(k,i)$ is replaced by a bump with altitude $-\mu(k,i)$ whose support is the set of members e of E with orders less than p that are in the support of the corresponding λ-bump and not in the support of this μ-bump. Now it follows from (12), the observation following (12), the observation following (13), and (11), that

$$1+\beta a > [1+\{1-[(1+\beta a)^2 - a^2]^{1/2}\}^2]^{1/2}.$$

This implies $\beta > (a^{-2} + (1/4)a^2)^{1/2} - a^{-1}$ and therefore

$$\beta > (1/4)a^3(4+a^4)^{-1/2}.$$

Since $a > \alpha$, it follows that (7) is satisfied if $\beta = (1/4)\alpha^3(4+\alpha^4)^{-1/2}$. □

Lemma 3.2. *The spaces* Ψ *and* Ψ_∞ *have the property that, for any* $\alpha > 0$ *and* $\beta = (1/2)\alpha(1+\alpha^2)^{-1/2}$,

$$||x+y|| \geq ||x|| + \beta||y|| \quad \textit{if} \quad ||y|| > \alpha||x|| \tag{14}$$

and (x,y) *is a skipping pair with respect to* $\{G_n\}$.

Proof. The proof is the same for both spaces. Suppose that (x,y) is a skipping pair. Let p be an integer for which $x \in \overline{\text{lin}}\,\{G_i : i < p\}$ and $y \in \overline{\text{lin}}\,\{G_i : i > p\}$. Without loss of generality, we assume that x and y belong to the corresponding linear space L, $||x|| = 1$, and $||y|| = a > \alpha$. If (14) is not satisfied for some β, then $||x+y|| < 1+\beta a$ and there exists a finite set $\{w_k\}$ for which $\Sigma w_k = x+y$ and

$$1+\beta a > \Sigma\varphi(w_k), \qquad \varphi(w_k) = \left[\sum_\nu \left\{\sum_{i=1}^\infty |u_k(\nu,i) - u_k(\nu,i+1)|\right\}^2\right]^{1/2},$$

where for each k there are finitely many infinite segments $\{S(\nu,k)\}$ whose first members are distinct members of the same order in E and $w_k = \Sigma_\nu u_k(\nu)$, where each $u_k(\nu) = \sum_{i=1}^\infty u_k(\nu,i)e_k(\nu,i)$ and $S(\nu,k)$ is the segment $\{e_k(\nu,i) : 1 \leq i < \infty\}$. If $S(\nu,k)$ has an $e_k(\nu,i)$ of order p, let this value of i be denoted by ρ. Otherwise, let $\rho = 0$. For each k and ν, we let

$$u_k(\nu) = \lambda_k(\nu) + \mu_k(\nu), \tag{15}$$

where $\mu_k(\nu, i)$ is constant for $1 \le i \le \rho$, $\mu_k(\nu, i) = u_k(\nu, i)$ if $i \ge \rho$, and $\lambda_k(\nu) = u_k(\nu) - \mu_k(\nu)$. Then $\lambda_k(\nu, i) = 0$ if $i \ge \rho$, $x = \Sigma_k \Sigma_\nu \lambda_k(\nu)$, and $y = \Sigma_k \Sigma_\nu \mu_k(\nu)$. Also,

$$\varphi(w_k) = \Bigg[\sum_\nu \Bigg\{ \sum_{i=1}^\infty |\lambda_k(\nu, i) - \lambda_k(\nu, i+1)| + \sum_{i=1}^\infty |\mu_k(\nu, i) - \mu_k(\nu, i+1)| \Bigg\}^2 \Bigg]^{1/2},$$

which implies

$$\varphi(w_k) \ge \Bigg[\sum_\nu \Bigg\{ \sum_{i=1}^\infty |\lambda_k(\nu, i) - \lambda_k(\nu, i+1)| \Bigg\}^2 + \sum_\nu \Bigg\{ \sum_{i=1}^\infty |\mu_k(\nu, i) - \mu_k(\nu, i+1)| \Bigg\}^2 \Bigg]^{1/2}. \tag{16}$$

Therefore,

$$\begin{aligned} 1 + \beta a &> \sum \varphi(w_k) \\ &\ge \Bigg[\Bigg\{ \sum_k \Bigg(\sum_\nu \Bigg\{ \sum_{i=1}^\infty |\lambda_k(\nu, i) - \lambda_k(\nu, i+1)| \Bigg\}^2 \Bigg)^{1/2} \Bigg\}^2 \\ &\qquad + \Bigg\{ \sum_k \Bigg(\sum_\nu \Bigg\{ \sum_{i=1}^\infty |\mu_k(\nu, i) - \mu_k(\nu, i+1)| \Bigg\}^2 \Bigg)^{1/2} \Bigg\}^2 \Bigg]^{1/2} \\ &\ge (||x||^2 + ||y||^2)^{1/2} = (1 + a^2)^{1/2}. \end{aligned}$$

This implies $\beta > (1 + a^{-2})^{1/2} - a^{-1} > (1/2)(1 + a^{-2})^{-1/2}$. Since $a > \alpha$, it follows that (14) is satisfied if $\beta = (1/2)a(1 + a^2)^{-1/2}$. $\square$

Lemma 3.3. *The spaces Ω and Ω_∞ have the property that, for any $\alpha > 0$ and $\beta = (1/2)\alpha(1 + \alpha^2)^{-1/2}$,*

$$||x + y|| \ge ||x|| + \beta||y|| \qquad \textit{if } \; ||y|| > \alpha||x||$$

and (x, y) is a skipping pair with respect to $\{G_n\}$.

Proof. The proof is the same for both spaces and similar to that for Lemma 3.2. We make the obvious changes that result from the new definition of $\varphi(w)$ until the functions $\lambda_k(\nu)$ and $\mu_k(\nu)$ have been defined. Recall that for Ω and Ω_∞,

$$\varphi(w_k) = \left[\sum_\nu \left(\sup \left\{ \sum_{i=1}^\infty [u_k(\nu, p_i) - u_k(\nu, p_{i+1})]^2 \right\} \right) \right]^{1/2}.$$

Then note that $\sup\{\sum_{i=1}^\infty [u_k(\nu, p_i) - u_k(\nu, p_{i+1})]^2\}$ is not increased if the sequence $\{p_i\}$ is restricted by requiring that $p_1 = \rho$ for some i. With λ_k and μ_k defined as in (15), it then follows as for (16) that

$$\varphi(w_k) \geq \left[\sum_\nu \sup \left\{ \sum_{i=1}^\infty [\lambda_k(\nu, p_i) - \lambda_k(\nu, p_{i+1})]^2 \right\} \right. \\ \left. + \sum_\nu \sup \left\{ \sum_{i=1}^\infty [\mu_k(\nu, p_i) - \mu_k(\nu, p_{i+1})]^2 \right\} \right]^{1/2}.$$

The concluding calculations are essentially the same as for Lemma 3.2. □

As was noted earlier, it follows from Lemma 2.2, the preceding three lemmas, and Theorem 2.1 that Λ, Φ, Ψ, and Ω have PCP (and therefore CPCP). It follows from the next lemma, Lemma 2.3, and Lemmas 3.1–3.3 that the hypotheses of Theorem 2.4 are satisfied for each of the spaces $\Lambda_\infty, \Phi_\infty, \Psi_\infty$, and Ω_∞, so that each space has CPCP and for each space the corresponding monotone decomposition $\{G_n\}$ is a BCSBD.

Lemma 3.4. *Each of the spaces $\Lambda_\infty, \Phi_\infty, \Psi_\infty$, and Ω_∞ satisfies the hypotheses of Lemma* 2.3.

Proof. The proof will be given for all these spaces simultaneously. Let X denote the space. We need to show that if K is a bounded convex nonempty subset of X, then K has the property that, for any $\Delta > 0$ and any weakly open convex set W with $W \cap K \neq \varnothing$, there is a weakly open convex set V which has the properties that $V \cap W \cap K \neq \varnothing$ and there are integers r and N such that, if $z \in V \cap W \cap K$ and $t > r$, then there exists $\xi \in X$ for which

$$||z - \xi|| < \Delta \quad \text{and} \quad \sum_1^\infty |a_i^t| \le N,$$

where $P_{(=t)}\xi = \sum_1^\infty a_i^t e_i^t$ with $\{e_i^t\}$ the orthonormal basis of G_t.

For an arbitrary $\varepsilon > 0$, we could use the type of argument in the beginning of the proof of Lemma 2.3 to show that, for any weakly open convex set W with $W \cap K \neq \varnothing$, there is a weakly open convex set $U \subset W$ and an integer r such that $U \cap K \neq \varnothing$, and, if $z \in U \cap K$, then

$$\operatorname{diam}[P_{(<r)}(U \cap K)] < \varepsilon, \qquad ||P_{(<r)}z|| > \sup\{||x|| : x \in U \cap K\} - \varepsilon.$$

Rather than complicating the proof unnecessarily, it will be understood that ε has been chosen *sufficiently small* and expressions in italics have meanings for which our conclusions are valid. Choose $z_0 \in U \cap K$ so that $||P_{(=r)}z_0||$ *nearly equals* $\sup\{||P_{(=r)}z|| : z \in U \cap K\} = \sigma$. Choose a finite subset $S = \{e_i^r : 1 \le i \le m\}$ of the basis vectors of G_r for which

$$||P_{(=r)}z_0|| \quad \textit{nearly equals} \quad \left[\sum_{i=1}^m (\alpha_i^r)^2\right]^{1/2},$$

where $P_{(=r)}z_0 = \sum_{i=1}^\infty \alpha_i^r e_i^r$. Choose $V \subset U$ so that $z_0 \in V$ and $[\sum_{i=1}^m (a_i^r)^2]^{1/2}$ *nearly equals* σ if $z \in V \cap K$ and $z = \sum_{i=1}^\infty a_i^r e_i^r$. Now let $z \in V \cap K$. *Approximate* z by η that belongs to L. *Approximate* $||\eta||$ by $\Sigma\varphi(w_k)$, where $\eta = \Sigma w_k$ and $w_k = \Sigma u_k(\nu)$. Each $u_k(\nu)$ is a function on a segment (a "bump" if X is Λ_∞ or Φ_∞). Discard all $u_k(\nu)$ whose supports are entirely on points of E with order r or greater. Since the projection of the sum of the remaining w_k's onto $\overline{\text{lin}}\,\{G_n : n < r\}$ is equal to $P_{(<r)}\eta$, the sum of all discarded $u_k(\nu)$'s is *small*. Therefore, we can assume without loss of generality that no such $u_k(\nu)$ exists, provided we replace the equality "$\eta = \Sigma w_k$" by "η is *nearly equal* to Σw_k." Then Σw_k is *nearly equal* to z.

Now note that $\Sigma\varphi(w_k)$ is not increased and is *decreased very little* if each $u_k(\nu)$ is modified to be zero at all points of S or followers of points of S, since the projection of the sum of the modified w_k's onto $\overline{\text{lin}}\,\{G_n : n < r\}$ is equal to $P_{(<r)}\eta$. Since the projection $P_{(\sim S)}\eta$ of

$P_{(=r)}\eta$ onto $\overline{\text{lin}}\,(\sim S)$ is *small*, ω is *nearly equal* to z if ω is obtained from Σw_k by replacing each $P_{(\sim S)}u_k(\nu)$ by zero. It then follows from Lemmas 3.1–3.3 that the norm of the sum over k and ν of the "part" of $u_k(\nu)$ that follows $\sim S$ is *small*. Replace ω by ξ, where ξ is the sum over k and ν of $u_k^*(\nu)$, where $u_k^*(\nu)$ is $u_k(\nu)$ with this "part" replaced by zero if the segment corresponding to $u_k(\nu)$ contains a point of $\sim S$, and $u_k^*(\nu) = u_k(\nu)$ otherwise. Then ξ is *nearly equal* to z. For each k, let $w_k^* = \Sigma_\nu u_k^*(\nu)$. Then for each w_k^* and $e_j^r \in S$, there is at most one ν for which the segment corresponding to $u_k^*(\nu)$ contains e_j^r. Therefore, for each $t \geq r$,

$$\varphi(w_k^*) \geq \left[\sum_{i=1}^{m} |h^t(k,j)|^2\right]^{1/2} \geq m^{-1/2}\sum_{i=1}^{m} |h^t(k,j)|,$$

where $u_k^*(\nu)$ corresponds to e_j^r and $h^t(k,j)$ is either 0 or the value of $u_k^*(\nu)$ at a basis vector of G_t. Then

$$\sum \varphi(w_k^*) \geq m^{-1/2}\sum_{k,j} |h^t(k,j)| \geq m^{-1/2}\sum |a_i^t|,$$

where $P_{(=t)}\xi = \Sigma a_i^t e_i^t$ with $\{e_i^t\}$ the orthonormal basis of G_t. Since $\Sigma\varphi(w_k^*) \leq \Sigma\varphi(w_k) \leq \Sigma\varphi(w_k) \leq 2\sigma$, the desired N can be $2m^{1/2}\sigma$. □

The first four columns of the table have been completed. No space that has CPCP has a subspace isomorphic with c_0. This completes the fifth column, except for Y, Z and Z_∞. But it can be shown that these spaces do not have subspaces isomorphic with c_0.

It follows from Theorem 1.1 that none of the spaces considered has a UBFDD, which completes column six.

The space X in the table must have an l_1-subspace, since otherwise it would have neither c_0 nor l_1-subspaces and therefore would be reflexive [**2**]. The rest of column 7 is left for the reader.

Since none of the spaces has a c_0-subspace, those spaces that have no l_1-subspaces cannot be subspaces of a space with an unconditional basis [**2**] and it follows from Theorem 1.1 that those spaces which fail PCP are not subspaces of a space with an unconditional basis. The spaces Ω

and Ω_∞ have J-subspaces, where J is the space described in [**11**]. Since J is not reflexive and has neither c_0 nor l_1-subspaces, it (and therefore Ω and Ω_∞) are not subspaces of a space with an unconditional basis.

Questions. 1. Let X be a Banach space that has a monotone BCSBD $\{G_n\}$ for which each G_n is isometric with l_2. Does X have CPCP? There is a space that fails CPCP and satisfies these conditions if "monotone" is replaced by "complemented" (S_*T_∞ of [**8**, Theorem VI.1]). The space Z_∞ fails CPCP, but its natural monotone decomposition is not a BCSBD.

2. It is known that X of Question 1 has CPCP if X also satisfies either (I) or (II):

(I) Conditions (i) and (ii) of Theorem 2.4.

(II) $\sum_{n=1}^\infty x_n$ converges whenever $x_n \in G_n$, $\sup_n ||\sum_{i=1}^n x_i|| < \infty$, and $\liminf_n ||x_n||_\infty = 0$, where $||x_n||_\infty$ is the sup norm of the sequence of coefficients of x_n when represented by the orthonormal basis of G_n [**8**, Proposition IV.3 and Theorem IV.5]. Does (I) imply (II) for such X? Is there some interesting necessary and sufficient condition for X to have CPCP? In particular, is there such an X that has CPCP and fails (II)? Suppose there is an E as described in Section 3 such that each initial segment of E has unit norm and the set of members of E with order n is an orthonormal basis for G_n. Then X fails PCP. For such an X, what are the answers to the preceding questions?

3. Let each member of E as described in Section 3 have infinitely many successors. Can a norm be defined on $\operatorname{lin}(E) = L$ so that the completion of L has CPCP, each initial segment of E has unit norm, the closed linear span of each branch of E is reflexive (or l_2) and the set of members of E with order n is an orthonormal basis for l_2? These hypotheses imply the space fails PCP.

4. The undetermined entries in the table.

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