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BLOW-UP BEHAVIOR FOR SEMILINEAR HEAT EQUATIONS: MULTI-DIMENSIONAL CASE

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ABSTRACT. This paper is concerned with the Cauchy problem:

$$u_t - \Delta u = F(u), \qquad (x,t) \in R^N \times (0,T)$$
$$u(x,0) = u_0(x)$$

where $u_0(x)$ is continuous, nonnegative and bounded, and $F(u) = u^p$ with $p > 1$ or $F(u) = e^u$. Assume that u blows up at $x = 0$ and $t = T$. In case $F(u) = u^p$, let $w(y,s) = (T-t)^{1/(p-1)}u(y(T-t)^{1/2},t)$, $s = -\log(T-t)$. We study the large time behavior of $w(y,s)$. In the radial case, we prove: if $w(y,s) \not\equiv \beta^\beta$ ($\beta = (p-1)^{-1}$), then either $w(y,s) = \beta^\beta(1-(2ps)^{-1}NH(y)) + o(1/s)$ where $H(y) = (2N)^{-1}|y|^2 - 1$ or there exists an $m \geq 3$, $k_m > 1$, constants C_i (not all zero) and polynomials $H_{m,i}$ of degree m, such that $w(y,s) = \beta^\beta(1 - e^{(1-m/2)s}\sum_{i=1}^{k_m} C_i H_{m,i}(y)) + o(e^{(1-m/2)s})$. The above convergence takes place in C^2_{loc} as well as in some weighted Sobolev space. For the nonradial solutions, we also obtain some results in the case $N = 2$. Similar results also hold in the case $F(u) = e^u$.

1. Introduction. This paper is concerned with nonnegative blowing up solutions of the initial value problem:

$$u_t = \Delta u + F(u) \quad \text{in} \quad R^N \times (0,T) \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in R^N \tag{1.2}$$

where $u_0(x)$ is continuous, nonnegative and bounded, and

$$F(u) = u^p \quad \text{with} \quad p > 1, \qquad \text{or} \quad F(u) = e^u. \tag{1.3}$$

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It is well-known that for suitably chosen initial data, the solution of (1.1)–(1.2) blows up in finite time. For instance, if $F(u) = u^p$ with $1 < p \le 1 + 2/N$, for any nontrivial solution u of (1.1)–(1.2), there exists a finite time T such that

$$\limsup_{t\to T}(\sup_{x\in R^N} u(x,t)) = +\infty \tag{1.4}$$

(see, e.g., [**6**].) We then call T the blow-up time. If, for some $x_0 \in R^N$, there exists a sequence $\{(x_n, t_n)\}$, such that $x_n \to x_0$, $t_n \to T$, and

$$\lim_{n\to\infty} u(x_n, t_n) = +\infty,$$

we call x_0 a blow-up point. There are many papers concerning the number of blow-up points, their location and the behavior of u near a blow-up point, we refer to [**4**] for a review of recent results.

Of particular interest is the study of asymptotic behavior of solutions as t approaches the blow-up time. In this direction, the pioneering work is Giga and Kohn [**9**], followed by [**10**] and [**11**]. They used the well-known change of variables: for any $a \in R^N$,

$$\begin{gathered} w_a(y,s) = (T-t)^{1/(p-1)}u(x,t), \qquad y = (x-a)(T-t)^{-1/2}, \\ s = -\log(T-t). \end{gathered} \tag{1.5}$$

One can check that:

$$\frac{\partial}{\partial s} w_a = \Delta w - \frac{y}{2}\cdot\nabla w_a - \frac{1}{p-1}w_a + w_a^p. \tag{1.6}$$

The question of studying the blow-up behavior of u near a blow-up point is thus transformed to studying the large time behavior of w. By using the “energy” methods, Giga and Kohn were able to prove that, if $p \le (N+2)/(N-2)$, then

$$w_a(y,s) \to \beta^\beta \quad \text{or} \quad 0, \qquad \beta = 1/(p-1), \tag{1.7}$$

uniformly on the sets $|y| \le R$ with $R > 0$ (if N is 1 or 2, then the conclusion is true for any $1 < p < \infty$). Moreover, if $w_a \to 0$, then a is not a blow-up point. Similar results have been established even for $p > (N+2)/(N-2)$, (see Theorem 2.1 below).

From now on, we shall always assume 0 is a blow-up point and $a = 0$; therefore, we shall suppress the subscript a. By (1.7),

$$\lim_{t\to\infty} (T-t)^{1/(p-1)} u(\xi(T-t)^{1/2}, t) = \beta^\beta$$

uniformly for $|\xi| < R$ with $R > 0$. This establishes the behavior of u in any space-time parabolas. It is natural to ask what happens beyond these parabolas. For example, what is the asymptotic shape of the curve where $(T-t)^{1/(p-1)}u$ is constant?

In the one-dimensional case, Galaktionov and Posachkov [**7**] used a formal argument to derive the ansatz

$$u(x,t) \sim (T-t)^{-1/(p-1)}\left[(p-1) + \frac{(p-1)^2}{4p(T-t)|\log(T-t)|}\right]^{-1/(p-1)}.$$

The counterpart of w is

$$w(y,s) = \beta^\beta\left[1 + \frac{(p-1)y^2}{4ps}\right]^{-1/(p-1)} + O(s^{-1}) \tag{1.8}$$

as $s \to \infty$. Moreover, at $y = 0$,

$$w(0,s) = \beta^\beta\left[1 + \frac{1}{2ps}\right] + o\left(\frac{1}{s}\right). \tag{1.9}$$

M. Herrero and J. Valázquez [**12**] and S. Fillipas [**3**] have independently given rigorous proofs of (1.9) under some conditions on the initial value.

In this paper, we shall extend (1.9) and related results to radial solutions of (1.1)–(1.2) for any dimension N. As to nonradial solutions, we also obtain some results for the case $N = 2$. Before we state our theorems, let us introduce some notations. For $1 \le q < \infty$, and any positive integer k, define

$$L^q_\rho(R^N) = \{f \in L^q_{\text{loc}}(R^N) : \int_{R^N} |f(x)|^q \rho(x)\,dx < \infty\},$$
$$H^k_\rho(R^N) = \{f \in L^2_\rho(R^N) : \text{ for any } j \in [0,k], f^{(j)} \in L^2_\rho(R^N)\}$$

where $\rho(x) \triangleq \exp(-|x|^2/4)$. From now on, the symbols "$||\cdot||$" and "$\langle\cdot,\cdot\rangle$" will denote the norm and the inner product in $L^2_\rho(R^N)$, respectively, and the symbol C will represent a positive constant, not necessarily the same at each occurrence. Now we can state our main results.

Theorem 1.1. *Assume that w is a bounded, nonnegative and radial solution of* (1.6) *with the property that* $\|w(\cdot,s)-\beta^\beta\|\to 0$ *as* $s\to\infty$. *Then one of the following cases must occur: either*

$$w(\cdot,s)\equiv\beta^\beta, \tag{1.10}$$

or, for $H(y)\triangleq(2N)^{-1}|y|^2-1$,

$$w(\cdot,s)=\beta^\beta-\frac{N\beta^\beta}{2ps}H(y)+o\left(\frac{1}{s}\right), \tag{1.11}$$

or there exists an $m\geq 3$, *some constants* C_i (*not all zero*) *and polynomials* $H_{m,i}$ *of degree* m, *such that*

$$w(\cdot,s)=\beta^\beta+\sum_{i=1}^{k_m}C_ie^{(1-m/2)s}H_{m,i}(y)+o(e^{(1-m/2)s})) \tag{1.12}$$

where convergence takes place in H^1_ρ *as well as in* C^2_{loc}.

If (1.11) happens, then we have the following theorem:

Theorem 1.2. *Let u be a solution of* (1.1)–(1.2) *with the following properties:*

(i) $F(u)=u^p$, $1<p<\infty$.

(ii) $u_0(x)$ *is a radial function and is monotone decreasing in* $|x|$.

(iii) w *is defined by* (1.5) *and* (1.11) *takes place.*

If $N\geq 3$ *and* $p\geq(N+2)/(N-2)$, *we add the assumption that* $\Delta u_0+u_0^p\geq 0$. *Then*

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(\xi((T-t)|\log(T-t)|)^{1/2},t)$$
$$=\beta^\beta\left[1+\frac{p-1}{4p}|\xi|^2\right]^{-1/(p-1)}$$

uniformly on compact sets $|\xi| \le R$ *with* $R > 0$.

Remark 1.3. Theorems 1.1 and 1.2 in the one-dimensional case were first proved in [**12**]. Actually, in that case, they showed that $u_0(x)$ is radial and has a single maximum at 0 which implies that (1.11) takes place. In the multi-dimensional case, we are unable to prove this fact.

Concerning the nonradial case, we have the following theorem.

Theorem 1.4. *Let* $N = 2$ *and* w *be a bounded and nonnegative solution of* (1.6) *with the property that* $||w(\cdot, s) - \beta^\beta|| \to 0$ *as* $s \to \infty$. *Then one of the following cases must occur: either*

$$w(\cdot, s) \equiv \beta^\beta,$$

or

$$0 < c \le s||w(\cdot, s) - \beta^\beta||_{H^1_\rho} \le C < \infty, \tag{1.13}$$

or there exists an $m \ge 3$, *some constants* C_i (*not all zero*) *and polynomials* $H_{m,i}$ *of degree* m, *such that*

$$w(\cdot, s) = \beta^\beta + e^{(1-m/2)s} \sum_{i=1}^{k_m} C_i H_{m,i}(\cdot) + o(e^{(1-m/2)s}) \qquad \text{as } s \to \infty$$

where convergence takes place in H^1_ρ *as well as* C^2_{loc}.

Corollary 1.5. *Assume* (1.13). *Then, for some positive constant* C,

$$(T-t)^{1/(p-1)} u(\xi[(T-t)|\log(T-t)|]^{1/2}, t) \ge \beta^\beta (1+C|\xi|^2)^{-1/(p-1)} \qquad \text{as } t \to T \tag{1.14}$$

uniformly on sets $|\xi| \le R$ *with* $R > 0$.

As to the case $F(u) = e^u$, we have parallel results. Let

$$w(y, s) = u(x, t) + \log(T-t), \qquad x = y(T-t)^{1/2}, \qquad s = -\log(T-t). \tag{1.15}$$

Then

$$w_s = \Delta w - \frac{y}{2} \cdot \nabla w + e^w - 1. \tag{1.16}$$

Theorem 1.6. *Let w be a bounded radial solution of* (1.16) *with the property that $||w(\cdot, s)|| \to 0$ as $s \to \infty$. Then we have the following alternatives: either*

$$w(\cdot, s) \equiv 0 \quad \text{for any } s > 0,$$

or

$$w(\cdot, s) + \frac{N}{2s}\left(\frac{|y|^2}{2N} - 1\right) = o(1/s), \tag{1.17}$$

or there exists an $m \geq 3$, some constants C_i (not all zero) and polynomials $H_{m,i}$ of degree m, such that:

$$w(\cdot, s) = \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i} + o(e^{(1-m/2)s}) \qquad \text{as } s \to \infty$$

where convergence takes place in H^1_ρ as well as in C^2_{loc}.

Theorem 1.7. *Let u be a solution of* (1.1)–(1.2) *with the following properties:*

(i) $F(u) = e^u$.

(ii) *$u_0(x)$ is radial and monotone decreasing in $|x|$. Moreover, $\Delta u_0 + e^{u_0} \geq 0$.*

(iii) *w is defined by* (1.15) *and* (1.17) *takes place.*

Then

$$\lim_{t \to T} \{u(\xi((T-t)|\log(T-t)|)^{1/2}, t) + \log(T-t)\} = -\log(1+|\xi|^2/4)$$

uniformly on compact subset $|\xi| \leq R$ with $R > 0$.

Theorem 1.8. *Let $N = 2$ and w be a bounded solution of* (1.16) *with the property that $|w(\cdot, s)|| \to 0$ as $s \to \infty$. Then we have the following alternatives*: *either*

$$w \equiv 0$$

or

$$0 < c \leq s||w(\cdot, s)|| \leq C < \infty, \tag{1.18}$$

or there exists an $m \geq 3$, some constants C_i (not all zero), such that

$$w(\cdot, s) = \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i} + o(e^{(1-m/2)s}) \qquad \text{as } s \to \infty.$$

Corollary 1.9. *Assume that* (1.18) *happens. Then, for some $C > 0$,*

$$u(\xi((T-t)|\log(T-t)|)^{1/2}, t) + \log(T-t) \geq -\log(1 + C|\xi|^2).$$

We shall only give the proofs for the case $F(u) = u^p$, since the proofs for the exponential case are the same with minor differences; we only need to change the function $f(v)$ in (2.3) to the function $\tilde{f}(v) = e^v - 1 - v$.

The rest of the paper is organized as follows. In Section 2, we shall review some known results. Then in Section 3 we shall prove some results for the general case (i.e., not necessarily radial solutions). Finally, Theorem 1.1 and Theorem 1.4 are established in Section 4 and Section 5, respectively. Theorem 1.2 can then be proved by adapting the argument in [**12**].

2. Preliminaries. We shall begin with the review of known results. Assume that u is a solution of (1.1) and (1.2) which blows up at 0 and $t = T$.

Theorem 2.1. (i) *Let $F(u) = u^p$ with $1 < p < \infty$ if $N = 1$ or 2, $p \leq (N+2)/(N-2)$ if $N \geq 3$. Then*

$$\lim_{t \to T} u(x(T-t)^{1/2}, t)(T-t)^{1/(p-1)} = \beta^\beta \tag{2.1}$$

uniformly on sets $|x| \le R$ *with* $R > 0$ (*see*[**9, 10, 11**]).

(ii) *Let* $F(u) = u^p$ *and* $u_0(x)$ *be radial and nonincreasing with the property that* $\Delta u_0 + u_0^p \ge 0$. *Then the conclusion of* (i) *is true for any* $1 < p < \infty$ (*see* [**1**]).

(iii) *Let* $F(u) = e^u$, $N \le 2$, *and* $\Delta u_0 + e^{u_0} \ge 0$. *Then*

$$\lim_{t \to T} u(x(T-t)^{1/2}, t) + \log(T-t) = 0 \tag{2.2}$$

uniformly on sets $|x| \le R$ *with* $R > 0$ (*see* [**14**]).

(iv) *Let* $F(u) = e^u$ *and* $u_0(x)$ *be radial and nonincreasing with the property that* $\Delta u_0 + e^{u_0} \ge 0$. *Then the conclusion of* (iii) *is true for any* N (see [**1**]).

Note that, under the assumptions of Theorem 2.1, the function w defined in (1.5) or in (1.15) is bounded.

Let $Lv \triangleq \rho^{-1}\nabla \cdot (\rho \nabla v) + v$ with $\rho = e^{-|y|^2/4}$. Then L is a self-adjoint operator on L^2_ρ. The eigenvalues of L are

$$\lambda = 1 - m/2, \qquad m = 0, 1, 2, \dots$$

and the corresponding eigenfunctions are

$$\begin{array}{lll} \text{for} & \lambda_0 = 1, & h_0 \\ \text{for} & \lambda_1 = 1/2, & h_1(y_i), \qquad i = 1, 2, \dots, N \\ & & (N \text{ distinct eigenfunctions}) \\ \text{for} & \lambda_2 = 0, & h_2(y_i), \qquad i = 1, 2, \dots, N \\ & & h_1(y_i)h_2(y_j), \qquad i \ne j, i, j = 1, 2, \dots, N \\ & & \left(N + \binom{N}{2} \text{ distinct eigenfunctions}\right) \\ & & \dots \end{array}$$

where $h_k(y) = H_k(y/2)$ and $H_k(x)$ is the standard k-th Hermite polynomial. These eigenfunctions form an orthogonal basis of L^2_ρ. For a proof, see [**3**].

We shall write $H_{m,i}$, $i = 1, \dots, k_m$ for the eigenfunctions corresponding to the eigenvalue $1 - m/2$ with the property $||H_{m,i}|| = 1$. Hence the set $\{H_{m,i}, m = 0, 1, \dots, i = 1, \dots, k_m\}$ forms an orthonormal basis of L^2_ρ.

Let $w(y,s) = (T-t)^{1/(p-1)}u(x,t)$, where y and s were defined in (1.5), and let $v(y,s) = w(y,s) - \beta^\beta$. From Theorem 2.1, $v \to 0$ as $s \to \infty$ uniformly on sets $|y| \le R$. Notice that w is bounded under our assumptions (see [**5**] and [**10**]). Hence we can use the dominated convergence theorem to obtain $v \to 0$ in L^2_ρ. Moreover, v solves

$$v_s = \Delta v - \frac{y}{2} \cdot \nabla v + v + f(v) \tag{2.3}$$

where $f(v) = (\beta^\beta + v)^p - (p-1)^{-p/(p-1)} - p(p-1)^{-1}v$. So $f(s) = O(s^2)$ as $s \to 0$. We can expand v in terms of Hermite polynomials:

$$v(y,s) = \sum_{m=0}^{\infty} \sum_{i=1}^{k_m} a_{m,i}(s) H_{m,i}(y)$$

and one can check that

$$\frac{d}{ds} a_{m,i} = \left(1 - \frac{m}{2}\right) a_{m,i} + \langle f(v), H_{m,i} \rangle. \tag{2.4}$$

Since u is nonnegative and w is bounded, we have $|v(y,s)| \le M$ and $v(y,s) \ge -\beta^\beta$ for $(y,s) \in R^N \times (0,\infty)$. Consequently, the hypotheses of the following theorem are satisfied by our function v.

Theorem 2.2. *Let v be a solution of* (2.4) *with the following properties*:

(i) *$v(y,s)$ exists for all time s and $v(y,s) \to 0$ as $s \to \infty$ uniformly on $|y| \le R$.*

(ii) *$|v(y,s)| \le M$ for all $(y,s) \in R^N \times (0,\infty)$.*

(iii) *$v(y,s) \ge -\beta^\beta$ for all $(y,s) \in R^N \times (0,\infty)$.*

Then either $||v(\cdot,s)|| \to 0$ exponentially fast, or

$$\sum_{m \ne 2} \sum_{i=1}^{k_m} a^2_{m,i}(s) = o\left(\sum_{i=1}^{k_2} a^2_{2,i}(s) \right) \tag{2.5}$$

as $s \to \infty$.

This theorem was proved by S. Fillipas in [**3**]. He also proved the following theorem.

Theorem 2.3. *If* $||v(\cdot, s)||$ *does not decay exponentially, then*

$$\frac{d}{ds}a_{2,j} = \frac{p}{2\beta^\beta}\langle\bigg(\sum_{i=1}^{k_2} a_{2,i}H_{2,i}\bigg)^2, H_{2,j}\rangle + o\bigg(\sum_{i=1}^{k_2} a_{2,i}^2\bigg). \tag{2.6}$$

If $\psi(s) \triangleq ||v(\cdot, s)||$ decays exponentially, then we shall see in Section 3 that either $v \equiv 0$ or (1.12) happens. If $\psi(s)$ does not decay exponentially, one of the following three cases must necessarily occur: either

$$\limsup_{s\to\infty} s\psi(s) = \infty, \tag{2.7}$$

or there exist positive constants C and c such that

$$0 < c \le \liminf_{s\to\infty}(s\psi(s)) \le \limsup_{s\to\infty}(s\psi(s)) \le C < \infty, \tag{2.8}$$

or

$$\liminf_{s\to\infty}(s\psi(s)) = 0 \quad \text{and} \quad \limsup_{s\to\infty}(\varepsilon^{\varepsilon s}\psi(s)) = \infty \tag{2.9}$$

for any $\varepsilon > 0$. We intend to show that the cases (2.7) and (2.9) do not occur. To this end, we need the following result (see [**2**]).

Lemma 2.4. *Assume that* $\psi(s)$ *is a nonnegative function such that* $\psi \in C^0([0,\infty))$, $\lim_{s\to\infty}\psi(s) = 0$, $\limsup_{s\to\infty} e^{\varepsilon s}\psi(s) = \infty$ *for any* $\varepsilon > 0$ (*respectively,* $\limsup_{s\to\infty} s\psi(s) = \infty$). *Then there exists a function* $\eta(s) \in C^\infty([0,\infty))$ *such that*

(i) $\eta > 0$, $\eta' < 0$, $\lim_{s\to\infty}\eta(s) = 0$.

(ii) $0 < \limsup_{s\to\infty}\psi(s)/\eta(s) < \infty$.

(iii) $\lim_{s\to\infty} e^{\varepsilon s}\eta(s) = \infty$ *for any* $\varepsilon > 0$ (respectively $\lim_{s\to\infty} s\eta(s) = \infty$).

(iv) $(\eta'/\eta)'$ *and* $(\eta''/\eta)'$ *belong to* $L^1(0,\infty)$.

(v) $\lim_{s\to\infty} \eta'(s)/\eta(s) = \lim_{s\to\infty} \eta''(s)/\eta(s) = 0$.

Following [**12**], we define

$$\tilde{v}(y,s) = \frac{v(y,s)}{\mu(s)} \tag{2.10}$$

where

$$\mu(s) = \begin{cases} \eta(s) & \text{if (2.7) or (2.9) holds, where } \eta(s) \\ & \text{is the corresponding function in Lemma 2.4} \\ 1/s & \text{if (2.8) holds.} \end{cases} \tag{2.11}$$

Expand $\tilde{v}$ in terms of Hermite polynomials:

$$\tilde{v} = \sum_{m=0}^{\infty}\sum_{i=1}^{k_m} b_{m,i} H_{m,i} \tag{2.12}$$

where

$$b_{m,i(s)} = \frac{a_{m,i}(s)}{\mu(s)}. \tag{2.13}$$

By studying the large time behavior of $\tilde{v}$, we shall be able to exclude (2.9) and in some special cases, (2.7) as well; see Sections 3, 4 and 5.

We shall also need the following results.

Lemma 2.5. *Assume that* v *solves* (2.3) *and* $|v| \le M < \infty$. *Then, for any* $r > 1$, $q > 1$, *and* $L > 0$, *there exists* $s_0^* = s_0^*(q,r)$ *and* $C = C(r,q,L)$, *such that*

$$||v(\cdot, s+s^*)||_{r,\rho} \le C||v(\cdot,s)||_{q,\rho}$$

for any $s > 0$ *and any* $s^* \in [s_0^*, s_0^* + L]$, *where* $||\cdot||_{r,\rho}$ *and* $||\cdot||_{r,q}$ *denote the usual norms in* $L^r_\rho(R^N)$ *and* $L^q_\rho(R^N)$, *respectively.*

Lemma 2.6. *Let ϕ be a solution of the initial value problem:*

$$\phi_s = \Delta\phi - \frac{y}{2}\cdot\nabla\phi + \frac{m}{2}\phi + h(y,s),$$
$$\phi(y,0) = \phi_0(y),$$

where $\phi \in L^2_\rho$ and $h(y,s) \in L^2_{\text{loc}}((0,\infty) : L^2_\rho(R^N))$. There exists a positive constant C such that, for any $S > 0$, there holds

$$\begin{aligned}||\phi(\cdot,S)||^2 + S||\nabla\phi(\cdot,S)||^2 + \int_0^S s^2||\Delta\phi(\cdot,s)||^2\,ds + \int_0^S s^2||\phi_s(\cdot,s)||^2\,ds \\ \leq C(S+S^2+e^{2mS})\left(||\phi_0||^2 + \int_0^S ||h(\cdot,s)||^2\,ds\right).\end{aligned}$$

Lemmas 2.5 and 2.6 were proved for the one-dimensional case in [**12**], but the arguments there can be easily adapted to apply to the multi-dimensional case.

3. The general case. As explained in Section 2, we are interested in studying the large time behavior of the function $\tilde{v} = v/\mu$ which satisfies

$$\tilde{v}_s = \Delta\tilde{v} - \frac{y}{2}\cdot\nabla\tilde{v} + \tilde{v} - \frac{\mu'}{\mu}\tilde{v} + \frac{f(\mu\tilde{v})}{\mu}. \tag{3.1}$$

Moreover, by Lemma 2.4 (ii)

$$||\tilde{v}(\cdot,s)|| \leq M < \infty. \tag{3.2}$$

For any sequence $s_j \to \infty$, define

$$\tilde{v}_j(y,s) = \tilde{v}(y, s_j + s).$$

By standard parabolic estimates (see [**13**]), it follows from (3.2) that

$$|\tilde{v}_j| \leq M(R) \qquad \text{on } B(R)\times(-R,\infty)$$

for sufficiently large s_j. Applying L^p and Schauder estimates, we obtain

$$|\tilde{v}_j|C^{2+\alpha}(B(R/2)\times(-R/2,\infty)) \leq M'(R).$$

Hence there exists a subsequence (again labeled as s_j) such that

$$\tilde{v}_j(y,s) \to \tilde{v}_\infty(y,s) \qquad \text{in } C^2(K) \tag{3.3}$$

for any compact subset $K \subset R^{N+1}$. By Lemma 2.4, it follows that $\tilde{v}_\infty$ solves

$$z_s = \Delta z - \frac{y}{2} \cdot \nabla z + z.$$

We wish to show that $\tilde{v}_\infty$ is independent of s. To this end, we need the following lemma.

Lemma 3.1. *For any $s_0 > 0$,*

$$\int_{s_0}^{+\infty} ||\tilde{v}_s(\cdot,s)||^2 \, ds < \infty. \tag{3.4}$$

The proof for $N = 1$ is due to Herrero and Velázquez [**12**]. For general N, the proof is similar but slightly different. Since the lemma is crucial for our argument, we provide the details here.

Proof of Lemma 3.1. Multiplying (3.1) by $\rho\tilde{v}_s$, and integrating over $R^N \times [s_0, s_1]$, we get

$$\begin{aligned}
\int_{s_0}^{s_1}\int_{R^N} \rho\tilde{v}_s^2 \, dx\, ds &= -\frac{1}{2}\int_{R^N} \rho|\nabla\tilde{v}(\cdot,s_1)|^2 + \frac{1}{2}\int_{R^N} \rho|\nabla\tilde{v}(\cdot,s_0)|^2 \\
&\quad + \int_{R^N} \rho\tilde{v}^2(\cdot,s_1) - \int_{R^N}\rho\tilde{v}^2(\cdot,s_0) \\
&\quad - \frac{1}{2}\int_{s_0}^{s_1}\int_{R^N} \rho\frac{\mu'}{\mu}(\tilde{v}^2)_s + \int_{s_0}^{s_1}\int_{R^N} \frac{f(\mu\tilde{v})}{\mu}\tilde{v}_s\rho \\
&\le \frac{1}{2}\int_{R^N} \rho|\nabla\tilde{v}(\cdot,s_0)|^2 + \int_{R^N}\rho\tilde{v}^2(\cdot,s_1) \\
&\quad - \frac{1}{2}\int_{s_0}^{s_1}\int_{R^N} \frac{\mu'}{\mu}(\tilde{v}^2)_s\rho + \int_{s_0}^{s_1}\int_{R^N}\frac{f(\mu\tilde{v})}{\mu}\tilde{v}_s\rho \\
&\triangleq J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Note that J_1 depends only on s_0. From (3.2), it follows that $J_2 \leq C$. By integration by parts,

$$J_3 = -\frac{\mu'(s_1)}{2\mu(s_1)} \int_{R^N} \rho \tilde{v}^2(y, s_1)\, dy + \frac{\mu'(s_0)}{2\mu(s_0)} \int_{R^N} \rho \tilde{v}^2(y, s_0)\, dy \\ + \frac{1}{2} \int_{s_0}^{s_1} \left(\frac{\mu'}{\mu}\right)' \left(\int_{R^N} \rho \tilde{v}^2\right) ds.$$

Hence J_3 is bounded as $s_1 \to \infty$ by (3.2) and Lemma 2.4. To estimate J_4, introduce

$$G(y,s) \triangleq \frac{1}{\mu}\left[\frac{(\beta^\beta + \mu\tilde{v})^{1+p}}{(1+p)\mu} - (p-1)^{-p/(p-1)}\tilde{v} - \frac{p\mu\tilde{v}^2}{2(p-1)} \right. \\ \left. - \frac{(p-1)^{-(p+1)/(p-1)}}{(p+1)\mu}\right],$$

$$g(y,s) \triangleq \frac{\mu'}{\mu^3}\left[-\frac{2(\beta^\beta + \mu\tilde{v})^{p+1}}{p+1} + (\beta^\beta + \mu\tilde{v})^p \mu\tilde{v} \right. \\ \left. + (p-1)^{-p/(p-1)}\mu\tilde{v} + \frac{2(p-1)^{-(p+1)/(p-1)}}{p+1}\right].$$

One readily verifies that

$$\frac{dG(y,s)}{ds} = \frac{f(\mu\tilde{v})}{\mu}\tilde{v}_s + g(y,s),$$

and

$$|G(y,s)| \leq C|v|^2, \qquad v = \mu\tilde{v},$$

$$|g(y,s)| \leq -C\frac{\mu'}{\mu^3}|v|^3.$$

Therefore, by (3.2) and Lemma 2.5, we have for some $L > 0$,

$$\begin{aligned}
|J_4| &= \left| \int_{R^N} \rho G(y, s_1)\, dy - \int_{R^N} \rho G(y, s_0)\, dy - \int_{s_0}^{s_1} \int_{R^N} g(y, s)\, dy\, ds \right| \\
&\le C\left(1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu^3}(s) \int_{R^N} \rho(y)|v|^3(y, s)\, dy\, ds\right) \\
&\le C\left(1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu^3}(s) \left[\int_{R^N} v^2(y, s - L)\rho(y)\, dy\right]^{3/2} ds\right) \\
&\le C\left(1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu_3}(s)\mu^3(s - L)\, ds\right) \\
&\le C\left(1 - \int_{s_0}^{s_1} \mu'(s)\, ds\right) = C(1 + \mu(s_0) - \mu(s_1))
\end{aligned}$$

where we used the fact $\mu(s - L)/\mu(s) \le M$ for any $s > L$ which follows easily from Lemma 2.4. The proof is complete. □

We return to the study of $\tilde{v}_j$. By Lemma 3.1,

$$\int_{-R}^{\infty} \int_{B(R)} \rho|\tilde{v}_{js}|^2\, dy\, ds = \int_{-R+s_j}^{\infty} \int_{B(R)} |\tilde{v}_s|^2 \rho\, dy\, ds \to 0$$

as $j \to \infty$. Hence

$$\int_{-R}^{\infty} \int_{B(R)} |\tilde{v}_{\infty s}|^2\, dy\, ds = 0.$$

It follows that $\tilde{v}_\infty$ is independent of s. We have thus proved

Lemma 3.2. *For any sequence $s_j \to \infty$, there exists a subsequence (again labeled as s_j), such that*

$$\tilde{v}_j(y, s) \to \tilde{v}_\infty(y) \qquad \text{in } C^2(K)$$

for any compact subset $K \subset R^{N+1}$. Furthermore, $\tilde{v}_\infty \in L^2_\rho$, and it solves

$$\Delta v - \frac{y}{2} \cdot \nabla v + v = 0. \tag{3.5}$$

Remark 3.3. If the solution $\tilde{v}_\infty$ of (3.5) were unique, we would have $\tilde{v}(y,s) \to \tilde{v}_\infty(y)$ as $s \to \infty$. Unfortunately, $\tilde{v}_\infty$ may not be unique since there exists a family of solutions of (3.5), i.e., $\sum_{i=1}^{k_2} \gamma_i H_{2,i}(y)$ for any $\gamma_i \in R^1$, $i = 1, \dots, k_2$. This forces us to use another approach by examining the dynamics of the coefficients $a_{2,i}$; see Sections 4 and 5.

Next we want to show that

$$\tilde{v}_j(y,s) \to \tilde{v}_\infty(y) \quad \text{in } L^2_\rho. \tag{3.6}$$

For this purpose, introduce

$$L^2_\rho(R^N \times (0,1)) = \left\{ g(y,s) : \int_0^1 \int_{R^N} g^2(y,s)\rho(y)\,dy\,ds < \infty \right\}.$$

Lemma 3.4. *The set* $K \triangleq \{\tilde{v}_j(y,s)\}$ *is precompact in* $L^2_\rho(R^N \times (0,1))$.

Proof. It follows from (3.2) that K is bounded in $L^2_\rho(R^N \times (0,1))$. By Lemma 2.5,

$$\int_{R^N} v^4(y,s)\rho(y)\,dy \le C\left(\int_{R^N} v^2(y,s-L)\rho(y)\,dy\right)^2.$$

After dividing by μ^4, we end up with

$$\int_{R^N} \tilde{v}^4(y,s)\rho(y) \le C\left(\int_{R^N} \tilde{v}^2(y,s-L)\rho(y)\right)^2.$$

Using the above inequality and (3.2), we can estimate the following integral:

$$\begin{aligned}
&\int_0^1 \int_{|y|>R} \tilde{v}_j^2(y,s)\rho(y)\,dy\,ds \\
&\quad \le \left(\int_{|y|>R} \rho(y)\,dy\right)^{1/2} \left(\int_0^1 \int_{|y|>R} \tilde{v}^4(y,s_j+s)\rho(y)\,dy\,ds\right)^{1/2} \\
&\quad \le C\left(\int_{|y|>R} \rho(y)\,dy\right)^{1/2} \left[\int_0^1 \left(\int_{R^N} \tilde{v}^2(y,s_j-L+s)\rho(y)\,dy\right)^2 ds\right]^{1/2} \\
&\quad \le C\left(\int_{|y|>R} \rho(y)\,dy\right)^{1/2} \to 0 \qquad \text{as } R \to \infty
\end{aligned} \tag{3.7}$$

uniformly in j. Next we multiply (3.1) by $\rho\tilde{v}$, then integrate over $R^N \times (0,1)$. By Lemma 2.5 and (3.2), it follows that (denote $\mu_j(s) = \mu(s_j + s)$)

$$
\begin{aligned}
\int_0^1 \int_{R^N} \rho|\nabla \tilde{v}_j|^2 &\leq \frac{1}{2}\int_{R^N} \rho \tilde{v}_j^2(y,0)\,dy \\
&\quad + \int_0^1 \int_{R^N} \rho\tilde{v}_j^2 - \int_0^1 \frac{\mu_j'}{\mu_j}\int_{R^N} \rho\tilde{v}_j^2 \\
&\quad + \int_0^1 \int_{R^N} \frac{f(\mu_j \tilde{v}_j)}{\mu_j}\rho\tilde{v}_j \\
&\leq C\left(1 + \int_0^1 \frac{|\mu_j'|}{\mu_j} + \int_0^1 \mu_j \int_{R^N} \rho|\tilde{v}_j|^3\right) \\
&\leq C\left[1 + \int_0^1 \mu_j \left(\int_{R^N} \rho\tilde{v}_j^2(y, s-L)\,dy\right)^{3/2}\right] \\
&\leq C
\end{aligned}
\tag{3.8}
$$

where we have used the fact that $f(s) \leq Cs^2$. Using the mean value theorem and noting (3.4) and (3.8), we obtain

$$
\int_0^1 \int_{R^N} \rho|\tilde{v}_j(y+h, s+t) - \tilde{v}_j(y,s)|^2\,dy\,ds \to 0 \tag{3.9}
$$

as $t \to 0$, $|h| \to 0$, uniformly in j. The conclusion of the lemma follows from (3.7) and (3.9) by standard results from real analysis.

By Lemma 3.4, for any sequence $s_j \to \infty$, there exists a subsequence (again labeled as s_j) such that

$$
\tilde{v}_j(y,s) \to \tilde{v}_\infty'(y,s) \qquad \text{in } L^2_\rho(R^N \times (0,1)).
$$

On the other hand, by Lemma 3.2, there exists another subsequence (again labeled as s_j) such that

$$
\tilde{v}_j(y,s) \to \tilde{v}_\infty(y)
$$

uniformly on compact subsets of R^N. Consequently,

$$
\tilde{v}_\infty'(y,s) = \tilde{v}_\infty(y),
$$

i.e.,

$$\tilde{v}_j(y,s) \to \tilde{v}_\infty(y) \qquad \text{in } L^2_\rho(R^N \times (0,1)). \tag{3.10}$$

We want to show further that (3.6) holds. Since L^2_ρ is a Hilbert space, we only need to check that

$$\int_{R^N} \rho(y)\tilde{v}_j^2(y,0)\,dy = \int_{R^N} \rho(y)\tilde{v}^2(y,s_j)\,dy \to \int_{R^N} \rho(y)\tilde{v}_\infty^2(y)\,dy. \tag{3.11}$$

Note that (3.10) implies

$$\int_0^1 \int_{R^N} \rho(y)\tilde{v}_j^2(y,s) \to \int_{R^N} \tilde{v}_\infty^2(y)\rho(y)\,dy \qquad \text{as } j \to \infty. \tag{3.12}$$

Hence the proof of (3.11) reduces to the following estimate:

$$\begin{aligned}
&\left| \int_{R^N} \rho(y)\tilde{v}^2(y,s_j) - \int_0^1 \int_{R^N} \rho(y)\tilde{v}_j(y,s)\,dy\,ds \right| \\
&\le \int_0^1 \int_{R^N} \rho(y)|\tilde{v}_j^2(y,s) - \tilde{v}_j^2(y,0)|\,dy\,ds \\
&= \int_0^1 \int_{R^N} \left| \int_0^1 2\tilde{v}_j \tilde{v}_{js}(y,ts)s\,dt \right| \rho(y)\,dy\,ds \\
&\le 2\int_0^1 s\Big(\int_0^1 \int_{R^N} \rho(y)\tilde{v}_j^2(y,ts)\,dy\,dt \Big)^{1/2} \\
&\quad \Big(\int_0^1 \int_{R^N} \rho(y)\tilde{v}_{js}^2(y,ts)\,dy\,dt \Big)^{1/2} ds \\
&\le C\int_0^1 s\Big(\int_0^1 \int_{R^N} \rho(y)\tilde{v}_s^2(y,s_j+ts)\,dy\,dt \Big)^{1/2} ds \to 0 \qquad \text{as } j \to \infty
\end{aligned}$$

by (3.4). To summarize, we have established:

Lemma 3.5. *For any sequence $s_j \to \infty$, there exists a subsequence (again labeled as s_j) such that*

$$\tilde{v}(y,s_j+s) \to \tilde{v}_\infty(y) \qquad \text{in } L^2_\rho(R^N)$$

and in $C^2(K)$ *for any compact subset* $K \subset R^N$. *Moreover,* $\tilde{v}_\infty$ *solves* (3.5) *and therefore belongs to the eigenspace corresponding to the eigenvalue* $\lambda_2 = 0$. *In particular,*

$$\tilde{v}_\infty = \sum_{i=1}^{k_2} \gamma_i H_{2,i}$$

for some constants γ_i, $i = 1, \dots, k_2$.

Now we are ready to study the large time behavior of $v(y,s)$. We begin wtih the case that $||v(\cdot,s)||$ decays exponentially, i.e.,

$$\psi(s) \leq C\varepsilon^{-\varepsilon s} \tag{3.13}$$

for some $\varepsilon > 0$.

Theorem 3.6. *Assume that* (3.13) *holds. Then either* $\psi \equiv 0$ *or there exists an* $m \geq 3$ *and constants* C_i, $i = 1, \dots, k_m$, *not all zero, such that*

$$||v(\cdot,s) - \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i}||_{H^1_\rho} = o(e^{(1-m/2)s}).$$

The proof given in [**12**] for the one-dimensional case works here with some trivial changes; hence, the details are omitted.

If $\psi(s)$ does not decay exponentially, then we know that there exist three possibilities (2.7)–(2.9).

Theorem 3.7. *The case* (2.9) *cannot occur.*

Proof. Let $V = \sum_{i=1}^{k_2} a_{2,i}^2$. We need to distinguish two types of Hermite polynomials of degree 2:

$$H_{2,i} = c_1\left(\frac{1}{2}y_i^2 - 1\right), \qquad i = 1, \dots, N,$$
$$H_{2,i} = c_2 y_j y_l, \qquad i = N+1, \dots, k_2, j \neq l, 1 \leq j, l \leq N.$$

For any fixed $i \le N$, assume that $H_{2,j(i)} = c_2 y_i y_1, \dots, H_{2,j(i)+N-1} = c_2 y_i y_N$. Using Theoem 2.3, we can easily check that

$$\frac{d}{ds} a_{2,i} = \nu_1 a_{2,i}^2 + \sum_{l=j(i)}^{j(i)+N-1} \nu_{2,l} a_{2,l}^2 + h_i, \qquad i = 1, \dots, N \tag{3.14}$$

$$\frac{d}{ds} a_{2,i} = \sum_{j \ne l} \nu_{j,l} a_{2,j} a_{2,l} + h_i, \qquad i = N+1, \dots, k_2 \tag{3.15}$$

where $h_i = o(\sum_{i=1}^{k_2} a_{2,i}^2)$, $\nu_1 = p(2\beta^\beta)^{-1} \langle H_{2,1}^2, H_{2,1} \rangle$, $\nu_{2,l}$ and $\nu_{j,l}$ are computable constants. Hence

$$\begin{aligned} \frac{d}{ds} V = 2 \sum_{i=1}^{k_2} a_{2,i} \frac{d}{ds} a_{2,i} &\ge -C \bigg(\sum_{i=1}^{k_2} a_{2,i}^2 \bigg)^{3/2} + h \\ &= -C V^{3/2} + h \end{aligned} \tag{3.16}$$

where $h = o(V^{3/2})$. Integrating (3.16), we get

$$-\frac{1}{V^{1/2}(s)} + \frac{1}{V^{1/2}(s_0)} \ge -C(s - s_0) + \int_{s_0}^{s} \tilde{h}(\tau)\, d\tau$$

or

$$-\frac{1}{sV^{1/2}(s)} + \frac{1}{sV^{1/2}(s_0)} \ge -C\bigg(1 - \frac{s_0}{s}\bigg) + \frac{1}{s} \int_{s_0}^{s} \tilde{h}(\tau)\, d\tau \tag{3.17}$$

where $\tilde{h}(\tau) \to 0$ as $\tau \to \infty$.

If (2.9) happens, there exists a sequence $s_j \to \infty$ such that $s_j \psi(s_j) \to 0$. Since $V^{1/2}(s) \le \psi(s)$, we also have that $s_j V^{1/2}(s_j) \to 0$. Setting $s = s_j$ in (3.17) and letting $j \to \infty$, we obtain a contradiction. The proof is complete.

4. The radial case. Throughout this section, we shall assume that $u(x,t)$ is a radial solution of (1.1)–(1.2); hence, $w(y,s)$ and $v(y,s)$ are all radial functions. We have shown that (2.9) cannot occur in

Section 3. If $v(y,s)$ is a radial function, we shall also show that (2.7) cannot happen.

By Lemma 3.5, for any sequence $s_j \to \infty$, there exists a subsequence (again labeled as s_j) such that:

$$\tilde{v}(y,s_j) \to \tilde{v}_\infty(y) = \sum_{i=1}^{k_2} \gamma_i H_{2,i}(y). \tag{4.1}$$

As we did in Section 3, we divide $H_{2,i}$ into two sets: $H_{2,i} = c_1(2^{-1}y_i^2 - 1)$, $i = 1, \ldots, N$; $H_{2,i} = c_2 y_j y_l$, $i = N+1, \ldots, k_2$, $j \neq l$, $1 \le j$, $l \le N$. Since $\tilde{v}(y,s_j)$ is a radial function, so is $\tilde{v}_\infty(y)$. It follows that

$$\gamma_1 = \cdots = \gamma_N, \qquad \gamma_{N+1} = \cdots = \gamma_{k_2} = 0. \tag{4.2}$$

Lemma 4.1. *There holds*

$$b_{2,i}(s) = \frac{a_{2,i}(s)}{\mu(s)} \to 0 \qquad \text{as } s \to \infty, \qquad i = N+1, \ldots, k_2.$$

Proof. Suppose that $b_{2,i_0}(s_j) \ge c > 0$ for some $s_j \to \infty$, $N+1 \le i_0 \le k_2$. Then there exists a subsequence (again labeled as s_j) such that $\tilde{v}(y,s_j) \to \tilde{v}_\infty(y) = \gamma \sum_{i=1}^N H_{2,i}(y)$ by (4.2). Hence $\langle \tilde{v}(y,s_j), H_{2,i_0}\rangle \to \langle \tilde{v}_\infty, H_{2,i_0}\rangle = 0$, i.e., $b_{2,i_0}(s_j) \to 0$, a contradiction. □

Next we shall remove the possibility (2.7).

Lemma 4.2. *The case* (2.7) *cannot occur.*

Proof. Assume that (2.7) is the case, i.e., $\limsup s\psi(s) = \infty$. By Lemma 2.4 (ii), there exists a sequence $s_j \to \infty$, such that

$$\|\tilde{v}(\cdot, s_j)\| = \psi(s_j)/\mu(s_j) \ge c > 0. \tag{4.3}$$

On the other hand, we have $\lim_{s\to\infty} s\mu(s) = \infty$ because of Lemma 2.4 (iii). Therefore,

$$\lim_{j\to\infty} s_j \psi(s_j) = \infty.$$

By Theorem 2.2,

$$\lim_{j\to\infty} s_j \left(\sum_{i=1}^{k_2} a_{2,i}^2(s_j) \right)^{1/2} = \infty. \tag{4.4}$$

For this sequence $s_j \to \infty$, we can find a subsequence (again labeled as s_j) such that

$$\tilde{v}(y, s_j) \to \tilde{v}_\infty(y) = \gamma \sum_{i=1}^{N} H_{2,i}(y) \qquad \text{in } L^2_\rho(R^N) \tag{4.5}$$

where $\gamma \neq 0$ because of (4.3). Furthermore, it follows from (4.5) that

$$b_{2,i}(s_j) = \frac{a_{2,i}(s_j)}{\mu(s_j)} \to \gamma \neq 0, \qquad i = 1, \dots, N, \tag{4.6}$$

$$b_{2,i}(s_j) = \frac{a_{2,i}(s_j)}{\mu(s_j)} \to 0, \qquad i = N+1, \dots, k_2 \tag{4.7}$$

as $j \to \infty$. In particular,

$$\frac{\sum_{i=N+1}^{k_2} a_{2,i}^2(s_j)}{\sum_{i=1}^{N} a_{2,i}^2(s_j)} \to 0 \qquad \text{as } j \to \infty. \tag{4.8}$$

By (4.4) and (4.8),

$$s_j \left(\sum_{i=1}^{N} a_{2,i}^2(s_j) \right)^{1/2} \to +\infty. \tag{4.9}$$

It follows from (4.6) and (4.9) that

$$s_j \left| \sum_{i=1}^{N} a_{2,i}(s_j) \right| \to \infty \qquad \text{as } j \to \infty. \tag{4.10}$$

We now take up an approach already used in Theorem 3.7. Let $V = \sum_{i=1}^N a_{2,i}$. By (3.14),

$$\begin{aligned} \frac{dV}{ds} &= \nu_1 \sum_{i=1}^N a_{2,i}^2 + \sum_{i>N} \nu_i a_{2,i}^2 + h \\ &\geq \varepsilon \biggl(\sum_{i=1}^N a_{2,i} \biggr)^2 + h = \varepsilon V^2 + h \end{aligned} \tag{4.11}$$

for some $\varepsilon > 0$ where $h = o(\sum_{i=1}^N a_{2,i}^2) = o(V^2)$. Integrating (4.11) yields

$$-\frac{1}{V(s)} + \frac{1}{V(s_0)} \geq \varepsilon(s - s_0) + \int_{s_0}^s g(t)\,dt \tag{4.12}$$

where $g(s) \to 0$ as $s \to \infty$. Setting $s = s_j$ in (4.12) and letting $j \to \infty$, we get a contradiction because of (4.10).

Combining Theorem 3.7 with Lemma 4.2, we conclude that if $\psi(s)$ does not decay exponentially, only (2.8) is possible. Next, we want to obtain the exact behavior of $a_{2,i}(s)$ as $s \to \infty$, $i = 1, \dots, N$. We begin with the following lemma.

Lemma 4.3. *Assume* (2.8). *There exists a* $\delta > 0$, $L > 0$, *such that*

$$\delta \leq \frac{a_{2,i}^2(s)}{\sum_{i=1}^N a_{2,i}^2(s)} \leq 1 \quad \textit{for any } s \in (L, \infty), \qquad i = 1, \dots, N.$$

Proof. Suppose for some $s_j \to \infty$, $1 \leq i_0 \leq N$, we have

$$\frac{a_{2,i_0}^2(s_j)}{\sum_{i=1}^N a_{2,i}^2(s_j)} \to 0 \qquad \text{as } j \to \infty.$$

From (2.8), it follows that $s_j a_{2,i_0}(s_j) \to 0$ as $j \to \infty$. On the other hand, by Lemma 3.5, there exists a subsequence (again labeled as s_j), such that $\tilde{v}(y, s_j) \to \tilde{v}_\infty(y)$ where $\tilde{v}_\infty \neq 0$ because of (2.8). Therefore,

$$\tilde{v}(y, s_j) = s_j v(y, s_j) \to \gamma \sum_{i=1}^N H_{2,i}(y) \qquad \text{in } L_\rho^2.$$

This, in turn, implies

$$s_j a_{2,i_0}(s_j) \to \gamma \neq 0,$$

a contradiction. □

Using Lemma 4.1 and Lemma 4.3, we can rewrite (3.14) as

$$\frac{d}{ds} a_{2,i} = \nu_1 a_{2,i}^2 + o(a_{2,i}^2), \qquad i = 1, \dots, N. \tag{4.13}$$

Integrating (4.13), we find that

$$-\frac{1}{a_{2,i}(s)} + \frac{1}{a_{2,i}(s_0)} = \nu_1(s - s_0) + \int_{s_0}^{s} h_i(t)\, dt, \qquad i = 1, \dots, N$$

where $h_i(s) \to 0$ as $s \to \infty$. Therefore,

$$\lim_{s\to\infty} s a_{2,i}(s) = -\frac{1}{\nu_1}, \qquad i = 1, \dots, N. \tag{4.14}$$

Combining (2.5), (4.8) and (4.14), we conclude that

$$\begin{aligned} v(y,s) &= \sum_{i=1}^{N} a_{2,i}(s) H_{2,i}(y) + \sum_{i=N+1}^{k_2} a_{2,i}(s) H_{2,i}(y) \\ &\quad + \sum_{m\neq 2} \sum_{i=1}^{k_m} a_{m,i} H_{m,i}(y) \\ &= -\frac{1}{\nu_1 s} \sum_{i=1}^{N} H_{2,i}(y) + o\left(\frac{1}{s}\right) \\ &= -\frac{N\beta^\beta}{2ps}\left(\frac{1}{2N}|y|^2 - 1\right) + o\left(\frac{1}{s}\right) \end{aligned} \tag{4.15}$$

where the convergence takes place in L^2_ρ. From Lemma 3.5, it follows that the convergence also takes place in $C^2_{\rm loc}$.

To finish the proof of Theorem 1.1, we need to extend the convergence in (4.15) to H^1_ρ, whichis required in the proof of Theorem 1.2. We begin with a lemma.

Lemma 4.4. *Suppose H_m and H_n are two different Hermite polynomials. Then*

$$\left\langle \frac{\partial}{\partial y_j} H_m, \frac{\partial}{\partial y_j} H_n \right\rangle = 0 \tag{4.16}$$

$$\left\| \frac{\partial}{\partial y_j} H_m \right\| = \left(\frac{m}{2}\right)^{1/2} \quad \textit{or} \quad 0. \tag{4.17}$$

Proof. Without loss of generality, assume that $H_m = h_{m_1}(y_1) \cdots h_{m_i}(y_i)$, $m_1 + \cdots + m_i = m$, and $y_j = y_1$. Note that h_{m_i} satisfies

$$h''_{m_i} - \frac{y}{2} h'_{m_i} + \frac{m_i}{2} = 0. \tag{4.18}$$

We compute

$$\begin{aligned} \left\| \frac{\partial}{\partial y_1} H_m \right\|^2 &= \int_{R^N} \rho \frac{\partial}{\partial y_1} (h_{m_1} \cdots h_{m_i}) \frac{\partial}{\partial y_1} H_m \\ &= -\int_{R^N} \frac{\partial}{\partial y_1} \left(\rho \frac{\partial}{\partial y_1} h_{m_1} \cdots h_{m_i} \right) H_m \\ &= \int_{R^N} \rho \frac{y_1}{2} h_{m_1} \cdots h_{m_i} H_m \\ &\quad - \int_{R^N} \rho \left(\frac{\partial^2}{\partial y_1^2} h_{m_1} \right) \cdots h_{m_i} H_m \qquad \text{(by (4.18))} \\ &= \frac{m_1}{2} \int_{R^N} \rho H_m^2 = \frac{m_1}{2} \qquad \text{(by (4.18))}. \end{aligned}$$

Similarly, one can check (4.16). Set

$$\begin{aligned} v - \sum_{i=1}^{N} a_{2,i} H_{2,i} &= \sum_{i=N+1}^{k_2} a_{2,i} H_{2,i} + \sum_{m=0}^{1} \sum_{i=1}^{k_m} a_{m,i} H_{m,i} \\ &\quad + \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} a_{m,i} H_{m,i} \\ &\triangleq I_1 + I_2 + I_3. \end{aligned} \tag{4.19}$$

We proceed to estimate various terms as follows:

$$\begin{aligned}||I_1||_{H^1_\rho} &= \sum_{i=N+1}^{k_2} a_{2,i}||\nabla H_{2,i}|| \\ &\le C\sum_{i=N+1}^{k_2} |a_{2,i}(s)| = o\left(\frac{1}{s}\right)\end{aligned}$$

where the last equality follows from (4.8) and (4.14). I_2 can be dealt with similarly. To estimate I_3, a different approach is needed. For any fixed $R > 0$, we can write

$$v(y,s) = S_L(R)v(y,s-R) + \int_{s-R}^{s} S_L(s-\tau)f(v(\cdot,\tau))\,d\tau \tag{4.20}$$

where S_L is the semigroup generated by L on L^2_ρ. Since $f(v) = \sum_{m=0}^{\infty}\sum_{i=1}^{k_m} < f(v),\, H_{m,i} > H_{m,i}$, we have

$$\begin{aligned}v(\cdot,s) = &\sum_{m=0}^{2}\sum_{i=1}^{k_m}\Bigg[a_{m,i}(s-R)e^{(1-m/2)R} \\ &\qquad + \int_{s-R}^{s} e^{(1-m/2)(s-\tau)}\langle f(v), H_{m,i}\rangle\,d\tau\Bigg]H_{m,i} \\ &+ \sum_{m=3}^{\infty}\sum_{i=1}^{k_m} a_{m,i}(s-R)e^{(1-m/2)R}H_{m,i} \\ &+ \sum_{m=3}^{\infty}\sum_{i=1}^{k_m} H_{m,i}\int_{s-R}^{s} e^{(1-m/2)(s-\tau)}\langle f(v), H_{m,i}\rangle\,d\tau \\ \triangleq\ & J_1 + J_2 + J_3.\end{aligned} \tag{4.21}$$

Comparing (4.19) with (4.21), we find that

$$I_3 = J_2 + J_3.$$

Let us estimate the H^1_ρ norm of J_2 and J_3. By Lemma 4.4,

$$||J_2||_{H^1_\rho} \le \sum_{m=3}^{\infty}\sum_{i=1}^{k_m} a_{m,i}(s-R)e^{(1-m/2)R}\left(1+\frac{m}{2}\right)^{1/2}.$$

We can easily check that

$$k_m \le C m^{N-1}. \tag{4.22}$$

Consequently,

$$\begin{aligned}||J_2||_{H^1_\rho} &\le \bigg(\sum_{m=3}^{\infty}\sum_{i=1}^{k_m}\Big(1+\frac{m}{2}\Big)e^{2(1-m/2)R}\bigg)^{1/2}\bigg(\sum_{m=3}^{\infty}\sum_{i=1}^{k_m}a_{m,i}^2(s-R)\bigg)^{1/2}\\ &\le C\bigg(\sum_{m=3}^{\infty}m^N e^{2(1-m/2)R}\bigg)^{1/2}\bigg\|v(\cdot,s-R)-\sum_{i=1}^{k_2}a_{2,i}(s-R)H_{2,i}\bigg\|.\end{aligned}$$

But

$$\sum_{3}^{\infty}m^N e^{2(1-m/2)R} = e^{-R}\sum_{j=0}^{\infty}(j+3)^N e^{-jR} \le C < \infty,$$

we conclude that

$$||J_2||_{H^1_\rho} \le C\bigg\|v(\cdot,s-R)-\sum_{i=1}^{k_2}a_{2,i}(s-R)H_{2,i}\bigg\| = o\Big(\frac{1}{s}\Big).$$

The estimate of J_3 is more involved. First we note that $f(s) \le Cs^2$; hence, $||f(v(\cdot,s))|| \le Cs^{-2}$. We shall next estimate the "$||\cdot||$" norm of J_3:

$$\begin{aligned}||J_3|| &\le \int_{s-R}^{s}\bigg\|\sum_{m=3}^{\infty}\sum_{i=1}^{k_m}e^{(1-m/2)(s-\tau)}\langle f(v),H_{m,i}\rangle H_{m,i}\bigg\|\,d\tau\\ &\triangleq \int_{s-R}^{s}g(\tau)\,d\tau.\end{aligned}$$

$g(\tau)$ can be estimated as follows:

$$\begin{aligned}g(\tau) &= \bigg[\int_{R^N}\rho\bigg(\sum_{m=3}^{\infty}\sum_{i=1}^{k_m}e^{(1-m/2)(s-\tau)}\langle f(v),H_{m,i}\rangle H_{m,i}\bigg)^2 dy\bigg]^{1/2}\\ &= \bigg(\sum_{m=3}^{\infty}\sum_{i=1}^{k_m}e^{2(1-m/2)(s-\tau)}\langle f(v),H_{m,i}\rangle^2\int_{R^N}\rho H_{m,i}^2\bigg)^{1/2}\end{aligned}$$

(recall $s - \tau > 0$)

$$\leq \left(\sum_{m=3}^{\infty} \sum_{i=1}^{k_m} \langle f(v), H_{2,i} \rangle^2 \right)^{1/2}$$
$$\leq ||f(v)|| \leq C \frac{1}{\tau^2}.$$

It follows that

$$||J_3|| \leq C/s^2. \tag{4.23}$$

We have thus proved that $||J_3(s)|| = o(1/s)$. Observe that J_3 solves

$$z_s = \Delta z - \frac{y}{2} \cdot \nabla z + z + \left(f(v) - \sum_{m=0}^{2} \sum_{i=1}^{k_m} \langle f(v), H_{m,i} \rangle H_{m,i} \right)$$
$$\triangleq Lz + h(y, s)$$

where

$$||h(\cdot, s)|| \leq ||f(v)|| \leq C/s^2. \tag{4.24}$$

Using Lemma 2.6, we obtain

$$R||\nabla J_3(\cdot, s)||^2 \leq C(R + R^2 + e^{2mR}) \left(||J_3(\cdot, s - R)||^2 + \int_{s-R}^{s} ||h(\cdot, \tau)||^2 \, d\tau \right) \tag{4.25}$$

for any fixed $R > 0$. The conclusion that $||J_3(s)||_{H_\rho^1} = o(1/s)$ follows from (4.23)–(4.25).

We summarize:

Theorem 4.5. *If $\psi(s)$ does not decay exponentially, then* (4.15) *holds, where convergence takes place in H_ρ^1 as well as in C_{loc}^2.*

Theorem 1.1 now follows from Theorem 3.6 and Theorem 4.6. The argument in [**12**] can then be used to prove Theorem 1.2. We only need to notice that if $S(t)$ is the semigroup associated with the heat operator, and if $v(y,s) = \sum_m \sum_i a_{m,i}(s) H_{m,i}(y)$, then $S_t v(y,s) = \sum_m \sum_i a_{m,i}(s)(1-t)^{m/2} H_{m,i}(y/(1-t)^{1/2})$. The details are omitted.

5. The case $N = 2$. In the case $N = 2$, if we can exclude (2.7), then Theorem 1.4 will be proven since we can argue as in Section 4 to prove (1.13).

Let us write $H_{2,1} = c_1(2^{-1}y_1^2 - 1)$, $H_{2,2} = c_1(2^{-1}y_2^2 - 1)$, $H_{2,3} = c_2 y_1 y_2$. In Section 4, we proved that

$$\left(\sum_{i=1}^{N} a_{2,i}\right)' \geq \varepsilon \left(\sum_{i=1}^{k_2} a_{2,i}^2\right)(1 + o(1)). \tag{5.1}$$

Although we assumed solutions were radial in Section 4, the proof there applies to the general case without any changes. In the case $N = 2$, (5.1) reads

$$\left(\sum_{i=1}^{2} a_{2,i}\right)' \geq \varepsilon \left(\sum_{i=1}^{3} a_{2,i}^2\right)(1 + o(1)) > 0. \tag{5.2}$$

Since $||v(\cdot,s)|| \to 0$ as $s \to \infty$, we see that $a_{2,1}(s) + a_{2,2}(s) \to 0$, and hence,

$$a_{2,1} + a_{2,2} < 0. \tag{5.3}$$

If (2.7) happens, as we showed before, there exists $s_j \to \infty$, such that

$$\tilde{v}(y, s_j) \to \tilde{v}_\infty(y) = \sum_{i=1}^{3} \gamma_i H_{2,i} \tag{5.4}$$

where the convergence takes place in $L^2_\rho(R^N)$ (see (4.5)), and

$$s_j \psi(s_j) \to \infty. \tag{5.5}$$

If we can show there exists another sequence τ_j such that

$$\tau_j |a_{2,1}(\tau_j) + a_{2,2}(\tau_j)| \to \infty, \tag{5.6}$$

then we will get a contradiction to (5.2) as in proving Lemma 4.2. Hence the proof of the fact that (2.7) cannot occur reduces to the construction of τ_j. We begin with a lemma.

Lemma 5.1. *There holds*

$$|a_{2,1} + a_{2,2}|(s) > |a_{2,3}|(s) \tag{5.7}$$

for s sufficiently large.

Proof. The $a_{2,i}$'s satisfy

$$\frac{d}{ds}a_{2,1} = \nu_1 a_{2,1}^2 + \frac{1}{2}\nu_1 a_{2,3}^2 + h_1, \tag{5.8}$$

$$\frac{d}{ds}a_{2,2} = \nu_1 a_{2,2}^2 + \frac{1}{2}\nu_1 a_{2,3}^2 + h_2, \tag{5.9}$$

$$\frac{d}{ds}a_{2,3} = \nu_1(a_{2,1} + a_{2,2})a_{2,3} + h_3 \tag{5.10}$$

where $h_i = o(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2)$, $i = 1, 2, 3$. Multiplying (5.10) by $\operatorname{sgn} a_{2,3}$, then adding (5.8) and (5.9) to it, we get

$$\begin{aligned}(a_{2,1} + a_{2,2} + |a_{2,3}|)' &\geq \nu_1(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2) \\ &\qquad - \nu_1|(a_{2,1} + a_{2,2})a_{2,3}| - \sum_{i=1}^{3}|h_i| \\ &\geq \delta(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2) - \sum_{i=1}^{3}|h_i| > 0\end{aligned}$$

for s sufficiently large. Since $a_{2,1} + a_{2,2} + |a_{2,3}| \to 0$ as $s \to \infty$, we see that

$$a_{2,1} + a_{2,2} + |a_{2,3}| < 0 \tag{5.11}$$

and (5.7) follows.

By Lemma 5.1 and (5.5),

$$s_j(a_{2,1}^2(s_j) + a_{2,2}^2(s_j))^{1/2} \to \infty. \tag{5.12}$$

Let us begin the proof of the existence of $\tau_j \to \infty$ such that (5.6) holds. Three cases can occur:

(i) $\gamma_1\gamma_2 = 0$. Let $\tau_j = s_j$ and (5.6) follows from (5.12).

(ii) $\gamma_1\gamma_2 > 0$. In this case $a_{2,1}(s_j)$ and $a_{2,2}(s_j)$ have the same sign when j is sufficiently large. We can again let $\tau_j = s_j$, and (5.6) follows from (5.12).

(iii) $\gamma_1\gamma_2 < 0$. Without loss of generality, assume that $\gamma_1 > 0$ and $\gamma_2 < 0$. Then $a_{2,1}(s_j) > 0$ and $a_{2,2}(s_j) < 0$ for j sufficiently large. From (5.4) and (5.12), it follows that

$$s_j a_{2,1}(s_j) \to \infty, \tag{5.13}$$

$$s_j |a_{2,2}(s_j)| \to \infty \tag{5.14}$$

as $j \to \infty$. Using the fact that $a_{2,1}(s_j)/\mu(s_j) \to \gamma_1$, $a_{2,2}(s_j)/\mu(s_j) \to \gamma_2$, we obtain that

$$\frac{a_{2,1}(s_j)}{(a_{2,1}^2(s_j) + a_{2,2}^2(s_j))^{1/2}} \geq c > 0.$$

Consequently,

$$a'_{2,1}(s_j) > 0 \qquad \text{(recall (5.8))}$$

for j sufficiently large. Since $a_{2,1}(s) \to 0$ as $s \to \infty$, there exists an $s_0 > s_j$ such that $a'_{2,1}(s_0) = 0$. Let

$$\tau_j = \inf\{s > s_j : a'_{2,1}(s) = 0\}.$$

Since $a'_{2,1}(s) \geq 0$ on (s_j, τ_j), we have

$$a_{2,1}(\tau_j) \geq a_{2,1}(s_j) > 0. \tag{5.15}$$

From (5.15) and (5.3), it follows that $a_{2,2}(\tau_j) \neq 0$. This assertion, combining with the facts that $a'_{2,1}(\tau_j) = 0$ and (5.8), gives us that

$$a_{2,1}^2(\tau_j) + a_{2,3}^2(\tau_j) = o(a_{2,1}^2(\tau_j) + a_{2,2}^2(\tau_j)) = o(a_{2,2}^2(\tau_j)). \tag{5.16}$$

From (5.15) and (5.16), we conclude

$$\tau_j |a_{2,1}(\tau_j) + a_{2,2}(\tau_j)| \geq \tau_j a_{2,1}(\tau_j) \geq s_j a_{2,1}(s_j) \to \infty \qquad \text{as } j \to \infty$$

by (5.13). $\square$

We summarize

Theorem 5.2. *In the case* $N = 2$, (2.7) *cannot occur.*

Theorem 1.4 now follows from Theorem 3.6 and Theorem 5.2.

Having established Theorem 1.4, we can argue as in [**12**] to prove Corollary 1.5. Indeed, all we need to prove (1.14) in that argument is

$$\sum_{i=1}^{k_2} a_{2,i}(s) H_{2,i}\left(\frac{x}{(T-t)^{1/2}}\right) \geq -\frac{C}{\log(T-t)} + o\left(\frac{1}{\log(T-t)}\right) \tag{5.17}$$

and (5.17) follows from (1.13).

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