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EXTENDED SPECTRAL RADIUS IN TOPOLOGICAL ALGEBRAS

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0. Introduction. In this note we discuss several definitions of spectral radius in locally convex topological algebras as introduced by Zelazko [**5**]. In particular, we present what is known about the equivalence or nonequivalence of these definitions in various classes of algebras. Some of these relations were already proven by Zelazko [**5**], some others are more recent [**1, 2, 3**], and some are new.

Spectral radius is one of the most important features of the Banach algebras theory. For an x in a commutative, complex Banach algebra $\mathcal{A}$ with unit e, we have many equivalent definitions:

$$R(x) = \lim_n \sqrt[n]{||x^n||},$$
$$R(x) = \sup\{|\lambda| : x - \lambda e \text{ is not invertible}\},$$
$$R(x) = \sup\{|F(x)| : F \text{ is a linear and multiplicative functional}\},$$
$$R(x) = \inf\left\{r : \text{ if the radius of convergence of } \sum_{n=0}^{\infty} \alpha_n \lambda^n \text{ is } r \right.$$
$$\left. \text{then } \sum_{n=0}^{\infty} \alpha_n x^n \text{ converges in } \mathcal{A}\right\}$$

and a number of others. Most of these definitions can be used to extend the notation of spectral radius to more general topological algebras; however, the extended definitions are no longer equivalent. In the next sections we discuss the relations between them.

1. Definitions and notation. A locally convex algebra $\mathcal{A}$ is a topological Hausdorff algebra which is a locally convex space. The topology on such an algebra can be introduced by a family $\{||\cdot||_\alpha\}$, $\alpha \in \Lambda$, of seminorms such that, for any α there is a β such that

$$||xy||_\alpha \leq ||x||_\beta ||y||_\beta, \tag{1}$$

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for all $x, y \in \mathcal{A}$.

A locally convex metrizable and complete algebra is called a B_0-algebra. In a B_0-algebra $\mathcal{A}$ the topology can be introduced by a sequence $(||\cdot||_i)$, $i = 1, 2, \ldots$ of seminorms satisfying

$$||x||_i \leq ||x||_{i+1} \tag{2}$$

and

$$||xy||_i \leq ||x||_{i+1}||y||_{i+1} \tag{3}$$

for $i = 1, 2, \ldots$ and for all $x, y \in \mathcal{A}$.

If, for a locally convex algebra $\mathcal{A}$, condition (1) can be replaced by

$$||xy||_\alpha \leq ||x||_\alpha ||y||_\alpha \tag{4}$$

for all $x, y \in \mathcal{A}$ and all $\alpha \in \Lambda$, then we call $\mathcal{A}$ locally multiplicatively convex (shortly m-convex).

Here are some simple examples of locally convex algebras:

Example 1.1. The algebra $\mathcal{E}$ of all entire functions of one complex variable with the pointwise operations and seminorms $||\cdot||_n$ defined by

$$||f||_n = \max_{|z|\leq n} |f(z)|$$

is an m-convex B_0-algebra.

Example 1.2. Let L^ω be the set of all measurable functions (or rather equivalence classes) on the unit interval $(0, 1)$ such that

$$||f||_n = \left(\int_0^1 |f(t)|^n \, dt \right)^{1/n} < \infty,$$

$n = 1, 2, \ldots$. By the Schwartz inequality, L^ω is a B_0-algebra under pointwise operations. It is not m-convex.

Example 1.3. Let $(\alpha_{\gamma,k})$, $\gamma \in \Gamma$, $0 \leq k \leq \infty$ be an infinite matrix of positive real numbers. Assume that for every $\gamma \in \Gamma$ there is $\gamma' \in \Gamma$ such that

$$\alpha_{\gamma,k+l} \leq \alpha_{\gamma',k} \cdot \alpha_{\gamma',l} \qquad \text{for all } k, l. \tag{5}$$

The matrix algebra $\mathcal{A}(\alpha_{\gamma,k})$ associated with the matrix $(\alpha_{\gamma,k})$ is the algebra of all formal complex power series $x = \sum_{k=0}^{\infty} x_k t^k$ such that

$$||x||_\gamma = \sum_{k=0}^{\infty} \alpha_{\gamma,k} |x_k| < \infty.$$

By (5), $\mathcal{A}(\alpha_{\gamma,k})$ is an algebra under Cauchy multiplication. If $\Gamma = \mathbf{N}$, $\mathcal{A}(\alpha_{n,k})$ is a B_0-algebra and if $\gamma' = \gamma$ in (5), it is m-convex.

The matrix algebra defined above will provide us later with a number of counterexamples.

We now give generalizations of the concepts of spectrum and spectral radius.

Definition 1.4. Let $\mathcal{A}$ be a complete complex locally convex algebra with a unit e. For an $x \in \mathcal{A}$ the extended spectrum of x is defined as

$$\Sigma(x) = \sigma(x) \cup \sigma_d(x) \cup \sigma_\infty(x),$$

where

$$\sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda e \text{ is not invertible in } \mathcal{A}\},$$
$$\sigma_d(x) = \{\lambda \in \mathbf{C} : t \to R(t,x) = (te - x)^{-1} \text{ is discontinuous at } t = \lambda\},$$

and

$$\sigma_\infty(x) = \begin{cases} \varnothing & \text{if } t \to R(1, tx) \text{ is continuous at } t = 0 \\ \infty & \text{otherwise.} \end{cases}$$

The extended spectral radius of x is defined as

$$R(x) = \sup\{|\lambda| : \lambda \in \Sigma(x)\}.$$

If $\mathcal{A}$ is a commutative complete complex m-convex algebra, then $\Sigma(x) = \sigma(x)$.

In [**1**] Zelazko proposed several alternative definitions of spectral radius in locally convex topological algebras. It is well known that all of them coincide for complex commutative Banach algebras.

Definition 1.5. Let $\mathcal{A}$ be a locally convex unital algebra, and let Λ be a family of continuous seminorms satisfying (1). For any $x \in \mathcal{A}$, we define

(a)
$$r_1(x) = \sup_{\alpha \in \Lambda} \limsup_n \sqrt[n]{\|x^n\|_\alpha}$$

(b)
$$r_2(x) = \sup\Big\{ \limsup \sqrt[n]{\|x^n\|} : \|\cdot\| \text{ is a continuous seminorm on } \mathcal{A}\}.$$

(c)
$$r_3(x) = \sup_{f \in \mathcal{A}^*} \limsup_n \sqrt[n]{|f(x^n)|},$$

where $\mathcal{A}^*$ is the set of all continuous functionals on $\mathcal{A}$.

(d)
$$r_4(x) = \sup_{f \in \mathfrak{M}(\mathcal{A})} |f(x)|,$$

where $\mathfrak{M}(\mathcal{A})$ is the set of all linear and multiplicative functionals on $\mathcal{A}$.

(e)
$$r_5(x) = \inf\{r : x - \lambda e \in G(\mathcal{A}) \text{ for all } |\lambda| > r\},$$

(f)
$$r_6(x) = \inf\Big\{ 0 < r \le \infty : \text{ there is a sequence of complex numbers } (\alpha_n) \text{ such that the radius of convergence of } \sum_{n=0}^{\infty} \alpha_n \lambda^n \text{ is } r \text{ and } \sum_{n=0}^{\infty} \alpha_n x^n \text{ converges in } \mathcal{A} \Big\}.$$

(g)
$$r_7(x) = \inf\Big\{ 0 < r \le \infty : \text{ for any sequence of complex numbers } (\alpha_n) \text{ such that the radius of convergence of } \sum_{n=0}^{\infty} \alpha_n \lambda^n \text{ is } r, \sum_{n=0}^{\infty} \alpha_n x^n \text{ converges in } \mathcal{A} \Big\}.$$

(h)
$$r_*(x) = \sup\Big\{\liminf_n \sqrt[n]{||x^n||} : ||\cdot|| \text{ is a continuous seminorm on } \mathcal{A}\Big\}.$$

It is easy to notice that $r_1(\cdot)$ and $r_2(\cdot)$ coincide for any locally convex topological algebra. On the other hand, definitions (d) and (e), though formally correct, can be meaningless in general as $\mathfrak{M}(\mathcal{A})$ can be empty (we assume here that $\sup\varnothing = -\infty$), see also the examples given in [**1**]. The following three theorems describe what we know about the relations between the spectral radii in various classes of algebras.

Theorem 1.6 [**1**]. *Let $\mathcal{A}$ be a complex, unital, commutative, complete m-convex algebra. Then, for any $x \in \mathcal{A}$, we have*

$$r_1(x) = r_2(x) = r_3(x) = r_4(x) = r_5(x) = r_6(x) = r_7(x) = r_*(x).$$

Theorem 1.7. *Let $\mathcal{A}$ be a B_0-algebra. Then, for any $x \in \mathcal{A}$ we have*

$$r_1(x) = r_2(x) = r_3(x) = r_7(x) \geq r_6(x) = r_*(x) \geq r_4(x).$$

We prove $r_6(x) = r_*(x)$ in Section 3, where we also give an example showing that the inequality $r_7(x) \geq r_6(x)$ can be strict. The remaining part of the theorem follows from a more general one.

Theorem 1.8. *Let $\mathcal{A}$ be a complex, unital, commutative, complete locally convex algebra. Then, for any $x \in \mathcal{A}$, we have*

$$r_1(x) = r_2(x) = r_3(x) = r_7(x) \geq r_6(x) \geq r_*(x) \geq r_4(x).$$

The first equation is trivial, the next two were proven by Zelazko [**1**], $r_7(x) \geq r_6(x)$ is obvious, $r_6(x) \geq r_*(x)$ was proven in [**2**], and the last one is an easy exercise. We will prove that the last three inequalities can be strict in general. However, there is an interesting class of locally convex, but not m-convex algebras, namely, the Φ-algebras defined by S. El-Helaly and T. Husain in [**4**], for which all the equalities from Theorem 1.6 hold true [**3**]. We discuss this in the next section.

2. Algebras with orthogonal basis and Φ-algebras. An algebra with orthogonal basis $(e_n)_{n=1}^{\infty}$ is a topological algebra $\mathcal{A}$ with the Schauder basis $(e_n)_{n=1}^{\infty}$ such that $e_n e_m = \delta_{nm} e_n$ for all $n, m \in \mathbf{N}$, where δ_{nm} is the Kronecker symbol. Since each $x \in \mathcal{A}$ can be written as $x = \sum_{n=1}^{\infty} \lambda_n e_n$ for a unique scalar sequence (λ_n), it follows that for each $n \in \mathbf{N}$, the functional $e_n^* : x \to \lambda_n$ is linear continuous and multiplicative, so $x = \sum_{n=1}^{\infty} e_n^*(x) \cdot e_n$. It can be easily proven that there is no other continuous nonzero linear multiplicative functional on $\mathcal{A}$ [**1**].

A simple example of a topological algebra with an orthogonal basis is the algebra of all complex sequences with pointwise algebraic operations and the topology of pointwise convergence; we denote it by s. For any topological algebra $\mathcal{A}$ with orthogonal basis there is a natural algebra isomorphism from $\mathcal{A}$ into s defined by $x \to \hat{x} \in s$, where $\hat{x}(n) = e_n^*(x)$, $n \in \mathbf{N}$.

We define now a special class of topological algebras with orthogonal basis [**4**].

Let Φ be a family of nonnegative functions on $\mathbf{N}$ satisfying the following conditions:

i) For each $n \in \mathbf{N}$, there exists a $\phi \in \Phi$ such that

$$\phi(n) \neq 0,$$

and

ii) For each $\phi \in \Phi$, there exists a $\psi \in \Phi$ such that

$$\phi(n) \leq \psi^2(n) \qquad \text{for all } n \in \mathbf{N}.$$

The collection of all such families of functions will be denoted by $\mathfrak{F}$.

For any Φ in $\mathfrak{F}$ we define a subalgebra $s(\Phi)$ of s, by

$$s(\Phi) = \{a = (a(n)) \in s : ||a||_\phi = \sup_n |a(n)\phi(n)| < \infty \\ \text{and } \lim_{n\to\infty} |a(n)|\phi(n) = 0 \text{ for all } \psi \in \Phi\}.$$

We say that $\Phi, \Psi \in \mathfrak{F}$ are equivalent if $s(\Phi) = s(\Psi)$. By Φ-algebra we mean any topological algebra isomorphic with $s(\Phi)$ for some Φ in $\mathfrak{F}$. It is easy to check that a Φ-algebra is complete and locally convex. It has a unit $e = \sum_{k=0}^{\infty} e_k = (1, 1, 1, \dots)$ if and only if each sequence from Φ is convergent to 0; in this case, it contains all bounded sequences.

Example 2.1. Let Φ be the set of all complex valued functions $\phi(n)$ defined on $\mathbf{N}$ such that

$$0 \le \phi(n) \le 1 \qquad \text{for all } n \in \mathbf{N} \tag{6}$$

and

$$\lim_{n\to\infty} \phi(n) = 0. \tag{7}$$

It is clear that the elements of Φ satisfy i) and ii). We show that the algebra $s(\Phi)$ coincides with the algebra l^∞ of all bounded sequences; however, the topology on $s(\Phi)$ is different from the sup topology of l^∞. The algebra $s(\Phi)$ is separable while l^∞, with sup norm, is not. Clearly, all bounded sequences are in $s(\Phi)$. Assume that $s(\Phi)$ contains an unbounded sequence x_0. We define ϕ by

$$\phi(n) = \min(1, (\max\{|x_0(k)| : k \le n\})^{-\frac{1}{2}}).$$

It is easy to check that $\phi \in \Phi$ and that $||x||_\phi = \infty$, which leads to a contradiction.

In [**4**], S. El-Helay and T. Husain asked if any complete locally convex algebra with orthogonal basis and a unit is a Φ-algebra. The following example gives a negative answer.

Example 2.2. Let Ψ be the set of all nonnegative, decreasing functions ψ on $\mathbf{N}$ such that $\lim_{n\to\infty} \psi(n) = 0$. Let

$$\mathcal{A} = \Big\{ x = (x(n)) : x(0) = 0 \text{ and } \\ ||x||_\psi = \sum_{n=0}^{\infty} |x(n+1) - x(n)|\psi(n) < \infty, \text{ for all } \psi \in \Phi \Big\}.$$

We show that $\mathcal{A}$ with pointwise operations is a locally convex complete algebra with a unit $e = (0, 1, 1, \dots)$. We need to prove that, for every $\psi \in \Psi$ there exists a $\phi \in \Psi$ such that, for some $K > 0$,

$$||x \cdot y||_\psi \le K ||x||_\phi \cdot ||y||_\phi, \qquad \text{for all } x, y \in \mathcal{A}.$$

Let $\psi \in \Psi$. We have

$$\begin{aligned} ||x \cdot y||_\psi &= \sum_{n=0}^{\infty} |x(n+1)y(n+1) - x(n)y(n)| \cdot \psi(n) \\ &\leq \sum_{n=0}^{\infty} |x(n+1) - x(n)| \cdot |y(n)| \cdot \psi(n) \\ &\quad + \sum_{n=0}^{\infty} |x(n+1)| \cdot |y(n+1) - y(n)| \cdot \psi(n) \end{aligned}$$

Now, let $\phi \in \Psi$ be such that $\phi^2(n) \geq \psi(n)$. Since ϕ is decreasing and $y(0) = 0$, we have

$$\begin{aligned} |y(n)|\phi(n) &\leq \sum_{j=0}^{n-1} |y(j+1) - y(j)| \cdot \phi(n) \leq \sum_{j=0}^{n-1} |y(j+1) - y(j)| \cdot \phi(j) \\ &\leq \sum_{j=0}^{\infty} |y(j+1) - y(j)| \cdot \phi(j) = ||y||_\phi. \end{aligned}$$

Similarly,

$$|x(n+1)|\phi(n) \leq \sum_{j=1}^{\infty} |x(j+1) - x(j)| \cdot \phi(j) = ||x||_\phi.$$

Hence, from (8), we get

$$\begin{aligned} ||x \cdot y||_\psi \leq &\sum_{n=1}^{\infty} \phi(n) \cdot |x(n+1) - x(n)| \cdot |y(n)| \cdot \phi(n) \\ &+ \sum_{n=1}^{\infty} |\phi(n)| \cdot |y(n+1) - y(n)| \cdot |x(n+1)| \cdot |\phi(n)| \\ \leq &\, ||x||_\phi \cdot ||y||_\phi + ||x||_\phi \cdot ||y||_\phi = 2||x||_\phi \cdot ||y||_\phi. \end{aligned}$$

Therefore, $\mathcal{A}$ is a locally convex algebra with an orthogonal basis consisting of vectors $e_n = (0, 0, \ldots, 1, 0, \ldots)$. It is easy to check that it is complete. We observe that $\mathcal{A}$ is not a Φ-algebra since it does not contain all bounded sequences, which are present in every $s(\Phi)$.

For example, the sequence $(0, 1, -1, 1, -1, \dots)$ is not in $\mathcal{A}$. It is easy to observe that $\mathcal{A}$ contains precisely all convergent complex sequences whose first term is 0. We do not know if all the spectral radii listed in Theorem 1.6 are equal for any algebra with orthogonal basis; we observe that they are equal for a smaller class of Φ-algebras.

Proposition 2.3. *Let $\mathcal{A} = s(\Phi)$ be a Φ-algebra. Then for any $x \in \mathcal{A}$ we have*

$$\begin{aligned} r_1(x) &= r_2(x) = r_3(x) = r_4(x) = r_5(x) \\ &= r_6(x) = r_7(x) = r_*(x) = \sup\{|\hat{x}(n)| : n \in \mathbf{N}\}. \end{aligned}$$

Proof. Let $x = (x(n)) \in s(\Phi)$. Then

$$\begin{aligned} \sqrt[k]{||x^k||_\phi} &= \{\sup\{|\phi(n)x^k(n)| : n \in \mathbf{N}\}^{1/k} \\ &= \sup\{\phi(n)^{1/k}|x(n)| : n \in \mathbf{N}\} \\ &\le \sup\{\phi(n)^{1/k} : n \in \mathbf{N}\} \cdot \sup\{|\hat{x}(n)| : n \in \mathbf{N}\}. \end{aligned}$$

Since, for all $\phi \in \Phi$, $\sup\{|\phi(n)| : n \in \mathbf{N}\} < \infty$, we get

$$r_2(x) = \limsup_k \sqrt[k]{||x^k||_\phi} \le \sup\{\hat{x}(n)| : n \in \mathbf{N}\} = r_4(x),$$

hence $r_2(x) \le r_4(x)$ and Theorem 1.7 gives the Proposition. $\square$

3. The extended spectral radius in B_0-algebras.

Theorem 3.1. *Let $(\mathcal{A}, (||\cdot||_n))$ be a complex B_0-algebra with a unit, then*

$$r_6(x) = r_*(x) \qquad \textit{for all } x \in \mathcal{A}.$$

Proof. Let $x \in \mathcal{A}$. We prove that $r_*(x) \le r_6(x)$; the proof works for any complex, unital locally convex algebra. The statement is obvious if $r_6(x) = \infty$ so assume that $r_6(x) < \infty$ and let $r > r_6(x)$. Hence, there exists a series $\sum_{n=0}^{\infty} b_n x^n$ convergent in $\mathcal{A}$, with $b_n \in \mathbf{C}$, and such that

$$1/r = \limsup_n \sqrt[n]{|b_n|}.$$

So, for any continuous seminorm $||\cdot||$, we have

$$1 \geq \limsup_n \sqrt{||b_n x^n||};$$

therefore,

$$1 \geq (1/r) \liminf_n \sqrt[n]{||x^n||}$$

and

$$r \geq \liminf_n \sqrt[n]{||x^n||}.$$

Hence, $r_*(x) \leq r_6(x)$.

It remains to prove that $r_6(x) \leq r_*(x)$; as before, we can assume that $r_*(x) < \infty$. Since $\mathcal{A}$ is a B_0-algebra and the topology on $\mathcal{A}$ is given by a countable family of seminorms, we can find two strictly increasing sequences (i_p) and (n_p) of positive integers such that

$$\lim_n \left(\sqrt[n_p]{||x^{n_p}||_{i_p}} \right) = r_*(x).$$

The radius of convergence of the series $\sum_{p=1}^{\infty} ||x^{n_p}||_{i_p} \lambda^{n_p}$ is $1/r_*(x)$. Let $0 < b < 1/r_*(x)$. Then the series $\sum_{p=1}^{\infty} ||x^{n_p}||_{i_p} b^{n_p}$ is convergent and, since $(||\cdot||_{i_p})$ is an increasing sequence of seminorms on $\mathcal{A}$, it follows that $\sum_{p=1}^{\infty} ||x^{n_p}||_i b^{n_p}$ is convergent for all $i \in \mathbf{N}$. The radius of convergence of $\sum_{p=1}^{\infty} b^{n_p} \lambda^{n_p}$ is $1/b$ so $1/b \geq r_6(x)$, and, since b was any real number such that $0 < b < 1/r_*(x)$, we get $r_6(x) \leq r_*(x)$. □

We now prove that $r_7(\cdot)$ and $r_6(\cdot)$ may not be equal for a B_0-algebra.

Theorem 3.2. *There is a B_0-algebra $\mathcal{A}$ such that, for any $0 \leq \alpha \leq \beta \leq \infty$, there is an x in $\mathcal{A}$ such that $r_7(x) = \beta$ and $r_6(x) = \alpha$.*

Proof. First we prove a lemma.

Lemma. *Let $(\alpha(n))_{n=0}^{\infty}$ be a sequence of real numbers such that*

$$\alpha(0) = 0, \tag{9}$$

$$\alpha(n) \leq \alpha(n+1) \leq \alpha(n) + 1, \tag{10}$$

$$\limsup_n \alpha(n)/n = 1, \tag{11}$$

and

$$\liminf_n \alpha(pn)/n = 0 \qquad \textit{for any } p \in \mathbf{N}. \tag{12}$$

Then there exists a sequence $(\beta(n))_{n=0}^{\infty}$ *satisfying relations* (9) *to* (12) *and*

$$\alpha(n+m) \le \beta(n) + \beta(m) \qquad \textit{for all } n, m = 0, 1, \dots . \tag{13}$$

Proof of the Lemma. We define $\beta(n)$ recursively.

$$\beta(0) = 0,$$

$$\beta(n) = \max\{\alpha(n), \alpha(n+1) - \beta(1), \dots , \\ \alpha(n + (n-1)) - \beta(n-1), \alpha(2n)/2\}.$$

Assume $n \le m$, then (13) is an immediate consequence of the definition of $\beta(m)$.

We prove (10). The inequality $\beta(n+1) \ge \beta(n)$ is obvious. It remains to show that $\beta(n+1) \le \beta(n) + 1$. Assume that

$$\beta(n) \ge \beta(n-1) + 2.$$

Hence, by the definition of $\beta(n)$ we have $2\beta(n) \ge \alpha(2n)$ and

$$\alpha(n+j) - \beta(j) \ge \beta(n-1) + 2, \qquad \text{for some } j = 0, \dots , n-1, \tag{14}$$

or

$$\alpha(2n) \ge 2 \cdot \beta(n-1) + 4 \ge \alpha(2n-2) + 4. \tag{15}$$

On the other hand we also have $\alpha(n+j) \le \alpha(n-1+j) + 1 \le \beta(n-1) + \beta(j) + 1$, which shows that (14) is not possible, and (15) contradicts (10). This contradiction proves that β satisfies (10).

Since β satisfies (10), we have $\beta(n) \leq n$; we also have $\alpha(n) \leq \beta(n)$, so β satisfies (11).

To prove that $\beta(n)$ satisfies (12), fix $\varepsilon > 0$. Let p be a positive integer. Since α satisfies (12) for $2p$, for any positive integer n_0 there is an integer $n \geq n_0$ such that $\alpha(2pn) \leq \varepsilon n$. By the definition of $\beta(n)$, $\beta(m) \leq \alpha(2m)$ so $\beta(pn) \leq \varepsilon n$, and this proves (12) for β and ends the proof of the lemma. $\square$

We now construct an example of a matrix algebra $\mathcal{A}_a = \mathcal{A}(\alpha_{p,n})$ of the formal power series of t such that $1 = r_6(t) < r_7(t) = a < \infty$.

Let $(\alpha_1(n))_{n=0}^{\infty}$ be any sequence which satisfies the relations (9) to (12); for example, it can be defined by $\alpha_1(0) = 0$ and

$$\alpha_1(n+1) = \begin{cases} \alpha_1(n) & \text{if } (2k)! \leq n < (2k+1)!, \\ \alpha_1(n)+1 & \text{if } (2k+1)! \leq n < (2k)!. \end{cases}$$

Then, let $(\alpha_2(n))_{n=0}^{\infty}$ be the sequence given by the Lemma for $\alpha = \alpha_1$; further, $(\alpha_3(n))_{n=0}^{\infty}$ is the sequence given by the Lemma for $\alpha = \alpha_2, \dots$, etc.

Fix $a > 1$, and put

$$\alpha_{p,n} = a^{\alpha_p(n)} \qquad p = 1, 2, \dots, \quad n = 0, 1, \dots .$$

Let $\mathcal{A}_a = \mathcal{A}(\alpha_{p,n})$, where $\mathcal{A}(\alpha_{p,n})$ is the matrix algebra defined in Example 1.3, and let t be the generator of $\mathcal{A}_\alpha$. We have

$$||t^n||_p = \alpha_{p,n}, \qquad p = 1, 2, \dots, \quad n = 0, 1, \dots .$$

Hence, from (11) and (12), we get

$$r_2(t) = \sup_p\{\limsup_n (\alpha_{p,n})^{1/n}\} = \sup_p\{\limsup_n a^{\alpha_p(n)/n}\} = a,$$

and

$$r_*(t) = \sup_p\{\liminf_n (\alpha_{p,n})^{1/n}\} = \sup_p\{\liminf_n a^{\alpha_p(n)/n}\} = a^0 = 1.$$

By Theorems 1.8 and 3.1, we get $1 = r_6(t) < a = r_7(t) < \infty$. $\square$

Let $0 < \alpha < \beta < \infty$, put $a = \alpha/\beta$, and let t be the generator of the algebra $\mathcal{A}_a$ we have just constructed. Then $r_7(\alpha t) = \beta$ and $r_6(\alpha t) = \alpha$. Now we employ a general method to construct a single B_0-algebra $\mathcal{A}$ such that, for any $0 \le \alpha \le \beta \le \infty$ there is an x in $\mathcal{A}$ such that $r_7(x) = \beta$ and $r_6(x) = \alpha$.

Let $\beta > 1$ and put

$$\mathcal{B} = \{x = (x_1, x_2, \dots) \in (\mathcal{A}_\beta)^{\mathbf{N}} : \\ \qquad ||x||_p = \sup_{n \in \mathbf{N}} ||n \cdot x_n||_p < \infty, p = 1, 2, \dots\}.$$

It is easy to check that $\mathcal{B}$ is a B_0-algebra. Let $\mathfrak{F}$ be a free ultrafilter of subsets of $\mathbf{N}$. Put

$$\mathcal{G} = \{x = (x_1, x_2, \dots) \in \mathcal{B} : \lim_{\mathfrak{F}} ||n \cdot x_n||_p = 1, p = 1, 2, \dots\}.$$

Recall that $\lim_{\mathfrak{F}} \gamma_n = \gamma$ means that, for any $\varepsilon > 0$, $\{n : |\gamma - \gamma_n| < \varepsilon\} \in \mathfrak{F}$. $\mathcal{G}$ is an ideal of $\mathcal{B}$ and we can define $\mathcal{B}_\beta = \mathcal{B}/\mathcal{G}$. Let t be the generator of the matrix algebra $\mathcal{A}_\beta$, and let $t_\beta = [(t/n)_{n=1}^\infty)]_{\mathcal{G}} \in \mathcal{B}_\beta$ be the coset of the sequence $(t/n)_{n=1}^\infty)$. We have $r_7(t_\beta) = \beta$ and $r_6(t_\beta) = 0$.

To end the construction, we let $\mathcal{A}$ be the l^∞-direct sum of the following family of B_0-algebras: $\{\mathcal{A}_a, \mathcal{B}_\beta : a > 1, \beta > 1\}$; this is an element of $\mathcal{A}$ that is of the form $\{(x_a, y_\beta\} : x_a \in \mathcal{A}_a, y_\beta \in \mathcal{B}_\beta, a > 1, \beta > 1\}$ and $||\{x_a, y_\beta\}||_p = \sup\{||x_a||_p, ||y_\beta||_p : x_a \in \mathcal{A}_a,\ y_\beta \in \mathcal{B}_\beta,$ $a > 1,\ \beta > 1\}$.

4. The extended spectral radius in locally convex algebras. In this section we give an example of a locally convex algebra $\mathcal{A}$ such that $r_6(\cdot) \neq r_*(\cdot)$.

Theorem 4.1. *There is a locally convex, complex, commutative, topological algebra $\mathcal{A}$ such that, for any $0 \le \alpha \le \beta \le \infty$ there is an x in $\mathcal{A}$ such that $r_6(x) = \beta$ and $r_*(x) = \alpha$.*

Proof. Put $A = \{(2n+1)! : n \in \mathbf{N}\}$, and let $\mathfrak{F}$ be a nonprime ultrafilter of subsets of A. For any $B \in \mathfrak{F}$ we define a sequence $\alpha_B(n)$ of positive integers by $\alpha_B(0) = 0$ and

$$\alpha_B(k+1) = \begin{cases} \alpha_B(k) & \text{if } (2n)! \le k < (2n+1)! \in B, \\ \alpha_B(k) + 1 & \text{otherwise.} \end{cases}$$

For a $B \subseteq A$, we define B^* by

$$B^* = \cup\{(2n!, (2n+2)!) \cap \mathbf{N} : (2n+1)! \in B\}.$$

Lemma 1. *For any $B \in \mathfrak{F}$, we have*

$$\alpha_B(0) = 0, \tag{16}$$

$$\alpha_B(k) \le \alpha_B(k+1) \le \alpha_B(k) + 1 \quad k \in \mathbf{N}, \tag{17}$$

$$\limsup_n \frac{\alpha_B(k)}{k} = 1, \tag{18}$$

$$\liminf_n \frac{\alpha_B(pk)}{k} = 0, \qquad p = 1, 2, 3, \dots, \tag{19}$$

$$\lim_{\substack{k \notin B^* \\ k \to \infty}} \frac{\alpha_B(k)}{k} = 1. \tag{20}$$

Moreover, for any $B_1, B_2 \in \mathfrak{F}$, we have

$$\alpha_{B_1 \cap B_2} \ge \max(\alpha_{B_1}, \alpha_{B_2}). \tag{21}$$

Proof of the Lemma. Formulas (16) and (17) are transparent, and (18) follows from (20). We prove (19) and (20). Note that, for any $p \in \mathbf{N}$, all but finitely many elements of B are of the form $p \cdot k$, so (19) will follow if we show that

$$\lim_{\substack{k \in B \\ k \to \infty}} \frac{\alpha_B(k)}{k} = 0. \tag{22}$$

Let $(2n+1)! \in B$. By the definition of α_B and (17), we have $\alpha_B((2n+1)!) = \alpha_B((2n)!) \le (2n)!$, so $\alpha_B((2n+1)!)/(2n+1)! \le 1/2n+1$. This gives (22).

To show (20), let $k \notin B^*$. We may assume that k is big enough so that $(0,k) \cap B \ne \varnothing$. Let n be the biggest integer such that $(2n-1)! < k$

and $(2n-1)! \in B$. We have $(2n-1)! < (2n)! \le k$; by the definition of α_B, we get

$$\frac{\alpha_B(k)}{k} \ge \frac{\alpha_B((2n-1)!)+(k-(2n-1)!)}{k} \ge 1 - \frac{(2n-1)!}{k} \ge 1 - \frac{1}{2n}$$

and this proves (20). $\square$

Lemma 2. *For any sequence α of real numbers which satisfies conditions* (17)–(21) *of Lemma* 1*, there is a sequence β of real numbers which also satisfies* (17)–(21) *and such that*

$$\alpha(n+m) \le \beta(n) + \beta(m), \qquad n, m \in \mathbf{N}.$$

Proof. The proof follows the same line as the proof of the Lemma in Theorem 3.2. $\square$

To end the proof of the theorem, fix $a > 1$. For any $B \in \mathfrak{F}$, we define a family Ω_B of sequences:

$$\Omega_B = \{(a^{\alpha_B^1(n)})_{n=1}^\infty, (a^{\alpha_B^2(n)})_{n=1}^\infty, \dots\},$$

where $\alpha_B^1(n) = \alpha_B(n)$, $(\alpha_B^2(n))_{n=1}^\infty$ is the sequence β given by Lemma 2, $(\alpha_B^3(n))_{n=1}^\infty$ is the sequence β given by Lemma 2 for $\alpha = \alpha_B^2$, etc. Put $\Omega = \cup_{B\in\mathfrak{F}}\Omega_B$. The family Ω satisfies (5) from Example 1.3 so we can define the matrix algebra $\mathcal{A}_a$ associated with Ω. This is the algebra of all formal power series $f = \Sigma\lambda_n x^n$, with the convolution multiplication and such that, for any sequence a in Ω, we have

$$||f||_a = \sum_{n=0}^{\infty} |\lambda_n| \cdot a(n) < \infty.$$

By (19),

$$r_*(x) = \sup_{a\in\Omega} \liminf_n ||x^k||^{1/k}) = 1.$$

We show that $r_6(x) = a > 1$. The inequality $r_6(x) \le a$ is obvious. Assume that $r_6(x) < a$. Let λ_k be a sequence of complex numbers such that the series $\Sigma\lambda_k x^k$ is convergent in $\mathcal{A}_u$ and that the radius of convergent of $\Sigma\lambda_k w^k$, $w \in \mathbf{C}$ is less than a. Hence,

$$\limsup_k |\lambda_k|^{1/k} > \zeta > 1/a.$$

Put $\Lambda = \{k \in \mathbf{N} : |\lambda_k| > \zeta^k\}$. Λ is an infinite set. Let $B \in \mathfrak{F}$ and assume that $\Lambda \backslash B^*$ is infinite. We have $\limsup_k ||\lambda_k x^k||_B^{1/k} \geq \zeta \lim_{\substack{k\to\infty \\ k\in\Lambda|B^*}} ||\lambda_k x^k||_B^{1/k} \geq \zeta \lim_{k\in\Lambda|B^*} ||x^k||_B^{1/k} = \zeta a > 1$. This contradicts the convergence of $\Sigma\lambda_k x^k$. Hence, for any $B \in \mathfrak{F}$, the set $\Lambda\backslash B^*$ is finite.

For any $B \in \mathfrak{F}$, B^* is a union of infinitely many disjoint segments of integers, say $B^* = I_1 \cup I_2 \cup I_3 \cup \dots$. Since $\Lambda\backslash B^*$ is finite and Λ is infinite, Λ must intersect with infinitely many segments from the family $\mathcal{Q} = \{I_1, I_2, I_3, \dots\}$. Let's divide $\mathcal{Q}$ into two disjoint families $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that Λ intersects with infinitely many segments from both families. Put $B_i = A \cap \cup_{I\in\mathcal{Q}_i} I$, $i = 1, 2$. Since $B_1 \cup B_2 = A$ and $\mathfrak{F}$ is an ultrafilter, one of these two sets must be in $\mathfrak{F}$, say $B_1 \in \mathfrak{F}$, then $\Lambda\backslash B_1^* \supseteq \Lambda \cap B_2^*$ so $\Lambda\backslash B_1^*$ is infinite. The contradiction proves that $r_6(x) = a$.

For any $a \geq 1$, we have constructed a locally convex topological algebra $\mathcal{A}_a$ and $x_a \in \mathcal{A}$ such that $r_*(x_a) = 1$ and $r_6(x_a) = a$. The theorem now follows by the same general arguments as those at the end of the proof of Theorem 3.2. □

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