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HANKEL FORMS OF ARBITRARY WEIGHT OVER A SYMMETRIC DOMAIN VIA THE TRANSVECTANT

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ABSTRACT. Invariant Hankel forms of higher weight are constructed corresponding to a symmetric domain. The construction generalizes the one in the case of the disk [**13**] and the ball [**16**].

0. Introduction. Hankel forms of arbitrary high weight are known in the case of the disk [**13**] and the ball [**16**]. In this paper we suggest a definition of Hankel form which, in principle, works for any symmetric domain.

We construct them with the aid of certain bilinear different covariants called transvectants. The use of this strange word (German: *Überschiebung*) is borrowed from classical invariant theory, where objects called transvectants were defined by P. Gordan [**7**]; a "rediscovery" appeared thus nearly 100 years later in [**13**]. Recall that classical invariant theory is mainly about the group $\mathrm{SL}(2, \mathbf{C})$. Thus, it is now a question of generalizing the transvectant to the case of an arbitrary semi-simple Lie group.

Roughly speaking, the success of our approach depends on the use of a "higher order version" of the Bergman kernel or, better, the Bergman operator (the fact that the Bergman kernel is the determinant of the Bergman operator).

The above is carried out in Section 2. Some auxiliary considerations are made in Sections 3 and 4; these sections are to some extent expository. Section 1 contains preliminary material. In Section 5 some concrete examples of transvectants are worked out and at the end a general formula stated as a theorem is mentioned.

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We remark that all this is just part of the vast project in which I have been engaged since 1983. The basic philosophy is as follows. If a group acts on a space of functions, then this action extends via conjugation also to operators and it is of interest to consider operators which behave nicely under the group action.

The present compilation is just a first step and much work remains to be done. In particular, we shall have very little to say about the new Hankel forms introduced. Our goal is simply to lay down here the purely formal foundations of the subject.

1. Basic facts about symmetric domains. A domain in complex space is called symmetric if for each of its points there exist a biholomorphic involution ("symmetry") which has the point as an isolated fixed point.

Symmetric domains were introduced by É. Cartan [**5**] in 1935.[1] In particular, Cartan gave a complete classification of irreducible bounded symmetric domains. One speaks of "Cartan domains." Roughly speaking, there are four main series of Cartan domains—these are the "classical" domains (types I–IV)—and two more "exceptional" domains (types V and VI in dimension 16 and 27 respectively).[2]

For example, the type $\mathrm{I}_{m,n}$ $(m \leq n)$ domain $\mathcal{D}(\mathrm{I}_{m,n})$ is realized by the set of all complex rectangular $m \times n$ matrices of operator norm less than 1. If $m = 1$ one has the ball and if both m and n are 1 we get the disk. In a similar way one obtains the type II_n and III_n domains using skew symmetric respectively symmetric matrices. Alternatively, one has in these cases a more geometric realization as certain Grassmannians. The Lie ball (type IV_n) has a realization as a certain open subset of a quadric lying in $\mathbf{P}^{n+1}$. The Lie ball is also related to the algebra Clifford numbers, exactly as types I–III correspond to the division algebras over the reals (complex numbers, quaternions, the reals themselves); types V and VI are connected with the Cayley-Graves algebra (octonions or octaves).

A Cartan domain has three important invariants: the *dimension* d, the *rank* r and the *genus* p. Everybody knows what dimension is. The rank can be defined as the largest number r such that a polydisk of dimension r can be (properly) imbedded into the domain. The only rank 1 case is the ball. The disk is specified by dimension 1 and then the rank is also 1, while the genus is 2. The definition of the genus—this involves the boundary structure—is more intricate so we have to refer to the literature. There are also two more sophisticated characteristics sometimes written a and b, for which we likewise must make an appeal to the literature. Let us just recall the relations

$$d = b + \frac{ar(r-1)}{2} + r, \qquad p = 2 + a(r-1) + b.$$

One can approach symmetric domains along various avenues:

a) the pedestrian way, case by case study [**11**];

b) the Lie approach [**10**];

c) the Jordan approach [**15**, **21**].

Each of these has it own virtues. So far I have myself always preferred a); this is the first time that I embark on avenue c), taking some advantage of the Jordan triple system structure. The little that is needed about this will be summarized in an appropriate location in Section 2.

As for the theory of function spaces on a symmetric domain, I refer Arazy's beautiful survey [**1**], and I found also his paper [**2**] helpful, although the latter addresses itself only to the tube type case (built on a Jordan algebra, rather than a Jordan triple system!).

2. Transvectants and Hankel forms. Let us fix a bounded symmetric domain $\mathcal{D}$ in $\mathbf{C}^d$. We endow $\mathbf{C}^d$ with the Hermitian norm $\|\cdot\|$ induced by the Bergman metric of $\mathcal{D}$, denoting the corresponding scalar product by $(\cdot,\bar{\cdot})$. We may clearly assume that the standard basis e_j $(j = 1, \ldots, d)$ is an orthonormal one, $e_1 = (1, 0, \ldots, 0)$ etc., $(e_j, \bar{e}_k) = \delta_{jk}$.

Let $A^{\alpha,2}(\mathcal{D})$ $(\alpha > -1)$ be the Dzhrbashyan space, i.e. $f \in A^{\alpha,2}(\mathcal{D})$ if and only if f is analytic in $\mathcal{D}$ and

$$\|f\|^2 = \int_{\mathcal{D}} |f(z)|^2 K^{-\frac{\alpha}{p}}(z, \bar{z}) d\varepsilon(z) < \infty,$$

where $K(z, \bar{w})$ is the Bergman kernel of $\mathcal{D}$ and $d\varepsilon$ the Euclidean measure in $\mathbf{C}^d$, p being the genus of $\mathcal{D}$. It is also expedient to put

$\lambda = \alpha + p$ and to have a special notation for the weighted measure $d\mu$, $d\mu(z) = K^{1-\lambda/p}(z,\bar{z})d\varepsilon(z)$. Then the above condition can be written

$$\|f\|^2 = \int_{\mathcal{D}} |f(z)|^2 \, d\mu(z) < \infty.$$

The condition $\alpha > -1$ or $\lambda > p-1$ guarantees the convergence of these integrals for sufficiently many functions f. Note that $K(z,\bar{w})$ is the reproducing kernel in $A^{\alpha,2}(\mathcal{D})$.

Remark. In order to indicate the dependence of the various data on the parameters, one may use a subscript, writing e. g. $\|f\|_\alpha$ and $d\mu_\alpha$ for $\|f\|$ and $d\mu$.

The automorphism group $\mathbf{G} = \operatorname{Aut}\mathcal{D}$ acts on $A^{\alpha,2}(\mathcal{D})$ via unitary maps according to the formula

$$U_\xi : f(z) \mapsto f(\xi(z))(\dot{\xi}(z))^{\lambda/p}, \qquad \xi \in \mathbf{G},$$

where $\dot{\xi}$ is the Jacobian of ξ (i.e. $\dot{\xi} = D\xi(z)/(Dz)$).

We would like to define bilinear forms H_φ on $A^{\alpha,2}(\mathcal{D}) \times A^{\alpha,2}(\mathcal{D})$ [3] depending on a tensor[4] "symbol" φ which behaves nicely ("covariantly") under the action of G:

$$H_\varphi(U_\xi f_1, U_\xi f_2) = H_{W_\xi \varphi}(f_1, f_2), \qquad \xi \in \mathbf{G},\ f_1, f_2 \in A^{\alpha,2}(\mathcal{D}),\ \varphi \in \mathcal{S},$$

where W is a suitable action on an appropriate space of symbols $\mathcal{S}$. We shall, informally, refer to such forms as Hankel forms. (Thus the transformation of a Hankel form is a Hankel form.) We seek them in the form

$$H_\varphi(f_1, f_2) = \int_{\mathcal{D}} (T(f_1,f_2)(z), \overline{\varphi(z)})_z \, d\iota(z),$$

where $d\iota$ is the G-invariant (Bergman) measure $\mathcal{D}$, $d\iota(z) = K(z,\bar{z})d\varepsilon(z)$, $(\cdot,\bar{\cdot})_z$ is a suitable G-invariant inner product (depending on $z \in \mathcal{D}$) and T is a bilinear differential covariant generalizing the transvectant in the case of the disc [**13**] and the ball [**16**]:

$$T(U_\xi f_1, U_\xi f_2) = W_\xi(T(f_1, f_2)),$$

W being a representation of the form $W = U \otimes U \otimes V$ with a suitable finite dimensional representation V. (The meaning of this is simply that the effect of W on symbols φ is:)

$$\varphi(z) \mapsto V\varphi(\xi z)(\dot{\xi}(z))^{2\lambda/p}, \qquad \xi \in \mathbf{G}.$$

Remark. The elements of $A^{\alpha,2}(\mathcal{D})$ should really be viewed as sections of a certain Hermitian line bundle L_λ over $\mathcal{D}$, on which $\mathbf{G}$ acts via fiber and metric preserving maps. Then $T(f_1, f_2)$ (as well as φ) is a section of a bundle of the type $L_\lambda \otimes L_\lambda \otimes M$, where M is a suitable vector bundle (cf. *infra*).

It is clear how to define T in the two "lowest" cases:

$$\begin{aligned} T_0(f_1, f_2) &= f_1 \cdot f_2, \\ T_1(f_1, f_2) &= df_1 \cdot f_2 - f_1 \cdot df_2. \end{aligned}$$

Lemma 0. *T_0 is covariant ($M =$ trivial).*

Lemma 1. *T_1 is covariant ($M = T'^*\mathcal{D} =$ complex cotangent bundle).*

Proof. Put $g_k = U_\xi f_k$, $k = 1, 2$. Then for $z \in \mathcal{D}$, $Z \in T'_z M$

$$dg_k(z; Z) = df_k(\xi(z); \xi_*(Z))(\dot{\xi}(z))^{\lambda/p} + f_k(\xi(z))\frac{\lambda}{p}(\dot{\xi}(z))^{\lambda/p-1}d\dot{\xi}(z; Z),$$

where ξ_* is the map on tangent vectors induced by ξ. It follows that

$$T_1(g_1, g_2)(z; Z) = T_1(f_1, f_2)(\xi(z), \xi_*(Z))(\dot{\xi}(z))^{2\lambda/p};$$

in brief:

$$T_1(g_1, g_2) = \xi^*(T_1(f_1, f_2))(\dot{\xi})^{2\lambda},$$

which is all that we need. $\square$

We would now like to define "higher" transvectants. We argue as follows. To establish the covariance of a bilinear differential expression

$T_1(f_1, f_2)$ *it suffices to do this when* f_1, f_2 *are reproducing kernels*, say, $f_1 = K^{\lambda/p}(\cdot, \bar{w}_1)$, $f_2 = K^{\lambda/p}(\cdot, \bar{w}_2)$ with $w_1, w_2 \in \mathcal{D}$.

We first illustrate this idea in concrete cases.

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{1,1})$ = disc, $d = 1$. Then according to [**13**] we have for each $s = 0, 1, 2, \ldots$ the bilinear differential covariant (transvectant)

$$T_s(f_1, f_2) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} \frac{D^j f_1}{(\lambda)_j} \cdot \frac{D^{s-j} f_2}{(\lambda)_{s-j}},$$
$$D = \frac{d}{dz},\ (\lambda)_j = \lambda(\lambda+1)\cdots(\lambda+j-1).$$

In this case $K(z, w) = (1 - z\bar{w})^{-2}$ and $p = 2$, so $f_1 = (1 - z\bar{w}_1)^{-\lambda}$, $f_2 = (1 - z\bar{w}_2)^{-\lambda}$ and we find

$$D^j f_1 = \frac{(\lambda)_j \bar{w}_1^j}{(1 - z\bar{w}_1)^{\lambda+j}}, \qquad D^{s-j} f_2 = \frac{(\lambda)_{s-j} \bar{w}_2^{s-j}}{(1 - z\bar{w}_2)^{\lambda+s-j}}.$$

Hence, by the binomial theorem,

$$T_s(f_1, f_2) = \frac{1}{(1 - z\bar{w}_1)^{\lambda})(1 - z\bar{w}_2)^{\lambda}} \Big(\frac{\bar{w}_1}{1 - z\bar{w}_1} - \frac{\bar{w}_2}{1 - z\bar{w}_2} \Big)^s$$
$$= \frac{(\bar{w}_1 - \bar{w}_2)^s}{(1 - z\bar{w}_1)^{\lambda+s}(1 - z\bar{w}_2)^{\lambda+s}}.$$

The covariance of the last expression is obvious.

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{1,d})$ = ball. This case is very similar (cf. [**16**]). We shall not enter into details here, because in Section 5 we shall treat this case from a general point of view.

Returning to the general case we can at least at once write down the formula

$$T_1(f_1, f_2) = \frac{\lambda}{p} f_1 f_2 \, d \log \frac{K(z, \bar{w}_1)}{K(z, \bar{w}_2)}. \tag{1}$$

By Lemma 1 we know that the differential involved here must behave in a covariant manner. However, it is possible to prove this also in a more direct fashion.

To do this, let us recall the formula [**15**, (2.10)]

$$K(z,\bar{w}) = \det B(z,\bar{w})^{-1}, \tag{2}$$

where $B(z,\bar{w})$ is the Bergman operator defined as [**15**, (2.9)]

$$B(z,\bar{w}) = \mathrm{id} - D(z,\bar{w}) + Q(z)Q(\bar{w}). \tag{3}$$

Here D and Q are certain Jordan theoretic data. More exactly, if $\{\cdot,\cdot,\cdot\}$ stands for the Jordan triple product on $\mathbf{C}^d \times \mathbf{C}^d \times \mathbf{C}^d$ defined by $\mathcal{D}$, then for $a \in \mathbf{C}^d$

$$\begin{aligned} D(z,\bar{w})a &= \{z\bar{w}a\}, \\ Q(z)\bar{a} &= \tfrac{1}{2}\{z\bar{a}z\}. \end{aligned}$$

Note that, for any given $z, w \in \mathbf{C}^d$, $D(z,w)$ is a linear operator on $\mathbf{C}^d$, while $Q(z)$ is an anti-linear one.

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{m,n})$ (type I). The elements of $\mathcal{D}$ are rectangular $m \times n$ matrices ($d = mn$, $p = m+n$) and

$$\begin{aligned} &\{z\bar{w}a\} = D(z,\bar{w})azw^*a + aw^*a \\ &Q(z)\bar{a} = za^*z. \end{aligned}$$

It follows that in this case

$$B(z,\bar{w})a = (1 - zw^*)a(1 - w^*z)$$

so (cf. [**11**, p. 84])

$$K(z,\bar{w}) = \det B(z,\bar{w})^{-1} = \det(1 - zw^*)^{-(m+n)}.$$

From (2) it follows that the differential in (1) can be written

$$-\mathrm{tr}\,[dB(z,\bar{w}_1)\cdot B^{-1}(z,\bar{w}_1) - dB(z,\bar{w}_2)\cdot B^{-1}(z,\bar{w}_2)].$$

We further record the formula [**15**, (2.10)]

$$B(\xi(z),\overline{\xi(w)}) = \xi_*(z)\cdot B(z,\bar w)\cdot \xi_*(w)^*. \tag{4}$$

(Here the upper $*$ denotes the adjoint of a matrix, while a dot $\cdot$ stands for matrix multiplication. Incidentally, the proof of (2) can be based on this formula, see [**15**, (2.12)].

Differentiating (4) with respect to z (keeping w fixed) we find

$$dB(\xi(z),\overline{\xi(w)}) = \xi_*(z)\cdot dB(z,\bar w)\cdot \xi_*(w)^* + d\xi_*(z)\cdot B(z,\bar w)\cdot \xi_*(w)^*.$$

On the other hand, we have, likewise by (4),

$$B^{-1}(\xi(z),\overline{\xi(w)}) = \xi_*(w)^{*-1}\cdot B^{-1}(z,\bar w)\cdot \xi_*(z)^{-1},$$

so that altogether if we multiply together

$$\begin{aligned} dB(\xi(z),\overline{\xi(w)})\cdot B^{-1}(\xi(z),\overline{\xi(w)}) \\ = \xi_*(z)\cdot dB(z,\bar w)\cdot B^{-1}(z,\bar w)\cdot \xi_*^{-1}(z) + d\xi_*(z)\cdot \xi_*^{-1}(z)\end{aligned}$$

or, in abbreviated but more readable form, with $B = B(z,w)$

$$\xi^*(dB\cdot B^{-1}) = \xi_*\cdot dB\cdot \xi_*^{-1} + d\xi_*\cdot \xi_*^{-1}.$$

We write this formula twice with w replaced by w_1 and w_2 and form the difference. Then the last terms drop out and we obtain, introducing the notation

$$\Omega \overset{\text{def}}{=} dB(z,\bar w_1)\cdot B^{-1}(z,\bar w_1) - dB(z,\bar w_2)\cdot B^{-1}(z,\bar w_2),$$

which we again may abbreviate to (with $B_1 = B(z,w_1)$, $B_2 = B(z,w_2)$)

$$\Omega = dB_1\cdot B_1^{-1} - dB_2\cdot B_2^{-1},$$

an end result which we can write in condensed form as

$$\boxed{\xi^*\Omega = \xi_*\cdot\Omega\cdot\xi_*^{-1}}.$$

(This time an upper $*$ is interpreted as pullback of exterior differential forms.) This formula again gives, taking the trace,

$$\boxed{\xi^*(\operatorname{tr}\Omega) = \operatorname{tr}\Omega}.$$

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{1,1}) =$ disc. Then, as we know,

$$\Omega = \frac{\bar{w}_1 - \bar{w}_2}{(1 - z\bar{w}_1)(1 - z\bar{w}_2)}\, dz.$$

As plainly (1) can be rewritten as

$$T_1(f_1, f_2) = \frac{\lambda}{p}\, f_1 f_2 \operatorname{tr} \Omega, \tag{5}$$

the above proves the desired covariance. (If $M = T'^{*}\mathcal{D}$, we must interpret V_ξ as ξ^*.)

Remark. The form $\operatorname{tr}\Omega$ can be expressed in terms of the *quasi-inverse* (see [**15**, Section 7.2]. Indeed, using formula JP31 (see [**15**, Appendix], we find that

$$\begin{aligned} dB(z, \bar{w}) &= -D(dz, \bar{w}) + Q(dz, z)Q(\bar{z}) \\ &= -D(dz, \bar{w}^z)B(z, \bar{w}), \end{aligned}$$

whence

$$\Omega = \operatorname{tr} D(dz, \bar{w}_2^z - \bar{w}_1^z) = (dz, \bar{w}_2^z - \bar{w}_1^z).$$

The point is now that this can be exploited to define higher weight generalizations. In particular, imitating what was done in [**16**], we are led to set for each $s = 0, 1, 2, \ldots$ (and still for $f_1 = K^{\lambda/p}(\cdot, \bar{w}_1)$, $f_2 = K^{\lambda/p}(\cdot, \bar{w}_2)$ with $w_1, w_2 \in D$)

$$T_s(f_1, f_2) = \text{const.}\, f_1 f_2 (\operatorname{tr}\Omega)^{\odot s}.$$

The exact value of the constant is here of no importance; the symbol $\odot$ stands for the symmetric tensor product. Thus now $M = (T'^{*}\mathcal{D})^{\odot}$.

Now we can also get rid of the restriction that f_1, f_2 be reproducing kernels. Indeed, for general f_1, f_2 in $A^{\alpha,2}(\mathcal{D})$ we set

$$\begin{aligned} T_s(f_1, f_2) = \text{const.} \iint_{\mathcal{D}\times\mathcal{D}} & K^{\lambda/p}(z, \bar{w}_1)K^{\lambda/p}(z, \bar{w}_2)(\operatorname{tr}\Omega)^{\odot s} \\ & \times f_1(w_1) f_2(w_2) d\mu_1(w_1) d\mu_2(w_2); \end{aligned} \tag{6}$$

this should be viewed as a definition. Its covariance is manifest.

But we can do even more. For we can equally well consider the case when the single parameter s is replaced by a "signature" $\mathbf{s} = (s_1, s_2, \dots, s_d)$, that is, a nonincreasing sequence of integers $s_1 \geq s_2 \geq \cdots \geq s_d \geq 0$, or a "partition" of "length" d. Namely, we observe that as a generalization of (4) we have

$$\boxed{\xi^*(\Omega \wedge \Omega \wedge \cdots \wedge \Omega) = \xi_* \cdot \Omega \wedge \Omega \wedge \cdots \wedge \Omega \cdot \xi_*^{-1}}.$$

Therefore we can as a generalization of (5) set

$$\begin{aligned} T_{\mathbf{s}}(f_1, f_2) = \text{const.}\, & f_1 f_2 (\operatorname{tr}\Omega)^{\odot(s_1-s_2)} \otimes (\operatorname{tr}\Omega\wedge\Omega)^{\odot(s_2-s_3)} \\ & \otimes \cdots (\operatorname{tr}\Omega\wedge\Omega\wedge\cdots\wedge\Omega)^{\odot s_d}. \end{aligned} \tag{6}$$

The covariance of this bilinear expression is immediate. This time we take $M = (T'^*\mathcal{D})^{\odot(s_1-s_2)} \otimes (T'^*\mathcal{D})^{\odot(s_2-s_3)} \otimes \cdots \otimes (T'^*\mathcal{D})^{\odot s_d}$.

This time we set for $f_1,\ f_2$ in $A^{\alpha,2}(\mathcal{D})$

$$\begin{aligned} T_{\mathbf{s}}(f_1, f_2) = \text{const.} \iint_{\mathcal{D}\times\mathcal{D}} & K^{\lambda/p}(z, \bar{w}_1) K^{\lambda/p}(z, \bar{w}_2) \\ & \times (\operatorname{tr}\Omega)^{\odot(s_1-s_2)} \otimes (\operatorname{tr}\Omega\wedge\Omega)^{\odot(s_2-s_3)} \\ & \otimes \cdots (\operatorname{tr}\Omega\wedge\Omega\wedge\cdots\wedge\Omega)^{\odot s_d} \\ & \times f_1(w_1) f_2(w_2)\, d\mu_1(w_1)\, d\mu_2(w_2); \end{aligned} \tag{7}$$

again it is clear that this is a covariant being.

Remark. In general, the corresponding Hankel forms denoted $H_\varphi^{\mathbf{s}}$, say, may not give an *irreducible* component in the spectral decomposition of the Hilbert space of Hilbert-Schmidt forms over $A^{\alpha,2}(\mathcal{D})$.

Remark. It is not quite clear what is the real bearing of this last generalization. For instance, it is easy to see that $\operatorname{tr}\Omega\wedge\Omega = 0$, so that one should put some restrictions on the signatures $\mathbf{s}$ to be used. In what follows, we shall therefore mainly have the case of the trivial signature $\mathbf{s} = (s, 0, \dots, 0)$ in mind.

One thing remains to be checked, namely that the transvectant is a local operator.[5] Because of the covariance already established it suffices to do the computation for $z = 0$. As $B(0, \bar{w}) = 1$, $dB(0, \bar{w}) = -D(dz, \bar{w})$ (see (3)), it is clear that the integral in (7) comes then as a sum of product of two integrals of the form

$$\int_{\mathcal{D}} \bar{w}^I f(w)\, d\mu(w), \tag{8}$$

where we have denoted by w^I a general monomial, $w^I = w_1^{i_1} w_2^{i_2} \dots w_d^{i_d}$ where $w_1, \dots, w_d$ now denote the coordinates of the vector w. Let $f(w) = \sum_I \hat{f}(I) w^I$ be the Taylor development of the analytic function f at the origin. Introduce also the Gram matrix

$$\Gamma_{IJ} = \int_{\mathcal{D}} w^I \bar{w}^J\, d\mu(w).$$

Let $|I|$ denote the length of the integer vector I, $|I| = i_1 + \cdots + i_d$. Then we know that $\Gamma_{IJ} = 0$ for $|I| \neq |J|$. It follows that each integral (8) comes as a finite sum

$$\sum_{\{J : |I| = |J|\}} \hat{f}(J) \Gamma_{IJ},$$

that is,

$$\sum_{\{J : |I| = |J|\}} \frac{f^{(J)}(0)}{J!}\, \Gamma_{IJ},$$

where $f^{(J)} = \partial^{|J|} f / \partial z_1^{j_1} \dots \partial z_d^{j_d}$, $J! = j_1! \dots j_d!$. Thus we have proved that there is a formula of the type

$$T_{\mathbf{s}}(f_1, f_2)(0) = \sum_{|I| + |J| = |\mathbf{s}|} a_{IJ} f_1^{(I)}(0) f_2^{(J)}(0).$$

But by the covariance already established we must have exactly the same formula for any point (just write 0 in place of z). Indeed, to see this we need only apply the formula already proved with f_1 and f_2 replaced by the functions $U_{\sigma_z} f_1$ and $U_{\sigma_z} f_2$, σ_z being the symmetry interchanging the points 0 and z. This substantiates our claim concerning the local character of the transvectant.

Remark. As far as we know, the Gram matrix $\{\Gamma_{IJ}\}$ has never been computed in the rank greater than 1 case (cf. [**11**]), so we cannot write down the explicit form of the transvectant in any such case. If the rank is 1 (the ball), we have an orthogonal basis of monomials so there is no problem. In Section 5 we suggest a slightly different approach, based on representation theory, which gives concrete results at least in some cases.

Remark. Note that the transvectant behaves covariantly even if we pass from $\mathbf{G}$ to its complexification $\mathbf{G}^c$, which is a complex Lie group.

3. Discussion. We have now defined on a symmetric domain $\mathcal{D}$ in $\mathbf{C}^d$ a covariant Hankel form on the Hilbert space $A^{\alpha,2}(\mathcal{D})$ corresponding to any given integer s or, more generally, a signature $\mathbf{s} = (s_1, \ldots, s_d)$. Our considerations have, however, been purely formal. The next issue on the agenda would now be to prove that they are Hilbert-Schmidt on a suitable space of tensor symbols φ. If we also knew that our procedure exhausts all irreducible covariant Hankel forms—this would be a completeness result—we would then have a Plancherel theorem for bilinear forms analogous to the one (implicit!) in [**13**] in the case of the disc (cf. also [**17**]). On the abstract level tensor products of representations in the discrete series have been investigated by Repka [**20**], so in principle it should be possible to read off the desired completeness from his results, but much work remains to be done. Writing out the corresponding orthogonal decomposition in terms of appropriately chosen bases, would then lead then to interesting considerations involving special functions (generalized hypergeometric functions); cf. [**18**].

Next, one would also ask the usual other questions pertaining to the "size" of a bilinear form (boundedness, compactness, membership Schatten-von Neumann (S_p-) classes); though usually it is the S_2-case that is the hardest. This would then similarly correspond to a Hausdorff-Young theorem for forms. None of this will be carried out here but will be left for the future. Note however that this, in particular, calls for developing a theory of invariant *tensor* spaces over a symmetric domain; so far only invariant *function* spaces have been studied and this theory is despite everything still an embryonic (Hilbert space) level (see [**1, 2**]).[6]

4. Other uses of transvectants. The covariant Hankel forms were, in Section 2, constructed with the aid of the transvectant. It turns out that the transvectant is a very useful tool in this type of analysis in general. In the present Section we point out three more instances of this.

4.1. Big Hankel operators. First, we remark that the study of Hankel forms is equivalent to the study of "small" Hankel operators (linear operators from $A^{\alpha,2}(\mathcal{D})$ to the space of conjugate-analytic functions $\overline{A^{\alpha,2}(\mathcal{D})}$). But we can apply our ideas equally well in the context of "big" Hankel operators (operators that map $A^{\alpha,2}(\mathcal{D})$ into its orthogonal complement $(A^{\alpha,2}(\mathcal{D}))^{\perp}$ in the Hilbert space $L^2(\mathcal{D}, d\mu)$. Following [**4**] let us consider operators whose kernels are of the form $K^{\lambda/p}(z_1, z_2)\overline{F(z_1, z_2)}$, where F is a function which is analytic in both its arguments. We subject F to the transformation rule $F(z_1, z_2) \mapsto F(\xi(z_1), \xi(z_2))$, $\xi \in \mathbf{G}$, so it is a "scalar" quantity. (Its restriction to the diagonal $\{z_1 = z_2\}$ is essentially (up to a conjugation) the Berezin covariant symbol; cf. [**3, 18**].) Let us return for a moment to the Hankel forms H_φ (Section 2), writing them now in the form

$$H_\varphi(f_1, f_2) = \int_{\mathcal{D}} (F_\varphi(z_1, z_2), f_1(z_1) \otimes f_2(z_2))_z \, d\mu(z_1) \, d\mu(z_2).$$

Here the transformation rule for F_φ, apparently, reads

$$F_\varphi(z_1, z_2) \mapsto F_{V_\xi \varphi}(\xi(z_1), \xi(z_2)) \, (\dot{\xi}(z_1))^\lambda (\dot{\xi}(z_2))^\lambda, \qquad \xi \in G.$$

Suppose now that the corresponding transvectant T viewed as a function of λ can be continued analytically all the way down to $\lambda = 0$. Then we can take $\lambda = 0$ in the above formula. Thus we get a candidate for a big Hankel operator. (For the case of the ball see [**16**].)

4.2. Orthogonal decomposition of $L^2(\mathcal{D}, d\mu)$. Our second application concerns the discrete parts the orthogonal decomposition of the Hilbert space $L^2(\mathcal{D}, d\mu)$ under the action of the group G. In the case of the disk and the ball this decomposition was written down explicitly in [**16**] and [**23**], respectively, and the discrete parts were investigated. We remark that our transvectants $T(f_1, f_2)$ as constructed in Section 3 make sense also if the function f_1 or f_2 are not necessarily analytic and we have the same covariance. Therefore we can corresponding to each transvectant T write down a discrete part, $A_T^{\alpha,2}(\mathcal{D})$, say, in the said decomposition (for a general domain $\mathcal{D}$) according to the following recipe: it consists of (tensor valued) functions f that are of the form

$$f = K^{-\lambda/p}\, T(g, K^{\lambda/p})$$

with g analytic. If $T = T_0$ (see Section 2), we get of course back the space $A^{\alpha,2}(\mathcal{D})$ itself.

4.3. Laguerre-Forsyth type invariants. In [**8**] it was indicated how the transvectant could be used to construct differential invariants connected with mth order *ordinary* differential operators on a Riemann surface. Every such operator L, say, determines a projective structure on the Riemann surface such that in terms of any projective coordinate z one has $Lf = d^m f/dz^m$ and terms of degree less than or equal to $m-2$. We can now recover the lower order terms by adding to the leading term $d^m f/dz^m$ terms of the form $M_l^{\Theta} f = T_l(f,\Theta)$, where T_l stands for the lth order transvectant and Θ is a differential form of order $m-l$, or weight $2m-2l$, $l = 0, 1, \ldots, m-2$.[7] The question now arises whether something similar can also be done in connection with total *partial* differential equations associated perhaps with a quotient of a symmetric domain, which is a natural higher dimensional generalization of the notion of a Riemann surface. This suddenly puts us in the realm of automorphic functions!

Remark. The leading part d^m/dz^m of the operator L is known as the *Bol operator* (see [**9**]). It can indeed be viewed as a special case of the transvectant. Namely, let us write down the expression for T_s when f_1 and f_2 change with different weights, say, ν_1 and ν_2 respectively (see [**9**]). By analytic continuation we can extend T_s to the case of arbitrary complex values of these parameters and T_s still transforms covariantly. Now the Bol operator (the case $\nu_1 = 1-s$) arises appears essentially as a residue. I wonder if this can be done also in higher dimensions also.

5. Concrete examples of transvectants and a general formula.[8] We describe a general technique for handling integrals of the type

$$(f,\bar{q}) = \int_{\mathcal{D}} \overline{q(w)} f(w)\, d\mu(w), \tag{1}$$

where f is an arbitrary holomorphic function and q a polynomial.

First we recall the so-called F-norm:[9]

$$\|f\|_F^2 = c_d \int_{\mathbf{C}^d} |f(z)|^2 e^{-\|z\|^2} d\varepsilon(z), \tag{2}$$

where c_d is a normalization constant chosen such that $\|1\|_F = 1$. It may be viewed as the limit of the whole family of norms $\|f\|_\alpha$. Indeed, changing scale and renormalizing the measure we can write

$$\|f\|_\alpha^2 = c_{d\alpha} \int_{\mathcal{D}_R} |f(z)|^2 \det\left(1 - \frac{D(z,\bar{z})}{R^2} + \frac{Q(z)Q(\bar{z})}{R^4}\right)^\alpha \frac{d\varepsilon(z)}{R^{2d}},$$

where $\mathcal{D}_R$ is the domain $\mathcal{D}$ blown up in the ratio R. If we recall that [**15**, (2.10)] $\|z\|^2 = \operatorname{tr} D(z,\bar{z})$, we see that if we pass to the limit ($\alpha \to \infty$, $R \to \infty$, $\alpha/R^2 \to 1$) this becomes (1).

We recall further that we have for the corresponding scalar product

$$(f,\bar{q})_F = \bar{q}\left(\frac{\partial}{\partial z}\right) f(0),$$

provided q is a polynomial.

Finally, we mention that if we consider the action of the isotropy group $\mathbf{K}$ of $\mathbf{G}$ at the origin 0 on polynomials then by a fundamental result due to Schmid (see, e.g., [**1**]) each irreducible component (subrepresentation) comes with multiplicity one. In other words, they are labeled by a signature $\mathbf{m} = (m_1, \ldots, m_r)$, a partition of length r, *not* d. Let $P_{\mathbf{m}}$ be the corresponding space of (homogeneous) polynomials. Because of the irreducibility the norms $\|f\| = \|f\|_\alpha$ and $\|f\|_F$ are proportional on $P_{\mathbf{m}}$:

$$\|f\|^2 = \frac{1}{a_{\mathbf{m}}} \|f\|_F^2,$$

where the proportionality factor $a_{\mathbf{m}} > 0$ will be considered as a function of $\lambda = \alpha + p$, writing $a_{\mathbf{m}} = a_{\mathbf{m}}(\lambda)$.

Having mentioned all of these facts, let us consider the orthogonal (or Peter-Weyl or Hua or Schmid) decomposition of f and q: $f = \sum f_{\mathbf{m}}$, $q = \sum q_{\mathbf{m}}$. Then we find using (2) that

$$\begin{aligned}(f, \bar{q}) &= \sum (f_{\mathbf{m}}, \bar{q}_{\mathbf{m}}) = \sum a_{\mathbf{m}}^{-1} (f_{\mathbf{m}}, \bar{q}_{\mathbf{m}})_F \\ &= \sum a_{\mathbf{m}}^{-1} (f, \bar{q}_{\mathbf{m}})_F = \sum a_{\mathbf{m}}^{-1} \bar{q}_{\mathbf{m}} \left(\frac{\partial}{\partial z} \right) f(0).\end{aligned}$$

In particular, this computation is applicable when q is a monomial, $q = z^I$ for some multi-index I.

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{1,d}) =$ ball. In this case $r = 1$, while $p = d + 1$ (as $a = 1$, $b = d$). Therefore to each degree of homogeneity there is only one signature and one irreducible component, implying by what has been said that the monomials are orthogonal. For the norm of a monomial one has the formula

$$\|z^I\|^2 = \frac{\Gamma(\lambda) I!}{\Gamma(\lambda + p)} = \frac{I!}{(\lambda)_m}, \qquad |I| = i_1 + \cdots + i_d = m, \ I! = i_1! \ldots i_d!.$$

On the other hand, it is well-known that

$$\|z^I\|_F^2 = |I!|, \qquad |I| = m.$$

Comparing the two expressions we conclude that in this case

$$a_m = a_m(\lambda) = (\lambda)_m \,(= \lambda(\lambda+1) \cdots (\lambda+m-1)), \quad \text{(Pochhammer symbol)}.$$

It follows that

$$\int_{\mathcal{D}} \bar{z}^I f(z) d\mu(z) = \frac{1}{(\lambda)_m} f^{(I)}(0).$$

It is now readily seen that

$$T_s(f_1, f_2) = \sum_{j=0}^{s} (-1)^s \binom{s}{j} \frac{d^j f_1}{(\lambda)_j} \frac{d^{s-j} f_2}{(\lambda)_{s-j}},$$

which is the formula given in [**16**] (explicitly at least for $s = 1, 2$).

Example. $\mathcal{D} = \mathcal{D}(\mathrm{I}_{2,2})$ = 4-dimensional matrix ball. In this case $p = 2$, while $p = 4$ (as $a = 2$, $b = 0$). We consider T_s in the simplest (nontrivial) case $s = 2$. Then there are only two competing signatures $(1, 1)$ and $(2, 0)$. In the general case of the domain $\mathcal{D}(\mathrm{I}_{n,n})$ the proportionality factor is (see [1])

$$a_{\mathbf{m}} = a_{\mathbf{m}}(\lambda) = \prod_{j=1}^{n} \left(\lambda - (j-1)\frac{a}{2} \right)_{m_j},$$

so in our case we have

$$a_{(1,1)} = (\lambda)_1 (\lambda - 1)_1 = \lambda(\lambda - 1), \tag{3}$$

$$a_{(2,0)} = (\lambda)_2 = \lambda(\lambda + 1). \tag{4}$$

Let us write the elements of $\mathcal{D} = \mathcal{D}(\mathrm{I}_{1,1})$, unimodular 2×2 matrices of operator norm less than 1 as

$$z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}.$$

Then a monomial $z^j z^k$ (with $j \leq k$) belongs to $P_{(1,1)}$ unless $(j, k) = (1, 4)$ or $(j, k) = (2, 3)$. In this case we thus have

$$\int_{\mathcal{D}} \bar{z}^j \bar{z}^k f(z)\, d\mu(z) = a_{1,1}^{-1} \frac{\partial^2 f(0)}{\partial z^j \partial z^k}.$$

On the other hand, the space $P_{(2,0)}$ is one dimensional and is spanned by the polynomial $z_1 z_4 - z_2 z_3$ (determinant). We have the orthogonal decomposition

$$z^1 z^4 = \frac{1}{2}(z^1 z^2 + z^3 z^4) + \frac{1}{2}(z^1 z^2 - z^3 z^4).$$

This gives

$$\int_{\mathcal{D}} \bar{z}^1 \bar{z}^4 f(z)\, d\mu(z) = a_{1,1}^{-1} \frac{\partial^2 f(0)}{\partial z^1 \partial z^4} + \frac{1}{2}(a_{2,0}^{-1} - a_{1,1}^{-1}) Hf,$$

where we have put

$$Hf = \frac{\partial^2 f}{\partial z^1 \partial z^4} - \frac{\partial^2 f}{\partial z^2 \partial z^3}.$$

In the same way we find (change of sign!)

$$\int_{\mathcal{D}} \bar{z}^2 \bar{z}^3 f(z)\, d\mu(z) = a_{1,1}^{-1} \frac{\partial^2 f(0)}{\partial z^1 \partial z^4} - \frac{1}{2}(a_{2,0}^{-1} - a_{1,1}^{-1}) Hf.$$

It is likewise easy to prove that

$$\int_{\mathcal{D}} \bar{z}^j f(z)\, d\mu(z) = a_1^{-1} \frac{\partial f(0)}{\partial z^j}.$$

Putting all this information together gives

$$\begin{aligned} T_2(f_1, f_2) &= a_{11}^{-1} d^2 f_1 \odot f_2 - 2a_1^{-2} df_1 \odot df_2 + a_{11}^{-1} f_1 \odot d^2 f_2 \\ &\quad + \frac{1}{2}(a_{2,0}^{-1} - a_{1,1}^{-1})(Hf_1 \odot f_2 - Hf_2 \odot f_1)(dz^1 dz^4 - dz^2 dz^3), \end{aligned}$$

where the numbers a are those from (3) and (4).

We end this Section by stating a general formula for T_s which in principle works for any bounded symmetric domain. We observe that we can write for each j

$$(dz, \bar{w})^j = \sum_{|\mathbf{m}|=j} q_{\mathbf{m}}(dz, \bar{w}), \tag{4}$$

where $q_{\mathbf{m}}$ is a polynomial in two arguments, which, when the first one is kept fixed, belongs to $P_{\mathbf{m}}$ in the second argument. (Here $(dz, \bar{w}) = \sum_{j=1}^n dz^j \bar{w}^j$ and we treat $dz^1, \dots, dz^n$ as *commuting* variables.) From (4) we readily derive

$$\boxed{(dz, \bar{w}_1 - \bar{w}_2)^s = \sum_{j=0}^{s} (-1)^s \binom{s}{j} \odot \sum_{|\mathbf{m}|=j} q_{\mathbf{m}}(dz, \bar{w}_1) \sum_{|\mathbf{n}|=s-j} q_{\mathbf{n}}(dz, \bar{w}_2)}\,.$$

This, apparently, yields the end result:

Theorem. *We have the following expression for the transvectant* T_s:

$$\boxed{T_s(f_1, f_2) = \sum_{j=0}^{s} (-1)^s \binom{s}{j} \sum_{|\mathbf{m}|=j} \frac{q_{\mathbf{m}}(dz\partial/\partial z) f_1}{a_{\mathbf{m}}(\lambda)} \odot \sum_{|\mathbf{n}|=s-j} \frac{q_{\mathbf{n}}(dz\partial/\partial z) f_2}{a_{\mathbf{n}}(\lambda)}}.$$

Example. Returning to the four-dimensional matrix ball $\mathcal{D}(\mathrm{I}_{2,2})$ we see that (4) in this case taking $j = 2$ gives us the decomposition

$$\begin{aligned}(dz, \bar{w})^2 = {} & [(dz, \bar{w})^2 - (\bar{w}^1 \bar{w}^4 - \bar{w}^2 \bar{w}^3)(dz^1 dz^4 - dz^2 dz^3)] \\ & + (\bar{w}^1 \bar{w}^4 - \bar{w}^2 \bar{w}^3)(dz^1 dz^4 - dz^2 dz^3),\end{aligned}$$

which of course yields the result in a previous example.

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ENDNOTES

1. The year I was born!

2. This terminology, apparently, is parallel to the one employed in group theory (classical groups versus exceptional groups).

3. The case when the factors are different has, if $\mathcal{D}$ is a disk, been studied by Zhang [**23**].

4. We use the word tensor informally in the sense of H. Weyl, to denote the elements of any vector space on which $\mathbf{G}$ acts.

5. Here I am obliged to Svante Janson for a helpful remark; at an earlier stage of this investigation I had thought, erroneously, that this might not be the case.

6. Recently, we received an interesting preprint from Pro. K. Zhu [**24**] where he does study Besov spaces in the context of symmetric domains; however, it is not yet clear that his spaces possess the right transformation properties. There is also a preprint by Hahn-Youssfi [**9**] on the same subject.

7. We have been informed by Prof. C. Itzykson that this is of interest also from the point of view of physics, in connection with so-called W-algebras; see e.g., [**12**]. See also [**6**], where such questions are considered in the "super" context.

8. In an earlier version of this paper, circulated as Mittag-Leffler report No. 25, 1990/91, Section 5 had a different content; the computations there (omitted now!) must be fixed up, as we overlooked the fact that $d^2 B \neq 0$.

9. The letter F can be read in various ways: Fis(c)her, Fock, ...

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