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A CHARACTERIZATION OF WEAK* DENTING POINTS IN $L^p(\mu, X)^*$

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ABSTRACT. Suppose that (Ω, Σ, μ) is a positive measure space and $1 < p,\ q < \infty$ with $1/p + 1/q = 1$. We prove that a point f in $L^p(\mu, X)^*$ is a weak* denting point of the unit ball $B_{L^p}(\mu, X)^*$ if and only if f is a unit vector in $L^q(\mu, X^*)$ and $f(t)/||f(t)||$ is a weak* denting point of the unit ball of X^* for almost all t in the support of f. This characterization yields: (1) $L^p(\mu, X)^*$ has property (G^*) if and only if X^* does; (2) every denting point of $B_{L^p(\mu,X)}$ is a weak* denting point of $B_{L^q(\mu,X^*)^*}$; and (3) a new proof of the fact that if X has the property (G), then X is p-average locally uniformly rotund for every p with $1 < p < \infty$.

Throughout, X will denote a nontrivial Banach space, X^* the dual space of X, B_X the unit ball and S_X the unit sphere of X, (Ω, Σ, μ) a positive measure space, p and q positive numbers with $1/p + 1/q = 1$, and $\operatorname{diam} K$ the diameter of K where $K \subset X$.

The extremal structure of the unit ball of the Lebesgue-Bochner function space $L^p(\mu, X)$ has been studied extensively. Characterizations have been obtained for extreme points, strongly extreme points, points of continuity, denting points, strongly exposed points, and locally uniformly rotund points of the unit ball of $L^p(\mu, X)$ (see, for example, [**2, 3, 4, 5, 6, 7, 10, 11**]). However, not much study has been done on the extremal structure of the unit ball of $L^p(\mu, X)^*$. The purpose of this note is to characterize a special kind of extreme point, namely, a weak* denting point, of the unit ball of $L^p(\mu, X)^*$ and to give some applications of the characterization.

Let's recall the definitions of a denting point and a weak* denting point. For a subset K of X, the slice of K determined by the functional x^* in X^* and $\delta > 0$ is the subset of K given by

$$S(x^*, K, \delta) = \{x \in K : x^*(x) > \sup x^*(K) - \delta\}.$$

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A point x is called a *denting point* of K if $x \in K$, and the family of all slices of K containing x is a neighborhood base of x with respect to the relative norm topology on K. For $K \subset X^*$, a slice of K is called a *weak* slice* if it is determined by some element in X and $\delta > 0$. A point x^* is called a *weak* denting point* of K if $x^* \in K$ and the family of all weak* slices of K containing x^* is a neighborhood base of x^* with respect to the relative norm topology on K. We use dent K (respectively, w^*-dent K) to denote the set of all denting points (respectively, weak* denting points) of K.

The symbol $L^p(\mu, X)$ denotes the Banach space of all measurable functions from Ω into X with finite p-norm, that is,

$$L^p(\mu, X) = \{f : \Omega \to X : f \text{ is measurable and } ||f||_p < \infty\},$$

where $||f||_p = [\int_\Omega ||f(t)||^p d\mu(t)]^{1/p}$. We use $L^p(\mu)$ to denote $L^p(\mu, X)$ whenever X is the field of real numbers.

The natural characterization of denting points of the unit ball in $L^p(\mu, X)$ [**7**] asserts that a point f in $L^p(\mu, X)$ is a denting point of $B_{L^p(\mu,X)}$ if and only if $||f||_p = 1$ and $f(t)/||f(t)||$ is a denting point of B_X for almost all t in the support of f (sup f). The space $L^q(\mu, X^*)$ is a subspace of $L^p(\mu, X)^*$ and, for any $f \in L^q(\mu, X^*)$ and $g \in L^p(\mu, X)$, the action of f on g is defined by $(f, g) = \int_\Omega (f(t), g(t))\, d\mu(t)$ [**1**]. We will show that a point f in $L^p(\mu, X)^*$ is a weak* denting point of $B_{L^p(\mu,X)^*}$ if and only if $f \in L^q(\mu, X^*)$ with $||f||_q = 1$ and $f(t)/||f(t)||$ is a weak* denting point of B_{X^*} for almost all t in sup f. This characterization yields: (1) $L^p(\mu, X)^*$ has property (G^*) if and only if X^* does; (2) every denting point of $B_{L^p(\mu,X)}$ is a weak* denting point of $B_{L^q(\mu,X^*)^*}$; and (3) a new proof of the fact that if X has property (G), then X is pALUR for every $1 < p < \infty$.

For $K \subset X^*$ (respectively, X), a point y in K is said to be a weak* point of continuity (respectively, point of continuity) of K if the relative weak* (respectively weak) and norm topologies on K coincide at y. We use w^*-pc K (respectively pc K, ext K) to denote the set of all weak* points of continuity (respectively points of continuity, extreme points) of K.

The space $L^q(\mu, X^*)$ norms $L^p(\mu, X)$, that is, $||f||_p = \sup\{(g, f) : g \in B_{L^q(\mu,X^*)}\}$ for every f in $L^p(\mu, X)$. By the Hahn-Banach theorem, if $K = B_{L^q(\mu,X^*)}$, then $\overline{K}^* = B_{L^p(\mu,X)^*}$, where $\overline{K}^*$ is the weak* closure of K in $L^p(\mu, X)^*$. Thus the following is a corollary of Proposition 2.9, [**4**].

Lemma 1. *The following are equivalent for an element f in $L^p(\mu, X)^*$.*

(1) *f is a weak* denting point of the unit ball of $L^p(\mu, X)^*$.*

(2) *f is a weak* denting point of the unit ball of $L^q(\mu, X^*)$.*

(3) *f is both an extreme point and a weak* point of continuity of the unit ball of $L^q(\mu, X^*)$.*

Suppose f is a nonzero element in $L^q(\mu, X^*)$. Define a measure $\mu_{f,q}$ by

$$\mu_{f,q}(E) = \int_E w(t)\, d\mu(t)$$

where

$$w(t) = \begin{cases} ||f(t)||^q & \text{if } t \in \sup f \\ 1 & \text{if } t \in \Omega \backslash \sup f. \end{cases}$$

Let I be the operator from $L^p(\mu_{f,q}, X)$ to $L^p(\mu, X)$ defined as follows:

$$I(h)(t) = \begin{cases} ||f(t)||^{q-1} h(t) & \text{if } t \in \sup f \\ h(t) & \text{if } t \in \Omega \backslash \sup f. \end{cases}$$

It is straightforward to prove the following lemma; a result which allows us to simplify to a great extent the original proof of the main result. We would like to thank the referee for recommending the use of this lemma in our proof.

Lemma 2. *The operator I is an isometric isomorphism and*

$$I^*(f)(t) = \begin{cases} f(t)/||f(t)|| & \textit{if } t \in \sup f \\ 0 & \textit{if } t \in \Omega \backslash \sup f. \end{cases}$$

Thus $I^*(f)$ is an element in $L^q(\mu_{f,q}, X^*)$ and $||I^* f(t)||$ is either 1 *or* 0 *for each t in Ω.*

Note that if f is a nonzero element in $L^p(\mu, X)$ and J from $L^p(\mu, X)$ to $L^p(\mu_{f,p}, X)$ is defined by

$$J(h)(t) = \begin{cases} h(t)/||f(t)|| & \text{if } t \in \sup f \\ h(t) & \text{if } t \in \Omega \backslash \sup f, \end{cases}$$

then J is an isometric isomorphism and

$$J(f)(t) = \begin{cases} f(t)/||f(t)|| & \text{if } t \in \sup f \\ 0 & \text{if } t \in \Omega \backslash \sup f. \end{cases}$$

Suppose that $K \subset X^*$ and $\varepsilon > 0$. We say that a point x^* is an ε-w^*-denting point of K if $x^* \in K$ and x^* is contained in some weak* slice of K of diameter less than ε. Let's use ε-w^*-dent K to denote the set of all ε-w^*-denting points of K. It is obvious that ε-w^*-dent K is the union of those weak* slices of K whose diameter is less than ε. Thus, ε-w^*-dent K is weak* open in K. We have w^*-dent $K = \cap_{n \geq 1} 1/n$-w^*-dent K.

Theorem 3. *An element f in $L^p(\mu, X^*)$ is a weak* denting point of $B_{L^p(\mu,X)^*}$ if and only if $f \in L^q(\mu, X^*)$ with $||f||_q = 1$ and $f(t)/||f(t)||$ is a weak* denting point of B_{X^*} for almost all t in* $\sup f$.

Proof (Part 1). Suppose $f \in w^*$-dent $B_{L^p(\mu,X)^*}$. By Lemma 1 the function f is a weak* denting point of $B_{L^q(\mu,X^*)}$, thus $f \in L^q(\mu, X^*)$ and $||f||_q = 1$. Thanks to Lemma 2, we may assume that $||f(t)||$ is either 1 or 0 for each t in Ω. It follows that $\mu(\sup f) = 1$.

Assume it is not true that $f(t) \in w^*$-dent B_{X^*} for almost all $t \in \sup f$. Let

$$F_n = \{t \in \sup f : f(t) \notin 1/n - w^* - \text{dent } B_{X^*}\}.$$

Then there is $n_0 \geq 1$ such that $\mu(F_{n_0}) \geq 0$. For simplicity of notation, let $E = F_{n_0}$ and let $\varepsilon = 1/n_0$. Since the set of simple functions in $L^p(\mu, X)$ is dense in $L^p(\mu, X)$ and f is a weak* denting point of $B_{L^q(\mu,X^*)}$, there is a slice S_0 of $B_{L^q(\mu,X^*)}$ determined by a simple function g and $\delta > 0$ such that $\operatorname{diam} S_0 < \varepsilon\mu(E)^{1/q}/6$ and $f \in S_0$. Thus, $g = \sum_{k=1}^m x_k \chi_{E_k}$ for some $x_k \in X$ for $k = 1, \ldots, m$, where $\{E_k\}_{1 \leq k \leq m}$ is a partition of Ω. We may assume that $||g||_p = 1$. Then $S_0 = \{h \in B_{L^q(\mu,X^*)} : (h, g) > 1 - \delta\}$ and $(f, g) > 1 - \delta$. There exists a positive number ε_1 such that

$$\varepsilon_1 < \varepsilon/6, \qquad \text{and} \qquad (f, g) - 2\varepsilon_1\mu(E)^{1/q} > 1 - \delta.$$

We may assume without loss of generality that $f(\Omega)$ is separable. Then there is a countable partition $\{A_n\}_{n \geq 1}$ of E with $\operatorname{diam} f(A_n) < \varepsilon_1$ for

$n \geq 1$. Choose t_n in A_n for each n. Suppose that S is a weak* slice of B_{X^*} containing $f(t_n)$. Since $\operatorname{diam} S \geq \varepsilon$, and since S is the convex hull of $S \cap S_{X^*}$, we conclude that $\operatorname{diam} S \cap S_{X^*} \geq \varepsilon$. In particular, for each $n \geq 1$ and $k = 1, \ldots, m$, we have $\operatorname{diam} S_{n,k} \cap S_{X^*} \geq \varepsilon$, where

$$S_{n,k} = \{x^* \in B_{X^*} : (x^*, x_k) > (f(t_n), x_k) - \varepsilon_1\}.$$

Thus, there exists $x^*_{nk} \in S_{X^*}$ such that

$$(*) \qquad (x^*_{nk}, x_k) > (f(t_n), x_k) - \varepsilon_1 \quad \text{and} \quad ||x^*_{nk} - f(t_n)|| \geq \varepsilon/3.$$

Define

$$f_1 = \chi_{\Omega \setminus E} f + \sum_{n \geq 1} f(t_n) \chi_{A_n} \quad \text{and} \quad f_2 = \chi_{\Omega \setminus E} f + \sum_{n \geq 1} \sum_{k=1}^{m} x^*_{nk} \chi_{An \cap E_k}.$$

Then $f_i \in B_{L^q(\mu, X^*)}$ because $f_i \in L^q(\mu, X^*)$ and $||f_i(t)|| = ||f(t)||$ for all t in Ω, where $i = 1, 2$. Since $\operatorname{diam} f(A_n) < \varepsilon_1$ for $n \geq 1$, we have $||f_1 - f||_q < \varepsilon_1 \mu(E)^{1/q}$. For any t in E, we have by $(*)$ that

$$||f_1(t) - f_2(t)|| \geq \varepsilon/3.$$

Thus

$$\begin{aligned} ||f_2 - f||_q &\geq ||f_2 - f_1||_q - ||f_1 - f||_q \\ &\geq \frac{\varepsilon \mu(E)^{1/q}}{3} - \varepsilon_1 \mu(E)^{1/q} \geq \frac{\varepsilon \mu(E)^{1/q}}{6}. \end{aligned}$$

So f_2 does not belong to S_0. On the other hand, it follows from $(*)$ and the choice of the number ε_1 that

$$\begin{aligned} (f_2, g) &= \int_{\Omega \setminus E} (f(t), g(t))\, d\mu(t) + \sum_{n \geq 1} \sum_{k=1}^{m} (x^*_{nk}, x_k) \mu(A_n \cap E_k) \\ &\geq \int_{\Omega \setminus E} (f(t), g(t))\, d\mu(t) + \sum_{n \geq 1} \sum_{k=1}^{m} [(f(t_n), x_k) - \varepsilon_1] \mu(A_n \cap E_k) \\ &= (f_1, g) - \varepsilon_1 \mu(E) = (f, g) - (f - f_1, g) - \varepsilon_1 \mu(E) \\ &\geq (f, g) - ||f - f_1||_q - \varepsilon_1 \mu(E) \geq (f, g) - \varepsilon_1 \mu(E)^{1/q} - \varepsilon_1 \mu(E) \\ &\geq (f, g) - 2\varepsilon_1 \mu(E)^{1/q} > 1 - \delta. \end{aligned}$$

Hence, f_2 belongs to S_0, a contradiction. This shows that $f(t) \in w^*$-dent B_{X^*} for almost all t in $\sup f$, and the first part of the proof is complete. $\square$

Before proving the sufficiency part of Theorem 3, we need to establish two more lemmas. Lemma 5 shows that denting points behave very much like strongly exposed points.

Lemma 4. *Let $f \in L^p(\mu)$ such that $f(t) \geq 0$ for all t in Ω. If $\{f_\lambda\}$ is a net in $L^p(\mu, X^*)$ such that the net $\{||f_\lambda(\cdot)||\}$ converges to f in $L^p(\mu)$, then there is a net $\{g_\lambda\}$ in $L^p(\mu, X^*)$ such that $||g_\lambda(t)|| = f(t)$ for all t and λ, and $\lim_\lambda ||f_\lambda - g_\lambda||_p = 0$.*

Proof. Pick x in S_X. For each λ, let $E_\lambda = \{t \in \Omega : f_\lambda(t) = 0\}$ and define

$$g_\lambda(t) = \begin{cases} f(t)x & \text{if } t \in E_\lambda \\ f(t)\frac{f_\lambda(t)}{||f_\lambda(t)||} & \text{if } t \in \Omega \backslash E_\lambda. \end{cases}$$

Then $||g_\lambda(t)|| = f(t)$ and $||f_\lambda(t) - g_\lambda(t)|| = ||\, ||f_\lambda(t)|| - f(t)||$ for all t and λ. Hence g_λ belongs to $L^p(\mu, X)$ and $\lim_\lambda ||f_\lambda - g_\lambda||_p = \lim_\lambda ||\, ||f_\lambda(\cdot)|| - f||_p = 0$, and the proof is thus complete. $\square$

For a set K in a dual Banach space X^*, we say that a point x^* in K is a weak*-weak* (respectively, weak*-weak) denting point of K if the family of all weak* slices of K containing x^* is a weak* (respectively weak) neighborhood base of x^* with respect to the relative weak* (respectively weak) topology on K.

Lemma 5. *Let x^* be a point of a convex set K in X^*. Then x^* is a weak*-weak* (respectively weak*-weak, weak*) denting point of K if and only if for every $\varepsilon > 0$ and every weak* (respectively weak, norm) neighborhood V of x^* in K, there exist x in S_X and $\delta > 0$ such that $x^* \in S(x, K, \varepsilon\delta)$ and $S(x, K, \delta) \subset V$.*

Proof. Suppose that x^* is a weak*-weak* (respectively, weak*-weak, weak*) denting point of K. Given $\varepsilon > 0$ and a weak* (respectively, weak, norm) neighborhood V of x^* in K. Since x^* is a weak*-weak*

(respectively weak*-weak, weak*) denting point of K, there are y in X and a number $\alpha > 0$ such that $x^* \in S(y, K, \alpha) \subset V$. Let $\beta = (x^*, y) - \sup y(K) + \alpha$, choose η with $0 < \eta < \min\{\beta/2, \varepsilon\beta/4\}$, and let $U = \{y^* \in K : |(y^* - x^*, y)| < \eta\}$. The set U is a weak* neighborhood of x^* in K. Hence, there are $x \in S_X$ and $\delta > 0$ such that

$$x^* \in S(x, K, \varepsilon\delta) \subset U.$$

It remains to show that $S(x, K, \delta) \subset S(y, K, \alpha)$ or equivalently $K \backslash S(y, K, \alpha) \subset K \backslash S(x, K, \delta)$.

Let $y^* \in K \backslash S(y, K, \alpha)$. Then $(y^*, y) \le \sup y(K) - \alpha$. We can choose $z^* \in K$ with $(z^*, x) > \sup x(K) - \varepsilon\delta/2$. It is clear that $z^* \in S(x, K, \varepsilon\delta) \subset U$. Thus $(z^* - x^*, y) < \eta$. Let $\lambda = 2\eta/\beta$. Then $0 < \lambda < 1$ and so $(1-\lambda)z^* + \lambda y^* \in K$. The element $(1-\lambda)z^* + \lambda y^*$ does not belong to U because

$$\begin{aligned}((1-\lambda)z^* + \lambda y^* - x^*, y) &= (1-\lambda)(z^* - x^*, y) + \lambda((y^*, y) - (x^*, y)) \\ &< (1-\lambda)\eta + \lambda(\sup y(K) - \alpha - (x^*, y)) \\ &= (1-\lambda)\eta - \lambda\beta = (1-\lambda)\eta - 2\eta < -\eta.\end{aligned}$$

Hence $(1-\lambda)z^* + \lambda y^* \notin S(x, K, \varepsilon\delta)$, that is,

$$((1-\lambda)z^* + \lambda y^*, x) \le \sup x(K) - \varepsilon\delta.$$

It follows that

$$\begin{aligned}(y^*, x) &\le \frac{1}{\lambda}(\sup x(K) - \varepsilon\delta - (1-\lambda)(z^*, x)) \\ &\le \frac{1}{\lambda}(\sup x(K) - \varepsilon\delta - (1-\lambda)(\sup x(K) - \varepsilon\delta/2)) \\ &< \sup x(K) - \varepsilon\delta/2\lambda = \sup x(K) - (\varepsilon\beta/4\eta)\delta \\ &< \sup x(K) - \delta.\end{aligned}$$

Therefore $y^* \in K \backslash S(x, K, \delta)$. The converse is obvious, and the proof is complete. □

Now we give the second part of the proof of Theorem 3.

Proof (Part 2). Let $\Omega_0 = \sup f$. By Lemma 2, we may assume that $||f(t)||$ is either 1 or 0 for each t in Ω. It follows that $\mu(\Omega_0) = 1$. We may also assume that $f(\Omega)$ is separable and $f(t) \in w^*$-dent B_{X^*} for all $t \in \Omega_0$. It is easy to verify that $f \in \operatorname{ext} B_{L^q(\mu,X^*)}$ (see [**11**]). By Lemma 1, we need only to show that $f \in w^*$-pc $B_{L^q(\mu,X^*)}$.

Towards this end, let $\{f_\lambda\}$ be a net in $B_{L^q(\mu,X^*)}$ such that w^*-$\lim_\lambda f_\lambda = f$. Because the space $L^q(\mu)$ is uniformly convex, we have $\lim_\lambda \|f_\lambda(\cdot)\| = \|f(\cdot)\|$ in $L^q(\mu)$. By Lemma 4, we may assume $\|f_\lambda(t)\| = \|f(t)\|$ for all t and λ. Fix ε with $0 < \varepsilon < 1$ and let $t \in \Omega_0$. Since $f(t)$ is a weak* denting point of B_{X^*}, there exist, by Lemma 5, an element x_t in S_X and $\delta_t > 0$ such that

$$f(t) \in S(x_t, B_{X^*}, \varepsilon\delta_t) \quad \text{and} \quad \operatorname{diam} S(x_t, B_{X^*}, \delta_t) < \varepsilon.$$

Since $f(\Omega_0)$ is separable and since $\{S(x_t, B_{X^*}, \varepsilon\delta_t)\}_{t\in\Omega_0}$ is an open covering of $f(\Omega_0)$, there is a sequence $\{t_n\}_{n\geq 1}$ in Ω_0 such that

$$f(\Omega_0) \subset \cup_{n\geq 1} S(x_n, B_{X^*}, \varepsilon\delta_n) \quad \text{and} \quad f(t_n) \in S(x_n, B_{X^*}, \varepsilon\delta_n),$$

where $x_n = x_{t_n}$ and $\delta_n = \delta_{t_n}$ for $n \geq 1$. Now, for $n \geq 1$, let

$$E_n = \{t \in \Omega_0 : f(t) \in S(x_n, B_{X^*}, \varepsilon\delta_n)\} \quad \text{and} \quad D_n = E_n \setminus \cup_{j=1}^{n-1} E_j.$$

Then $\{D_n\}$ is a partition of Ω_0. For each n and λ, let

$$D_{n\lambda} = \{t \in D_n : (f_\lambda(t), x_n) \leq 1 - \delta_n\}.$$

We claim that, for each $n \geq 1$, there is a λ_n such that $\mu(D_{n\lambda}) \leq \varepsilon\mu(D_n)$ for all $\lambda \geq \lambda_n$.

We may assume that $\mu(D_n) \neq 0$. Since w^*-$\lim_\lambda f_\lambda = f$, we have

$$\lim_\lambda \int_{D_n} (f_\lambda(t), x_n)\, d\mu(t) = \int_{D_n} (f(t), x_n)\, d\mu(t) > (1 - \varepsilon\delta_n)\mu(D_n).$$

Choose λ_n such that $\lambda \geq \lambda_n$ implies that

$$\int_{D_n} (f_\lambda(t), x_n)\, d\mu(t) > (1 - \varepsilon\delta_n)\mu(D_n).$$

Since

$$\begin{aligned}\int_{D_n} (f_\lambda(t), x_n)\, d\mu(t) &= \int_{D_n\setminus D_{n\lambda}} (f_\lambda(t), x_n)\, d\mu(t) + \int_{D_{n\lambda}} (f_\lambda(t), x_n)\, d\mu(t)\\ &\leq \mu(D_n \setminus D_{n\lambda}) + (1 - \delta_n)\mu(D_{n\lambda})\\ &= \mu(D_n) - \delta_n\mu(D_{n\lambda}),\end{aligned}$$

we have $(1-\varepsilon\delta_n)\mu(D_n) < \mu(D_n) - \delta_n\mu(D_{n\lambda})$. So $\mu(D_{n\lambda}) \le \varepsilon\mu(D_n)$ for $\lambda \ge \lambda_n$ as claimed.

Choose $n_0 \ge 1$ so that $\mu(\cup_{n>n_0} D_n) < \varepsilon$. Let $\lambda_0 \ge \lambda_1, \dots, \lambda_{n_0}$. If $t \in D_n$, then $f(t) \in S(x_n, B_{X^*}, \varepsilon\delta_n)$, and if $t \in D_n \backslash D_{n\lambda}$ then $(f_\lambda(t), x_n) > 1-\delta_n$ and so $f_\lambda(t) \in S(x_n, B_{X^*}, \delta_n)$. Since $\operatorname{diam} S(x_n, B_{X^*}, \delta_n) < \varepsilon$, we have for all $\lambda \ge \lambda_0$,

$$\begin{aligned} ||f_\lambda - f||_q^q &= \sum_{n=1}^{n_0} \bigg(\int_{D_n \backslash D_{n\lambda}} ||f_\lambda(t) - f(t)||^q \, d\mu(t) \\ &\qquad\qquad + \int_{D_{n\lambda}} ||f_\lambda(t) - f(t)||^q \, d\mu(t) \bigg) \\ &\quad + \int_{\cup_{n>n_0} D_n} ||f_\lambda(t) - f(t)||^q \, d\mu(t) \\ &\le \sum_{n=1}^{n_0} (\varepsilon^q \mu(D_n \backslash D_{n\lambda}) + 2^q \mu(D_{n\lambda})) \\ &\quad + 2^q \mu(\cup_{n>n_0} D_n) \\ &\le \sum_{n=1}^{n_0} (\varepsilon^q \mu(D_n) + 2^q \varepsilon \mu(D_n)) + 2^q \varepsilon \\ &\le \varepsilon^q + 2^q\varepsilon + 2^q\varepsilon < 5^q \varepsilon. \end{aligned}$$

Thus, $||f_\lambda - f||_q < 5\varepsilon^{1/q}$ for all $\lambda \ge \lambda_0$. Therefore, $f \in w^*\text{-pc}\, B_{L^q(\mu, X^*)}$ and the proof of Theorem 3 is complete. □

Corollary 6. *An element in $L^p(\mu, X)$ is a denting point of the unit ball of $L^p(\mu, X)$ if and only if it is a weak* denting point of the unit ball of $L^q(\mu, X^*)^*$.*

Proof. Since $B_{L^p(\mu,X)}$ is a subset of $B_{L^q(\mu,X^*)^*}$, the first statement implies the second. Conversely, suppose that f is a denting point of the unit ball of $L^p(\mu, X)$. Then $f(t)/||f(t)||$ is a denting point of the unit ball of X for almost all t in $\sup f$ [**7**]. But every denting point of B_X is a weak* denting point of $B_{X^{**}}$ (see [**4**, Proposition 2.9]). Thus, f is, by Theorem 3, a weak* denting point of the unit ball of $L^q(\mu, X^*)^*$ and the proof is complete. □

Remark 7. Corollary 6 is an improvement of the characterization of denting points of the unit ball of $L^p(\mu, X)$ in [**4**] in the following sense: the corollary implies that every denting point in $L^p(\mu, X)$ has a norm neighborhood base consisting of slices determined by elements in $L^q(\mu, X^*)$, which is in general a proper subspace of $L^p(\mu, X)^*$ [**1**].

An analog of Corollary 6 for points of continuity is also true.

Corollary 8. *An element in $L^p(\mu, X)$ is a point of continuity of the unit ball of $L^p(\mu, X)$ if and only if it is a weak* point of continuity of the unit ball of $L^q(\mu, X^*)^*$.*

Proof. Since $B_{L^p(\mu,X)}$ is a subset of $B_{L^q(\mu,X^*)^*}$, the first statement implies the second. Now suppose that $f \in \text{pc } B_{L^p(\mu,X)}$. We may assume that the space $L^p(\mu, X)$ is of infinite dimension. It follows that $||f||_p = 1$. By Lemma 2 and Lemma 4 we may assume that $\mu(\Omega) = 1$ and $||f(t)|| = 1$ for all t in Ω (see the second part of the proof of Theorem 3 for details). Thus, there is an at most countable partition $\{E_n\}_{0 \le n < m}$ of Ω such that μ is atom-free on E_0 and, for each n with $0 < n < m$, the set E_n is an atom. Let $A_1 = E_0$, and let $A_2 = \Omega \backslash E_0$. Let μ_i be the restriction of μ to A_i. Then, in the obvious way,

$$L^p(\mu, X) = L^p(\mu_1, X) \oplus_p L^p(\mu_2, X),$$
$$L^q(\mu, X^*) = L^q(\mu_1, X^*) \oplus_q L^q(\mu_2, X^*),$$

and

$$L^q(\mu, X^*)^* = L^q(\mu_1, X^*)^* \oplus_p L^q(\mu_2, X^*)^*.$$

It is obvious that the function $(f|_{A_i})/||f|_{A_i}||_p$ is a point of continuity of $B_{L^p(\mu_i,X)}$ for $i = 1, 2$, and that $L^p(\mu_2, X)^* = L^q(\mu_2, X^*)$ since $f \in \text{pc}\, B_{L^p(\mu,X)}$ and μ_2 is purely atomic. Thus, $(f|_{A_2})/||f|_{A_2}||_p \in w^*$-pc $B_{L^p(\mu_2,X)}$ regarded as a subset of $L^q(\mu_2, X^*)^*$. Since μ_1 is atom-free, by [**4**, Corollary 2.4], the function $(f|_{A_1})/||f|_{A_1}||_p$ belongs to dent $B_{L^p(\mu_1,X)}$. By Corollary 6, the function $(f|_{A_1})/||f|_{A_1}||_p$ is a weak* denting point of $B_{L^q(\mu_1,X^*)^*}$ and thus a weak* point of continuity of $B_{L^q(\mu_1,X^*)^*}$. It is now straightforward to verify that f is a weak* point of continuity of $B_{L^p(\mu,X)}$ regarded as a subset of $L^q(\mu_2, X^*)^*$. Since

$B_{L^p(\mu,X)}$ is a norm closed and weak* dense in $B_{L^q(\mu,X^*)^*}$, we conclude by Proposition 2.9 in [**4**] that $f \in$ weak*-pc $B_{L^q(\mu,X^*)^*}$. This completes the proof. $\square$

Recall that X has property (G) if every unit vector in X is a denting point of B_X. Similarly, the dual space X^* is said to have property (G^*) if every unit vector in X^* is a weak* denting point of B_{X^*}.

Corollary 9. *The dual space X^* has property (G^*) if and only if $L^p(\mu, X)^*$ has property (G^*).*

Proof. It is obvious that if the space $L^p(\mu, X)^*$ has property (G^*), then X^* has property (G^*). Suppose that X^* has property (G^*). Then X^* has the RNP [**9**], so $L^p(\mu, X)^* = L^q(\mu, X^*)$ [**1**]. By Theorem 3, the space $L^p(\mu, X)^*$ has property (G^*). This completes the proof. $\square$

Let $1 < p < \infty$, and let λ be the Lebesgue measure on $\Omega = [0, 1)$. A Banach space X is said to be p-average locally uniformly rotund (pALUR) [**8**] if, for every $x \in X$ and $f_n \in L^p(\lambda, X)$ such that $\lim_n ||x + f_n||_p = ||x||$ and $\int_\Omega f_n(t)\, d\lambda(t) = 0$, then $\lim_n ||f_n||_p = 0$. It was proved in [**8**] that property (G) and pALUR are equivalent. Here we give a new proof that the property (G) implies pALUR.

Corollary 10. *If X has the property* (G)*, then X is* (pALUR) *for every $1 < p < \infty$.*

Proof. Let $x \in X$ and $f_n \in L^p(\lambda, X)$ such that $\lim_n ||x + f_n||_p = ||x||$ and $\int_\Omega f_n(t)\, d\lambda(t) = 0$. Without loss of generality, we may assume that $||x|| = 1$. Then $x \in \operatorname{dent} B_X$, or equivalently, $x \in w^*$-dent $B_{X^{**}}$. By Theorem 3, the constant function $g = x\chi_\Omega$ is a weak* denting point of the unit ball of $L^q(\lambda, X^*)^*$. Define for each $n \geq 1$ a measurable function g_n from $[0, 1)$ to X by the formula

$$g_n(t) = f_n(2^n t - i + 1) \quad \text{if} \quad \frac{i-1}{2^n} \leq t < \frac{i}{2^n} \quad \text{and} \quad 1 \leq i \leq 2^n.$$

Then $||g + g_n||_p = ||x + f_n||_p$ and $\int_E g_n(t)\, d\lambda(t) = (1/2^n) \int_\Omega f_n(t)\, d\lambda(t) = 0$ whenever E is of the form $[(i-1)/2^n, i/2^n)$. It follows that

$\lim_n ||g + g_n||_p = 1$ and that the sequence $\{g + g_n\}_{n=1}^{\infty}$ converges to g in the weak* topology on $L^q(\mu, X^*)^*$. Thus $\{g + g_n\}_{n=1}^{\infty}$ converges to g in L^p-norm, that is, $\lim_n ||g_n||_p = 0$. Hence, $\lim_n ||f_n||_p = 0$ since $||f_n||_p = ||g_n||_p$. This completes the proof. □

REFERENCES

1. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, Rhode Island, 1977.

2. P. Greim, *An extremal vector-valued L^p-function taking no extremal vectors as values*, Proc. Amer. Math. Soc. **84** (1982), 65–68.

3. ———, *A note on strongly extreme and strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **93** (1985), 65–66.

4. Z. Hu and B-L. Lin, *RNP and CPCP in Lebesgue-Bochner function spaces*, Illinois J. Math. **37** (1993), 329–347.

5. ———, *Strongly exposed points in Lebesgue-Bochner function spaces*, Proc. Amer. Math. Soc. **120** (1994), 1159–1165.

6. J.A. Johnson, *Strongly exposed points in $L^p(\mu, X)$*, Rocky Mountain J. Math. **10** (1980), 517–519.

7. B-L. Lin and P-K. Lin, *Denting points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **97** (1986), 629–633.

8. B-L. Lin, P-K. Lin and S. Troyanski, *Some geometric and topological properties of the unit sphere in a Banach space*, Math. Ann. **274** (1986), 613–616.

9. I. Namioka and R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **13** (1974), 735–750.

10. M. Smith, *Rotundity and extremity in $l^p(X_i)$ and $L^p(\mu, X)$*, Contemp. Math., Amer. Math. Soc. **52** (1983), 143–162.

11. K. Sundaresan, *Extreme points of the unit cell in Lebesgue-Bochner function spaces*, Colloq. Math. **22** (1970), 111–119.

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