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REALIZATIONS OF FINITE DIMENSIONAL ALGEBRAS OVER THE RATIONALS

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Introduction. The realization problem is recurrent in abelian group theory: Given a ring R, when can R be realized as $R \simeq \mathrm{End}\,(G)$ for G a certain type of abelian group or module? In this note we will be interested in a specific form of the realization problem: Given a finite dimensional vector space V over the rationals Q and a Q-algebra $A \subseteq \mathrm{End}\,(V)$, when can A be realized as $A = Q\mathrm{End}\,(G)$ for G an additive subgroup of V with $QG = V$? Here we are identifying $\mathrm{End}\,(G)$ with a subring of $\mathrm{End}\,(V)$ in the usual way.

A related question arose in [**4**] and [**1**]. In general, if G is a mixed abelian group with torsion subgroup T, there is a natural homomorphism $\theta : \mathrm{End}\,(G) \to \mathrm{End}\,(G/T)$. In [**1**], Albrecht, Goeters and Wickless investigated the image of θ for G in a class $\mathcal{G}$ of groups in which G/T is always a finite dimensional Q-vector space and also the image of θ is a finite dimensional Q-algebra. As above, assume A is a subalgebra of $\mathrm{End}\,(V)$ where V is a finite dimensional Q-space. If there exists a group $G \in \mathcal{G}$ and an isomorphism $G/T \simeq V$ such that the image of the induced composition

$$\mathrm{End}\,(G) \to \mathrm{End}\,(G/T) \to \mathrm{End}\,(V)$$

is precisely A, then A is said to be $\mathcal{G}$-realizable. We shown in Section 2 that a Q-subalgebra A of $\mathrm{End}\,(V)$ can be $\mathcal{G}$-realized if and only if A can be realized as $Q\mathrm{End}\,(G)$ for G a full locally free subgroup of V (Theorem 2.4). In Section 3 we show that if A can be realized by any group, then A can be realized by a locally free group (Theorem 3.5). This result answers in the affirmative a conjecture made in [**5**]. In Section 4, we show that the algebras A that can be realized are plentiful. Some examples are included to illustrate the usefulness of the theory.

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1. Definitions and preliminaries. All groups are, of course, abelian; and the torsion-free groups have finite rank. The symbol T always denotes the torsion subgroup of a mixed group that should be clear from the context. Then T_p is the p-primary component of T. A torsion-free group G is *locally free* if the localization G_p at each rational prime p is a free module over Z_p, the localization of the integers Z at the prime p. Locally free groups have played an important role in the theory of torsion-free groups, dating at least back to Warfield [**6**].

The p-adic integers and p-adic numbers will be denoted by $\hat{Z}_p$ and $\hat{Q}_p$, respectively. If G is a torsion-free group or ring, denote $\hat{G}_p = \hat{Z}_p \otimes_Z G$. As usual, we identify G with the subgroup $Z \otimes G$ of $\hat{G}_p$. We also regard G as a subgroup of its divisible hull QG, so that everything lives inside $\hat{Q}_p \otimes G$.

In Section 2 we will work with a class $\mathcal{G}$ of mixed groups defined in [**1**] as follows.

Definition 1.1. The *class* $\mathcal{G}$ consists of all groups of the form $H = G \oplus W$, where W is finite and G is a mixed group, with torsion subgroup $T = \oplus T_p$, that satisfies the following conditions.

(a) The natural inclusion of T in ΠT_p extends to a pure embedding of G into ΠT_p; and

(b) G/T is a nonzero finite rank group; and

(c) G contains a maximal-rank free subgroup X that projects onto each T_p via the natural projection $\Pi T_p \to T_p$.

Remark. In view of condition (b) of Definition 1.1, if X and X' are any two maximal-rank free subgroups of G, then there is a positive integer n such that $nX \subseteq X'$ and $nX' \subseteq X$. Thus, condition (c) could be restated as follows: If X is any maximal-rank free subgroup of G, then X projects onto T_p for almost all primes p.

It is shown in [**1**] that $\mathcal{G}$ is precisely the class of reduced mixed groups H of finite torsion-free rank such that H is self-small and H/T is divisible. In this setting, the image of the map $\theta : \operatorname{End}(H) \to \operatorname{End}(H/T)$ is always a finite dimensional Q-algebra A. The algebra A has a number of useful attributes. For example, the flat dimension of H as an End (H)-module is equal to the flat dimension of H/T as an A-module [**1**, Theorem 3.1]. The paper [**4**] initiated the study of algebras A that occur as the image of θ, and offers other applications.

2. Equivalent realization problems. In this section we show that the mixed group realization problem of [**1**] is equivalent to the locally free realization problem studied in [**5**]. The equivalence sheds light on questions considered in both papers. To formulate our statements more concisely, we introduce some new definitions. Let V be a finite dimensional Q-vector space. A subgroup G of V is called *full* in V provided $QG = V$. If A is a subalgebra of End (V), we say that A is *$\mathcal{G}$-realizable* if there exists a mixed group $H \in \mathcal{G}$ and an isomorphism $H/T \simeq V$ such that A is equal to the image of End (H) under the induced map End $(H) \to$ End $(H/T) \to$ End (V).

A is *locally free realizable* if there exists a locally free full subgroup G of V such that

$$A = Q\mathrm{End}\,(G) = Q\{\varphi \in \mathrm{End}\,(V) \mid \varphi G \subseteq G\}.$$

For our purposes, the finite summand W of $H = G \oplus W$ in $\mathcal{G}$ is irrelevant. Indeed, if A is $\mathcal{G}$-realized by H, then A is $\mathcal{G}$-realized by G. Therefore, all groups G chosen from $\mathcal{G}$ will be assumed to satisfy conditions (a), (b) and (c) of Definition 1.1.

Let $G \in \mathcal{G}$, and let X be a maximal-rank free subgroup of G. Then $F = (X + T)/T$ is canonically isomorphic to X. By definition of $\mathcal{G}$, there is an embedding of G in ΠT_p giving projection maps $G \to T_p$. Thus, for almost all p, there is an epimorphism $F \to T_p$ given by the composition of the isomorphism $F \to X$ and the projection $X \to T_p$ (see the remark following Definition 1.1). Since T_p is p-local, for almost all p there is an induced epimorphism $\pi_p : F_p \to T_p$.

Definition 2.1. Let V be a finite dimensional Q-vector space, and suppose that $G \in \mathcal{G}$ with $\nu : G/T \to V$ an isomorphism. Let X be a maximal-rank free subgroup of G and $F = (X + T)/T$. Then an element $\varphi \in$ End (V) is *ν-realized* by G if and only if for almost all primes p there is a homomorphism $\varphi(p) : T_p \to T_p$ such that the following diagram is commutative.

$$
\begin{tikzcd}
F_p \arrow[r, "\pi_p"] \arrow[d, "\nu^{-1}\varphi\nu"'] & T_p \arrow[d, "\varphi(p)"] \\
F_p \arrow[r, "\pi_p"'] & T_p
\end{tikzcd}
\tag{2.1p}
$$

Note that, given $\varphi \in \operatorname{End}(V)$, the map $\nu^{-1}\varphi\nu$ is an endomorphism of G/T. If $F = (X+T)/T$ is the full free subgroup of G/T from above, then $(\nu^{-1}\varphi\nu F + F)/F$ is finite. It follows that, for almost all primes p, $\nu^{-1}\varphi\nu F_p \subseteq F_p$. Thus, for any $\varphi \in \operatorname{End}(V)$, the left vertical arrow is a well-defined map for almost all primes p. It is easy to check that ν-realizability is independent of the choice of the maximal-rank free subgroup X of G. Indeed, if X' is another maximal-rank free subgroup of G and $F' = (X'+T)/T$, then F and F' are finite rank full free subgroups of G/T. Thus, $F_p = F'_p$ for almost all primes p. The next lemma collects some useful facts from [**4**]. A proof is included for the reader's convenience.

Lemma 2.2. *For each prime p, let T_p be a reduced p-group and suppose G is a pure subgroup of ΠT_p containing $T = \oplus T_p$. Then T is the torsion subgroup of G and*:

(a) *G/T is divisible.*

(b) *Each endomorphism of G lifts uniquely to an element of* $\operatorname{End}(\Pi T_p)$.

(c) *If G has finite torsion-free rank, then a map $\lambda \in \Pi\operatorname{End}(T_p)$ represents an endomorphism of G if and only if $\lambda(X) \subseteq G$ for some maximal-rank free subgroup X of G.*

Proof. It is immediate that T is the torsion subgroup of G. Part (a) is routine. For (b), note that since G/T is divisible and G is reduced, the restriction map defines a monomorphism from $\operatorname{End}(G)$ onto a subring of $\operatorname{End}(T)$. But $\operatorname{End}(T) = \operatorname{End}(\oplus T_p) = \Pi(\operatorname{End} T_p) = \operatorname{End}(\Pi T_p)$. Thus, we can regard $\operatorname{End}(G)$ as a subring of $\operatorname{End}(\Pi T_p)$. In this setting $\operatorname{End}(G)$ is simply the subring of $\operatorname{End}(\Pi T_p)$ consisting of all maps which send G into G. Since G is pure in ΠT_p and contains $\oplus T_p$, it is easy to show that $\operatorname{End}(G)$ is a pure subring of $\operatorname{End}(\Pi T_p)$.

(c) Since $T = \oplus T_p$ is the torsion subgroup of ΠT_p, we have $\lambda(T) \subset T \subset G$ for each $\lambda \in \mathrm{End}\,(\Pi T_p)$. Suppose also that $\lambda(X) \subset G$ for some maximal-rank free subgroup $X \subset G$. Let $g \in G$ be an element of infinite order. By the maximality of rank (X) there exists a positive integer m such that $mg \in X$. Thus, $m\lambda(g) = \lambda(mg) \in G$. Since G is pure in ΠT_p, it follows that $m\lambda(g) = mg'$ with $g' \in G$. Hence, $\lambda(g) - g' \in T$ and $\lambda(g) = g' + [\lambda(g) - g']$ is in G. We have shown that $\lambda(G) \subseteq G$, as required.

Lemma 2.3. *Let V be a finite dimensional Q-vector space.*

(a) *Suppose $G \in \mathcal{G}$ and $\nu : G/T \to V$ is a fixed isomorphism. Then $\varphi \in \mathrm{End}\,(V)$ is ν-realizable if and only if $\nu^{-1}\varphi\nu \in \theta\mathrm{End}\,(G)$, where $\theta : \mathrm{End}\,(G) \to \mathrm{End}\,(G/T)$ is the canonical homomorphism.*

(b) *If A is a subalgebra of $\mathrm{End}\,(V)$, then A is $\mathcal{G}$-realized by $G \in \mathcal{G}$ if and only if there exists an isomorphism $\nu : G/T \to V$ such that A is precisely the set of ν-realizable elements in $\mathrm{End}\,(V)$.*

Proof. (a) Let X be a maximal-rank free subgroup of G, and set $F = (X + T)/T$. Suppose that $\nu^{-1}\varphi\nu = \theta\phi$ for some $\phi \in \mathrm{End}\,(G)$. By Lemma 2.2(b), ϕ may be written $\phi = \Pi\varphi(p)$ with $\varphi(p) \in \mathrm{End}\,(T_p)$. It is routine to check that the maps $\varphi(p)$ fulfill the conditions of Definition 2.1, so that φ is ν-realizable. Conversely, we show that each ν-realizable element of $\mathrm{End}\,(V)$ belongs to $\theta\mathrm{End}\,(G)$. Let $\varphi \in \mathrm{End}\,(V)$ be ν-realizable with maps $\varphi(p)$ making the diagram (2.1p) commute for almost all primes p. If $x \in X \subseteq G \subseteq \Pi T_p$, then we may represent x as $x = \Pi x(p)$ with $x(p) \in T_p$. The diagrams (2.1p) imply the equation

$$\nu^{-1}\varphi\nu(x + T) = \Pi\varphi(p)x(p) + T.$$

Indeed, a diagram chase shows that if $\nu^{-1}\varphi\nu(x + T) = y + T$ for some $y = \Pi y(p)$, then $y(p) = \varphi(p)x(p)$ for almost all p.

Define a map $\phi = \Pi\varphi(p) \in \mathrm{End}\,(\Pi T_p)$. Observe that for all $x = \Pi x(p)$ in X, we have $\phi(x) = \Pi\varphi(p)x(p)$, so that $\phi(x) + T = \nu^{-1}\varphi\nu(x + T)$. Furthermore, since $(X+T)/T$ is a finite rank full free subgroup of G/T, there exists a positive integer m with $m\nu^{-1}\varphi\nu[(X+T)/T] \subseteq (X+T)/T$. Thus, $m\phi(X) \subseteq X + T \subset G$. But G is pure in ΠT_p so that $\phi(X) \subseteq G$. By Lemma 2.2(c), $\phi(G) \subseteq G$, that is, ϕ represents an endomorphism of G. Finally, the equality $\phi(x) + T = \nu^{-1}\varphi\nu(x+T)$ shows that $\theta\phi$ agrees with $\nu^{-1}\varphi\nu$ on $F = (X+T)/T$. Hence, $\theta\phi = \nu^{-1}\varphi\nu$ on $V = QF$ and $\nu^{-1}\varphi\nu \in \theta\mathrm{End}\,(G)$, as desired.

Part (b) is an immediate consequence of (a) and the definitions. $\square$

We are ready for the main theorem of this section.

Theorem 2.4. *Let V be a finite dimensional Q-space and A a subalgebra of* $\mathrm{End}\,(V)$. *The following are equivalent*:

(a) *A is $\mathcal{G}$-realizable.*

(b) *A is locally free realizable.*

Proof. (a) $\to$ (b). Suppose that A is realized by $G \in \mathcal{G}$, with accompanying isomorphism $\nu : G/T \to V$. Since it will greatly simplify our discussion to do so, we identify V with G/T and set $\nu = 1$. Thus, A becomes a subalgebra of $\mathrm{End}\,(G/T)$. The skeptical reader can easily convert the following proof to the general setting.

Our task is to construct a full locally free subgroup $H \subset V$ with $Q\mathrm{End}\,(H) = A$. To begin, let X be a maximal-rank free subgroup of G, and $F = (X+T)/T \simeq X$. Choose R to be a full free subring of A such that $RF \subseteq F$. Write $T = \oplus_{p\in S} T_p$, where S is the set of primes p such that $T_p \neq 0$. By Lemma 2.3 each $r \in R \subset A$ is ν-realizable. That is, there are endomorphisms $r(p)$ of T_p such that for almost all p, the diagram (2.1p) (reproduced below with $\nu = 1$) is commutative.

$$\begin{array}{ccc} F_p & \xrightarrow{\pi_p} & T_p \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r(p)} \\ F_p & \xrightarrow[\pi_p]{} & T_p \end{array} \tag{2.1p}$$

Because each π_p is an epimorphism, the commutativity of (2.1p) implies that the endomorphism $r(p)$ is uniquely determined by r. It follows that, for $r, s \in R$,

$$(r+s)(p) = r(p) + s(p), \quad \text{and} \quad (rs)(p) = r(p)s(p)$$

whenever p is a prime such that (2.1p) commutes for both r and s. Since R is a finitely generated Z-module, we may in fact conclude that, for almost all $p \in S$, there is an induced R-module structure on T_p given by $rx = r(p)x$ for $r \in R$ and $x \in T_p$. Denote by S' the set of all primes $p \in S$ for which T_p becomes an R-module in this way. Our remarks to this point show that $S \backslash S'$ is finite. Since $RF \subseteq F$, each localization F_p is an R-module. Moreover, for $p \in S'$, the induced R-module structure on T_p makes $\pi_p : F_p \to T_p$ an R-epimorphism.

For $p \in S'$, let $N_p = \ker \pi_p$. Then N_p is an R-submodule of F_p with $p^{k(p)}F_p \subseteq N_p$ for some nonnegative integer $k(p)$ (since $G \in \mathcal{G}$ each T_p is finite). Thus,

$$F_p \subseteq p^{-k(p)}N_p \subseteq p^{-k(p)}F_p \subset V = G/T.$$

For $p \in S'$ define $H_p = p^{-k(p)}N_p$; and for all other primes put $H_p = F_p$. Let $H = \cap H_p$. Then H is a torsion-free group of rank n with $F \subseteq H \subset V$. The group H is locally free since, for $p \in S'$, $H_p/F_p = p^{-k(p)}N_p/F_p \subseteq p^{-k(p)}F_p/F_p$ is finite; while for $p \notin S'$, $H_p = F_p$.

We claim that $Q\mathrm{End}\,(H) = A$. Indeed, for $p \in S'$, N_p is an R-submodule of F_p so that $H_p = p^{-k(p)}N_p$ is an R-submodule of $p^{-k(p)}F_p$. Thus, $RH_p \subseteq H_p$ for all primes p in the set S'. Moreover, $RF \subseteq F$ implies $RH_p \subseteq H_p$ for all primes p not in S'. It follows that $R \subseteq \mathrm{End}\,(H)$ and hence that $A = QR \subseteq Q\mathrm{End}\,(H)$. On the other hand, let φ be an element of $\mathrm{End}\,(V)\backslash A$. Then the map φ is not ν-realizable ($\nu = 1$) by Lemma 2.3(b). Since $\varphi(F_p) \subseteq F_p$ for almost all p, it follows that, for infinitely many primes p, φ does not induce an endomorphism $\varphi(p)$ on T_p making diagram (2.1p) commute. This assertion is equivalent to the fact that N_p is not a φ-invariant subgroup of V for an infinite set of primes p. But then $\varphi H_p \not\subseteq H_p$ for infinitely many p and $\varphi \notin Q\mathrm{End}\,(H)$. Thus, H realizes A.

(b) $\to$ (a). Suppose that there exists a full locally free $H \subset V$ with $Q\mathrm{End}\,(H) = A$. Using the techniques of [**1**], it suffices to show that A can be $\mathcal{G}$-realized under the additional assumption that H has no rank one summand of type equal to outer type H. Choose a maximal free subgroup $F \subseteq H$. As before, choose a full free subring R of A such that $RF \subseteq F$. Since R is a finitely generated subring of $Q\mathrm{End}\,(H)$, we can assume without loss of generality that $RH \subseteq H$. Thus, for all primes p, we have F_p a full R-submodule of the R-module H_p. Since H is locally free, H_p/F_p is finite. Consequently, for each prime p, there exists a nonnegative integer $k(p)$ with $F_p \subseteq H_p \subseteq p^{-k(p)}F_p$. We have an exact sequence of R-modules:

$$(\dagger) \quad 0 \to H_p/F_p \to p^{-k(p)}F_p/F_p \to p^{-k(p)}F_p/H_p \cong F_p/p^{k(p)}H_p \to 0,$$

where all of the maps are the natural ones.

For each prime p, let T_p be the finite p-group $F_p/p^{k(p)}H_p$. Let $G \in \mathcal{G}$ be the pure subgroup of ΠT_p generated by $\oplus T_p$ and canonical image of F in $\Pi T_p = \Pi F_p/p^{k(p)}H_p$. Then there is a natural isomorphism ν from $G/T = G/(\oplus T_p)$ to $QF = V$. Note that $\nu^{-1}F$ is then a maximal-rank free subgroup of G/T. This represents a slight change of notation.

To complete the proof, it suffices by Lemma 2.3(b) to show that A coincides with the set of ν-realizable elements in End(V). Since, for all p, $p^{k(p)}H_p$ is an R-submodule of F_p, each element of R induces a legitimate endomorphism $r(p)$ on $T_p = F_p/p^{k(p)}H_p$. That is, there is a commutative diagram,

$$\text{(2.1p)} \qquad \begin{array}{ccc} \nu^{-1}F_p & \xrightarrow{\pi_p} & T_p \\ {\scriptstyle \nu^{-1}r\nu}\downarrow & & \downarrow{\scriptstyle r(p)} \\ \nu^{-1}F_p & \xrightarrow[\pi_p]{} & T_p \end{array}$$

This shows that R, and hence $A = QR$, are contained in the set of ν-realizable elements of End(V). Suppose $\varphi \in$ End$(V)\backslash A$. We will show that φ is not ν-realizable. First, $\varphi \notin Q$End$(H) = A$. Since H is locally free, it follows that $\varphi H_p \not\subseteq H_p$ for infinitely many p. But F is finite rank free, so $(\varphi F + F)/F$ is finite. Hence, $\varphi F_p \subseteq F_p$ for almost all p. Thus, for infinitely many p, the map φ induces a natural endomorphism of the group $p^{-k(p)}F_p/F_p$ such that H_p/F_p is not φ-invariant. In view of the exact sequence $(\dagger)$, for these p there cannot be an endomorphism $\varphi(p)$ on $T_p = F_p/p^{k(p)}H_p$ which makes (2.1p) commute. Thus, φ is not ν-realizable, and the proof is complete. $\square$

3. Locally free realizability. In this section we show that if A is a subalgebra of End(V) with $A = Q$End(H) for some full subgroup H of V, then H can be chosen locally free. We begin with a proposition due to J.W.S. Cassels.

Proposition 3.1 [**3**, Chapter 5, Theorem 1.1]. *Let K be a finitely generated field extension of the rational numbers Q. Then K embeds into the p-adic numbers $\hat{Q}_p$ for infinitely many primes p.*

We are grateful to the referee for providing a reference for Proposition 3.1. The next lemma records a well-known result in the theory of torsion-free groups. Let H be a torsion-free group, and let R be a subring of $\operatorname{End}(H)$. Then $\hat{H}_p = \hat{Z}_p \otimes H$ becomes a module over $\hat{R}_p = \hat{Z}_p \otimes R$ in the usual way. Also standard is the fact that any endomorphism of QH may be regarded (uniquely) as a $\hat{Q}_p$-endomorphism of $Q\hat{H}_p$. We denote the divisible subgroup of $\hat{H}_p$ by $\operatorname{div}(\hat{H}_p)$.

Lemma 3.2. *Suppose that A is a Q-algebra and H is a finite rank torsion-free group with $Q\operatorname{End}(H) = A$. Then, for each $\varphi \in \operatorname{End}(QH) \backslash A$, one of the following conditions must hold:*

(a) *$\varphi(\hat{H}_p) \not\subseteq \hat{H}_p$ for infinitely many primes p, or*

(b) *$\varphi(\operatorname{div} \hat{H}_p) \not\subseteq \operatorname{div} \hat{H}_p$ for some prime p.*

We also list for reference a combination of Lemma 1.4 and Proposition 1.5 from [**5**].

Lemma 3.3. *Let V be a finite dimensional Q-space, A a subalgebra of $\operatorname{End}(V)$ and R a full free subring of A. The following are equivalent.*

(a) *There is a full locally free subgroup H of V such that $Q\operatorname{End}(H) = A$.*

(b) *For each $\varphi \in \operatorname{End}(V) \backslash A$, there exist infinitely many primes p and elements $w(p) \in V$ such that $\varphi(w(p)) \notin R_p w(p)$.*

(c) *For each $\varphi \in \operatorname{End}(V) \backslash A$, there exist infinitely many primes p and elements $w(p) \in \hat{V}_p$ such that $\varphi(w(p)) \notin \hat{R}_p w(p)$.*

The next proposition will be used in the proof of the main theorem of this section, as well as in Section 4.

Proposition 3.4. *Let V be a finite dimensional Q-vector space, A*

a subalgebra of End(V) *and* $\varphi \in \text{End}(V)\backslash A$. *Suppose that, for some prime* p, *there exists* $w \in \hat{V}_p$ *such that* $\varphi w \notin \hat{A}_p w = (\hat{Z}_p \otimes A)w$. *Then, for infinitely many primes* q *there exists* $w(q)$ *in* $\hat{V}_q$ *such that* $\varphi w(q) \notin \hat{A}_q w(q)$.

Proof. Write $w = \Sigma \alpha_i \otimes h_i \in \hat{V}_p = \hat{Q}_p \otimes V$, with $\alpha_i \in \hat{Q}_p$, $h_i \in V$, and let K be the finitely generated extension of Q (contained in $\hat{Q}_p$) generated by $\{\alpha_i\}$. By Proposition 3.1, K may be embedded in $\hat{Q}_q$ for infinitely many primes q. For such a prime q, we may identify K with a subfield K' of $\hat{Q}_q$, whereby $w = \Sigma \alpha_i \otimes h_i$ is identified with an element w' of $K' \otimes V \subseteq \hat{Q}_q \otimes V = \hat{V}_q$. With this identification, $\varphi w' \notin (K' \otimes A)w'$, because $\varphi w \notin (K \otimes A)w \subseteq \hat{A}_p w$. Suppose that $\varphi w' \in \hat{A}_q w' = (\hat{Q}_q \otimes A)w'$. Then $\varphi w' \in (K' \otimes V) \cap ((\hat{Q}_q \otimes A)w')$. Write $\hat{Q}_q = K' \oplus L$ as K'-modules, and let π denote projection onto K'. The idempotent π induces a K'-projection of $\hat{Q}_q \otimes V$ onto $K' \otimes V$. Then $\varphi w' = \pi \varphi w' \in \pi[(K' \otimes V) \cap ((\hat{Q}_q \otimes A)w')] = (K' \otimes V) \cap (K' \otimes A)w' = (K' \otimes A)w'$, a contradiction. We have shown that there are infinitely many primes q for which there is a $w(q) \in \hat{V}_q$ with $\varphi w(q) \notin \hat{A}_q w(q)$. $\square$

Theorem 3.5. *Let* V *be a finite dimensional* Q*-space and* A *a subalgebra of* End(V). *Then* A *is realizable by a full subgroup of* V *if and only if* A *is locally free realizable.*

Proof. The "if" direction is obvious. Conversely, suppose that A is realizable by a full subgroup H of V. That is, H is a full subgroup of V with $A = Q\text{End}(H)$. In particular, there is a full free subring R of A with $RH \subseteq H$. In view of Lemma 3.3, to show that A is realizable by a locally free submodule of V, it suffices to show that for $\varphi \in \text{End}(V)\backslash A$, there exist infinitely many primes p such that $\varphi w \notin \hat{R}_p w$ for some $w = w(p)$ in $\hat{V}_p$. Given such a φ, there is nothing to show if condition (a) of Lemma 3.2 holds. Thus, we may assume that $\varphi(\text{div}\, \hat{H}_p) \not\subseteq \text{div}\, \hat{H}_p$ for some prime p. In particular, there exists $w \in \hat{H}_p$ such that $\varphi w \notin (\hat{Q}_p \otimes R)w = \hat{A}_p w$. By Proposition 3.4, there are infinitely many primes q for which there exists $w(q) \in \hat{V}_q$ with $\varphi w(q) \notin \hat{A}_q w(q)$. Plainly, for such a $w(q)$, $\varphi w(q) \notin \hat{R}_q w(q)$. From

Lemma 3.3, it follows that A is locally free realizable. □

A special case of Theorem 3.5 confirms a conjecture made in [**5**]: Every algebra that can be realized by a quotient divisible group can be realized by a locally free group.

4. Realizable algebras. We conclude with some results, an example and a conjecture on the set of all realizable subalgebras of $\operatorname{End}(V)$, V a fixed Q-vector space. In view of Theorem 2.4, "realizable" can be taken to mean either $\mathcal{G}$-realizable or locally free realizable. Thus, the results of [**5**] and [**1**] can be combined to show that the realizable subalgebras form a large subset of the set of all subalgebras of $\operatorname{End}(V)$.

If $\dim V = n$, let $\{u_{ij} : 1 \leq i, j \leq n\}$ be a subset of $\operatorname{End}(V)$ corresponding to a complete set of matrix units: $u_{ij}u_{kl} = \delta_{jk}u_{il}$. Suppose U is a subset of $\{u_{ij}\}$ such that the Q-vector space spanned by $U \cup \{1\}$ is a Q-subalgebra of $\operatorname{End}(V)$. That is, the subspace spanned by U is closed under products. We call such an algebra a *subalgebra generated by matrix units.*

Proposition 4.1. *Let V be a finite dimensional Q-vector space and*

$$\mathcal{A} = \{A \subseteq \operatorname{End}(V) : A \text{ is realizable}\}.$$

(a) *$\mathcal{A}$ is closed under conjugation by any invertible element of* $\operatorname{End}(V)$.

(b) *$\mathcal{A}$ contains all semisimple subalgebras of* $\operatorname{End}(V)$. *More generally, $\mathcal{A}$ contains all subalgebras of* $\operatorname{End}(V)$ *that satisfy the double centralizer condition in* $\operatorname{End}(V)$.

(c) *$\mathcal{A}$ contains all subalgebras of* $\operatorname{End}(V)$ *generated by matrix units.*

Proof. (a) If A is locally free realized by $H \subseteq V$, then $\varphi A \varphi^{-1}$ is locally free realized by φH.

(b) [**5**, Theorem 2.1].

(c) [**1**, Theorem 4.2]. □

Curiously, the set of realizable subalgebras is not closed under algebra isomorphism, as the next example shows.

Example 4.2. Let $V = Q^4$, and let A be the subalgebra of $\mathrm{End}\,(V)$ defined as the set of all rational matrices of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ w & x & a & 0 \\ y & z & 0 & a \end{bmatrix} \quad \text{with } z = w - x + 2y.$$

Example 2.6 of [**5**] shows that A is not realizable. However, the algebra A is isomorphic to the algebra A' consisting of all rational matrices of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ w & x & a & 0 \\ y & 0 & 0 & a \end{bmatrix}.$$

The algebra A' is a subalgebra generated by matrix units and is therefore realizable by Proposition 4.1(c).

Example 4.2 tempts us to offer a final conjecture.

Conjecture. *Let V be a finite dimensional vector space over Q. Then every subalgebra of* $\mathrm{End}\,(V)$ *is isomorphic to a realizable subalgebra of* $\mathrm{End}\,(V)$.

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