ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 4, Fall 1996

THE FUNDAMENTAL GROUP OF WHITNEY BLOCKS

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ABSTRACT. Let X be a Peano continuum. Let $C(X)$ be the hyperspace of subcontinua of X, and let $\mu : C(X) \to \mathbf{R}$ be a Whitney map. In this paper we prove: Theorem A. If $0 \leq Q < R < S < T \leq \mu(X)$, then there exists a surjective homomorphism $\phi : \pi_1(\mu^{-1}(Q,R)) \to \pi_1(\mu^{-1}(S,T))$, where $\pi_1(Y)$ means the fundamental group of Y. Theorem B. If $0 \leq S < T \leq \mu(X)$, then $\pi_1(\mu^{-1}(S,T))$ is finitely generated. Theorem C. X is a simple closed curve if and only if $\pi_1(\mu^{-1}(S,T))$ is a nontrivial group for every $0 \leq S < T \leq \mu(X)$.

0. Introduction. Throughout this paper X will denote a continuum (a nonempty, compact, connected metric space) with metric $\boldsymbol{d}$. Let $C(X)$ denote the hyperspace of all subcontinua of X with the Hausdorff metric $\mathcal{H}$. A map is a continuous function. A *Whitney map* for $C(X)$ is a map $\mu : C(X) \to \mathbf{R}$ such that (a) $\mu(\{x\}) = 0$ for every $x \in X$, (b) If $A, B \in C(X)$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$, and (c) $\mu(X) = 1$. A *Whitney block* for $C(X)$, respectively a *Whitney level* for $C(X)$, is a set of the form $\mu^{-1}(S,T)$, respectively $\mu^{-1}(T)$, where $0 \leq S < T \leq 1$. The fundamental group of a space Y is denoted by $\pi_1(Y)$.

Hyperspaces are acyclic (see [**13**, Theorem 1.2]). For Whitney levels, the situation is different; the following observation was made by J.T. Rogers, Jr., in [**11**]: "As we go higher into the hyperspace, no new one-dimensional holes are created, and perhaps some one-dimensional holes are swallowed." This intuitive statement has found several formulations.

In [**12**, Theorem 5], J.T. Rogers, Jr., proved:

Theorem. *If μ is a Whitney map for $C(X)$ and $0 \leq s \leq t \leq 1$, then there exists a monomorphism*

$$\gamma^* : H^1(\mu^{-1}(t)) \longrightarrow H^1(\mu^{-1}(s))$$

Received by the editors on April 29, 1994.
AMS (MOS) *Subject Classification.* Primary 54B20.
Key words and phrases. Hyperspaces, Whitney blocks, Whitney levels, fundamental group.

where $H^1(Y)$ denotes the reduced nth Alexander-Čech cohomology group of Y.

In [**4**, Theorem A], the author proved:

Theorem. *If X is a Peano continuum, μ is a Whitney map for $C(X)$ and $0 \leq s \leq t \leq 1$, then*

$$r(\mu^{-1}(t)) \leq r(\mu^{-1}(s))$$

where $r(Y)$ denotes the multicoherence degree of Y.

Two-dimensional holes behave in a different way. A. Petrus [**11**] showed that if D denotes the unit disk in the Euclidean plane, then there exists a Whitney level $\mu^{-1}(T)$ for $C(D)$ (for an appropriate Whitney map μ) such that there exists a retraction of $\mu^{-1}(T)$ onto a 2-sphere. Related to these topics, in [**3**], the author obtained a characterization of dendroids in terms of n-connectedness of Whitney levels.

In this paper we study the fundamental group of Whitney blocks. We prove that if X is a Peano continuum and μ is a fixed Whitney map for $C(X)$, then:

Theorem A. *If $0 \leq Q < R < S < T \leq \mu(X)$, then there exists a surjective homomorphism $\phi : \pi_1(\mu^{-1}(Q,R)) \to \pi_1(\mu^{-1}(S,T))$.*

Theorem B. *If $0 \leq S < T \leq 1$, then $\pi_1(\mu^{-1}(S,T))$ is finitely generated.*

Theorem C. *The following assertions are equivalent*:

(a) *X is a simple closed curve*,

(b) *$\pi_1(\mu^{-1}(S,T))$ is a nontrivial group for every $0 \leq S < T \leq 1$.*

(c) *For each $R < 1$, there exist $R < S < T \leq 1$ such that $\pi_1(\mu^{-1}(S,T))$ is a nontrivial group.*

1. Preliminary constructions. We will identify X with $F_1(X) =$

$\{x\} \in C(X) : x \in X\}$. The unit closed interval in the real line $\mathbf{R}$ is denoted by I. If $A, B \in C(X)$ and $A \subset B$, an *order arc* from A to B is a map $\gamma : I \to C(X)$ such that $\gamma(0) = A$, $\gamma(1) = B$ and $r \leq s$ implies $\gamma(r) \subset \gamma(s)$. The existence of order arcs is guaranteed by [**10**, Theorem 1.08]. If $\alpha, \beta : I \to Y$ are two paths in a space Y such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, the notation $\alpha \simeq \beta$ (in Y) means that there exists a homotopy $F : I \times I \to Y$ such that: (a) $F(r, 0) = \alpha(r)$ and $F(r, 1) = \beta(r)$ for every $r \in I$, and (b) $F(0, s) = \alpha(0)$ and $F(1, s) = \alpha(1)$ for each $s \in I$. If $A \in C(X)$ and $\varepsilon > 0$, define $N(\varepsilon, A) = \{p \in X : \text{ there exists } a \in A \text{ such that } \boldsymbol{d}(p, a) < \varepsilon\}$.

If $\mathcal{A} = \mu^{-1}(T)$ is a Whitney level for $C(X)$, we define $\mathcal{A}^- = \mu^{-1}[0, T]$. Using order arcs (see [**10**, Theorem 1.08]), it is easy to show that $\mathcal{A}^- = \{A \in C(X) : \text{ There exists } B \in \mathcal{A} \text{ such that } A \subset B\}$, so $\mathcal{A}^-$ does not depend on μ.

Lemma 1.1. *Let $\mathcal{A}$ be a Whitney level for $C(X)$. Let $\alpha, \beta : I \to \mathcal{A}^-$ be two paths such that $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$ and $[(\cup\{\alpha(r) : r \in I\}) \cup (\cup\{\beta(r) : r \in I\})] \in \mathcal{A}^-$. Then $\alpha \simeq \beta$ (in $\mathcal{A}^-$).*

Proof. Let $\mathcal{A} = \mu^{-1}(T)$. Let S be the unit circle in the Euclidean plane $\mathbf{R}^2$. Let $S^- = \{(x, y) \in S : y \leq 0\}$ and $S^+ = \{(x, y) \in S : y \geq 0\}$. Let $\rho : \mathbf{R}^2 \to \mathbf{R}$ be the projection in the first coordinate. Define $\rho_1 = \rho \mid S^- : S^- \to [-1, 1]$ and $\rho_2 = \rho \mid S^+ : S^+ \to [-1, 1]$. Then ρ_1 and ρ_2 are homeomorphisms. Since we are identifying S with $F_1(S) = \{\{z\} \in C(S) : z \in S\}$, we may consider the maps $\rho_1^{-1}, \rho_2^{-1} : [-1, 1] \to C(S)$. Since $C(S)$ is homeomorphic to a disk, then there exists a map $F : [-1, 1] \times I \to C(S)$ such that $F(r, 0) = \rho_1^{-1}(r)$ and $F(r, 1) = \rho_2^{-1}(r)$ for every $r \in [-1, 1]$ and $F(-1, s) = (-1, 0)$ and $F(1, s) = (1, 0)$ for every s.

Define $\gamma : S \to \mathcal{A}^-$ by:

$$\gamma(z) = \begin{cases} \alpha((\rho_1(z) + 1)/2) & \text{if } z \in S^- \\ \beta((\rho_2(z) + 1)/2) & \text{if } z \in S^+. \end{cases}$$

Then γ is a map.

Define $G : I \times I \to \mathcal{A}^-$ by $G(r, s) = \cup\{\gamma(z) : z \in F(2r - 1, s)\}$. Then G is a map. Since $G(r, s) \subset \cup\{\gamma(z) : z \in S\} \subset (\cup \mathrm{Im}\, \alpha) \cup (\cup \mathrm{Im}\, \beta)$, then $G(r, s) \in \mathcal{A}^-$ for every (r, s). Notice that $G(r, 0) = \gamma(\rho_1^{-1}(2r - 1)) =$

$\alpha(r)$ and $G(r,1) = \beta(r)$ for every r. Since $G(0,s) = \gamma(-1,0) = \alpha(0)$ and $G(1,s) = \alpha(1)$ for every s, therefore $a \simeq \beta$ (in $\mathcal{A}^-$). $\square$

Theorem 1.2. *Let $\mathcal{A} = \mu^{-1}(T)$ be a Whitney level for $C(X)$. If $\alpha : I \to \mathcal{A}^-$ is a path such that $\alpha(0), \alpha(1) \in \mathcal{A}$, define $\alpha^{\mathcal{A}} : I \to \mathcal{A}$ by $\alpha^{\mathcal{A}}(r) = \cup\{\alpha(t) : r \leq t \leq q(r)\}$, where $q(r) \in [r,1]$ is taken in such a way that $\mu(\alpha^{\mathcal{A}}(r)) = T$. Then*

(a) *$\alpha^{\mathcal{A}}$ is well defined,*

(b) *$\alpha^{\mathcal{A}}(0) = \alpha(0)$ and $\alpha^{\mathcal{A}}(1) = \alpha(1)$,*

(c) *$\alpha^{\mathcal{A}}$ is continuous,*

(d) *If $\alpha, \beta : I \to \mathcal{A}^-$ are two paths such that $\alpha(0) = \beta(0) \in \mathcal{A}$ and $\alpha(1) = \beta(1) \in \mathcal{A}$ and $a \simeq \beta$ (in $\mathcal{A}^-$), then $\alpha^{\mathcal{A}} \simeq \beta^{\mathcal{A}}$ (in $\mathcal{A}$).*

(e) *If $\alpha, \beta : I \to \mathcal{A}^-$ are two paths such that $\alpha(0), \beta(1) \in \mathcal{A}$ and $\alpha(1) = \beta(0) \in \mathcal{A}$, then $(\alpha\beta)^{\mathcal{A}} = \alpha^{\mathcal{A}}\beta^{\mathcal{A}}$ (here we are considering the usual product of paths).*

Proof. (a), (b) and (c) are easy to prove.

(d) Let $F : I \times I \to \mathcal{A}^-$ be a map such that $F(r,0) = \alpha(r)$ and $F(r,1) = \beta(r)$ for every r, and $F(0,s) = \alpha(0)$ and $F(1,s) = \alpha(1)$ for every s. Define $F^{\mathcal{A}} : I \times I \to \mathcal{A}$ by $F^{\mathcal{A}}(r,s) = \cup\{F(t,s) : r \leq t \leq q(r,s)\}$, where $q(r,s) \in [r,1]$ is chosen in such a way that $\mu(F^{\mathcal{A}}(r,s)) = T$. It is easy to check that $F^{\mathcal{A}}$ is a homotopy between $\alpha^{\mathcal{A}}$ and $\beta^{\mathcal{A}}$ such that $F^{\mathcal{A}}(0,s) = \alpha^{\mathcal{A}}(0)$ and $F^{\mathcal{A}}(1,s) = \alpha^{\mathcal{A}}(1)$ for all s.

(e) Suppose that, for each $r \in I$,

$$(\alpha\beta)^{\mathcal{A}}(r) = \cup\{\alpha\beta(t) : r \leq t \leq q_1(r)\},$$

$$\alpha^{\mathcal{A}}(r) = \cup\{\alpha(t) : r \leq t \leq q_2(r)\},$$

and

$$\beta^{\mathcal{A}}(r) = \cup\{\beta(t) : r \leq t \leq q_3(r)\}.$$

Let $r \in [0,1/2]$. If $q_1(r) \geq 1/2$, then $(\alpha\beta)^{\mathcal{A}}(r) \supset \cup\{\alpha\beta(t) : r \leq t \leq 1/2\} = \cup\{\alpha(2t) : r \leq t \leq 1/2\} = \cup\{\alpha(t) : 2r \leq t \leq 1\} \supset \cup\{\alpha(t) : 2r \leq$

$t \leq q_2(2r)\} = \alpha^{\mathcal{A}}(2r)$. Since $(\alpha\beta)^{\mathcal{A}}(r)$, $\alpha^{\mathcal{A}}(2r) \in \mathcal{A}$, then $(\alpha\beta)^{\mathcal{A}}(r) = \alpha^{\mathcal{A}}(2r)$. If $q_1(r) \leq 1/2$, then $(\alpha\beta)^{\mathcal{A}}(r) = \cup\{\alpha(t) : 2r \leq t \leq 2q_1(r)\}$ which contains or is contained in $\cup\{\alpha(t) : 2r \leq t \leq q_2(r)\} = \alpha^{\mathcal{A}}(2r)$, then $(\alpha\beta)^{\mathcal{A}}(r) = \alpha^{\mathcal{A}}(2r)$. Hence, $(\alpha\beta)^{\mathcal{A}}(r) = \alpha^{\mathcal{A}}(2r)$ for every $r \in [0, 1/2]$. Similarly, $(\alpha\beta)^{\mathcal{A}}(r) = \beta^{\mathcal{A}}(2r-1)$ for each $r \in [1/2, 1]$. Therefore, $(\alpha\beta)^{\mathcal{A}} = \alpha^{\mathcal{A}}\beta^{\mathcal{A}}$. $\square$

Theorem 1.3. *Let $\mathcal{A} = \mu^{-1}(T)$ be a Whitney level for $C(X)$. Let $\alpha : I \to \mathcal{A}$ be a map. Suppose that there exists a point p in the set $\cap\{\alpha(r) : r \in I\}$. Let γ_0, respectively γ_1, be an order arc from $\{p\}$ to $\alpha(0)$, respectively from $\{p\}$ to $\alpha(1)$. Then $\alpha \simeq (\gamma_0^{-1}\gamma_1)^{\mathcal{A}}$ (in $\mathcal{A}$).*

Proof. Notice that $p \in (\gamma_0^{-1}\gamma_1)(r)$ for every $r \in I$. Then α and $(\gamma_0^{-1}\gamma_1)^{\mathcal{A}}$ are paths in the space $C_p(T) = \{A \in \mathcal{A} : p \in A\}$. As was proved by Lynch in [**7**], this space is an AR. Therefore, $\alpha \simeq (\gamma_0^{-1}\gamma_1)^{\mathcal{A}}$ (in $C_p(T) \subset \mathcal{A}$). $\square$

Theorem 1.4. *Suppose that X is a Peano continuum. Let $\mathcal{B} = \mu^{-1}(R, S)$ be a Whitney block for $C(X)$. Let $\mathcal{A} = \mu^{-1}(T)$ be a Whitney level, where $R < T < S$. Fix $A_0 \in \mathcal{A}$, $a \in A_0$ and let γ be an order arc from $\{a\}$ to A_0. Let $\alpha : I \to \mathcal{B}$ be a path such that $\alpha(0) = \alpha(1) = A_0$. Then there exists a map $\sigma : I \to X$ such that $\sigma(0) = \sigma(1) = a$ and $\alpha \simeq (\gamma^{-1}\sigma\gamma)^{\mathcal{A}}$ (in $\mathcal{B}$).*

Proof. From [**1**] and [**9**], we may assume that the metric $\boldsymbol{d}$ for X is a convex metric and diameter of $X = 1$. Define $K : C(X) \times I \to C(X)$ by $K(A, r) = \{p \in X :$ there exists $a \in A$ such that $\boldsymbol{d}(a, p) \leq r\}$. Clearly, (see [**10**, 0.65.3]) K is continuous and $K(A, 0) = A$ and $K(A, 1) = X$ for every $A \in C(X)$.

Fix $T_0 \in (R, S)$ such that $\operatorname{Im}\alpha \subset \mu^{-1}[0, T_0)$. For each $s \in I$, $\mu(K(\alpha(s), 0)) = \mu(\alpha(s)) < T_0$ and $\mu(K(\alpha(s), 1)) = \mu(X) > T_0$, then there exists $r(s) \in I$ such that $\mu(K(\alpha(s), r(s))) = T_0$. We will show that $r(s)$ is unique. Suppose, on the contrary, that there exist $r_1 < r_2$ such that $\mu(K(\alpha(s), r_1)) = T_0 = \mu(K(\alpha(s), r_2))$. Since $K(\alpha(s), r_1) \subset K(\alpha(s), r_2)$, then $K(\alpha(s), r_1) = K(\alpha(s), r_2)$. Since $\mu(X) > T_0$, we may choose a point $q \in X - K(\alpha(s), r_2)$. Let $p \in \alpha(s)$

be such that $\boldsymbol{d}(p,q) = \min\{\boldsymbol{d}(x,q) : x \in \alpha(s)\}$. The convexity of $\boldsymbol{d}$ implies that there exists an isometry $\rho : [0, \boldsymbol{d}(p,q)] \to X$ (see [**10**, 0.65.3]) such that $\rho(0) = p$ and $\rho(\boldsymbol{d}(p,q)) = q$. Since $q \notin K(\alpha(s), r_2)$, $0 \le r_1 < r_2 < \boldsymbol{d}(p,q)$. Let $y = \rho(r_2)$, then $d(p,y) = d(\rho(0), \rho(r_2)) = r_2$, so $y \in K(\alpha(s), r_2) = K(\alpha(s), r_1)$. Then there exists an $x \in \alpha(s)$ such that $\boldsymbol{d}(x,y) \le r_1$. Thus $\boldsymbol{d}(x,q) \le \boldsymbol{d}(x,y) + \boldsymbol{d}(y,q) \le r_1 + \boldsymbol{d}(p,q) - r_2 < \boldsymbol{d}(p,q)$ which contradicts the choice of p. This contradiction proves that $r(s)$ is unique.

It is easy to show that r is continuous and, from the choice of T_0, $r(s) > 0$ for each $s \in I$. Let $r_0 = \min\{r(s) : s \in I\} > 0$.

Set $\mathcal{C} = \mu^{-1}(T_0)$. Define $\beta : I \to \mathcal{C}$ by $\beta(s) = K(\alpha(s), r(s))$. Then β is continuous. Choose $\delta > 0$ such that $|s - t| < \delta$ implies that $\mathcal{H}(\alpha(s), \alpha(t)) < r_0$.

Fix a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of I such that, for every $i = 1, \dots, m$, $s_i - s_{i-1} < \delta$. For each $i \in \{0, 1, \dots, m\}$, choose a point $p_i \in \alpha(s_i)$, with $p_0 = p_m = a$.

Let $i \in \{0, 1, \dots, m-1\}$. Let $s \in [s_i, s_{i+1}]$. Then $\mathcal{H}(\alpha(s), \alpha(s_i)) < r_0$. This implies that $p_i \in K(\alpha(s), r_0) \subset \beta(s)$. Then we may choose an order arc $\gamma_s^{(i)}$ from $\{p_i\}$ to $\beta(s)$. We need to choose $\gamma_{s_0}^{(0)}$ in a more precise way, we define this order arc by:

$$\gamma_{s_0}^{(0)}(s) = \begin{cases} \gamma(2s) & \text{if } s \in [0, 1/2], \\ K(\alpha(0), (2s-1)r(0)) & \text{if } s \in [1/2, 1]. \end{cases}$$

Set $\gamma_0 = \gamma_{s_0}^{(0)}$.

From Theorem 1.3,

$$\begin{aligned} \beta &\simeq (\beta \mid [s_0, s_1])(\beta \mid [s_1, s_2]) \cdots (\beta \mid [s_{m-1}, s_m]) \\ &\simeq ((\gamma_{s_0}^{(0)})^{-1} \gamma_{s_1}^{(0)})^{\mathcal{C}} ((\gamma_{s_1}^{(1)})^{-1} \gamma_{s_2}^{(1)})^{\mathcal{C}} \cdots ((\gamma_{s_{m-1}}^{(m-1)})^{-1} \gamma_{s_m}^{(m-1)})^{\mathcal{C}} \\ &= ((\gamma_{s_0}^{(0)})^{-1} \gamma_{s_1}^{(0)} (\gamma_{s_1}^{(1)})^{-1} \gamma_{s_2}^{(1)} \cdots (\gamma_{s_{m-1}}^{(m-1)})^{-1} \gamma_{s_m}^{(m-1)})^{\mathcal{C}} \\ &\simeq ((\gamma_0)^{-1} \gamma_{s_1}^{(0)} (\gamma_{s_1}^{(1)})^{-1} \gamma_{s_2}^{(1)} \cdots (\gamma_{s_{m-1}}^{(m-1)})^{-1} \gamma_{s_m}^{(m-1)} (\gamma_0)^{-1} \gamma_0)^{\mathcal{C}} \end{aligned}$$

(in $\mathcal{C}$).

Let $i \in \{1, \dots, m\}$. Since $\boldsymbol{d}$ is convex, $N(r_0, \alpha(s_i))$ is an open connected subset of X, then $N(r_0, \alpha(s_i))$ is path connected. Since $p_{i-1} \in N(r_0, \alpha(s_i))$, there exists a map $\sigma_i : I \to N(r_0, \alpha(s_i)) \subset \beta(s_i)$

such that $\sigma_i(0) = p_{i-1}$ and $\sigma_i(1) = p_i$. Since $\cup \operatorname{Im}(\gamma_{s_i}^{(i-1)}(\gamma_{s_i}^{(i)})^{-1}) \subset \beta(s_i)$, Lemma 1.1 implies that $\gamma_{s_i}^{(i-1)}(\gamma_{s_i}^{(i)})^{-1} \simeq \sigma_i$ (in $\mathcal{C}^-$).

Define $\sigma = \sigma_1\sigma_2\cdots\sigma_m$. Then $\beta \simeq (\gamma_0^{-1}\sigma_1\cdots\sigma_m\gamma_0)^{\mathcal{C}} = (\gamma_0^{-1}\sigma\gamma_0)^{\mathcal{C}}$ (in $\mathcal{C}$).

Define $G : I \times I \to \mathcal{B}$ by

$$G(s,t) = \begin{cases} K(\alpha(0), 3st(r(0))) & \text{if } 0 \le s \le 1/3, \\ K(\alpha(3s-1), t(r(3s-1))) & \text{if } 1/3 \le s \le 2/3, \\ K(\alpha(0), 3(1-s)t(r(0))) & \text{if } 2/3 \le s \le 1. \end{cases}$$

Then G is continuous. Notice that

$$G(s,1) = \begin{cases} \gamma_0((3s+1)/2) & \text{if } 0 \le s \le 1/3, \\ \beta(3s-1) & \text{if } 1/3 \le s \le 2/3 \\ \gamma_0((4-3s)/2) & \text{if } 2/3 \le s \le 1, \end{cases}$$

and the map $\zeta(s) = G(s,0)$ is such that $\zeta \simeq \alpha$ (in $\mathcal{B}$). Therefore $\alpha \simeq (\gamma_0 \mid [1/2,1])\beta(\gamma_0 \mid [1/2,1])^{-1}$ (in $\mathcal{B}$). Set $\gamma^* = \gamma_0 \mid [1/2,1]$. Then $\alpha \simeq \gamma^*\beta\gamma^{*-1} \simeq \gamma^*(\gamma_0^{-1}\sigma\gamma_0)^{\mathcal{C}}\gamma^{*-1}$ (in $\mathcal{B}$).

Define $\lambda : I \times I \to C(X)$ by

$$\lambda(s,t) = \begin{cases} \gamma_0(t(1-3s)) & \text{if } 0 \le s \le 1/3, \\ \sigma(3s-1) & \text{if } 1/3 \le s \le 2/3, \\ \gamma_0(t(3s-2)) & \text{if } 2/3 \le s \le 1. \end{cases}$$

Then λ is continuous and $\mu(\lambda(s,t)) \le \mu(\gamma_0(t))$ for every (s,t).

Define $L : I \times I \to B$ by

$$L(s,t) = \begin{cases} \gamma_0((3ts+1)/2) & \text{if } 0 \le s \le 1/3, \\ \cup\{\lambda(u,(t+1)/2) : 3s-1 \le u \le q(s,t)\} & \\ \quad \text{where } q(s,t) \in [3s-1,1] \text{ is chosen} & \\ \quad \text{in such a way that } \mu(L(s,t)) = & \\ \quad \mu(\gamma_0((t+1)/2)) & \text{if } 1/3 \le s \le 2/3 \\ \gamma_0((3t(1-s)+1)/2) & \text{if } 2/3 \le s \le 1. \end{cases}$$

Clearly $L(s,t) \in \mathcal{B}$ for every (s,t) and L is continuous.

Notice that, for each s, $\lambda(s,1/2) = (\gamma_0 \mid [0,1/2])^{-1}\sigma(\gamma_0 \mid [0,1/2])(s) = \gamma^{-1}\sigma\gamma(s)$.

Since

$$L(s,0)=\begin{cases} A_0 & \text{if } 0\le s\le 1/3 \\ & \text{or } 2/3\le s\le 1, \\ \cup\{\gamma^{-1}\sigma\gamma(u): 3s-1\le u\le q(s,0)\}, & \\ \quad\text{where } \mu(L(s,0))= & \\ \quad\mu(\gamma_0(1/2))=T & \text{if } 1/3\le s\le 2/3. \end{cases}$$

Then the map $\zeta_1(s)=L(s,0)$ is such that $\zeta_1\simeq(\gamma^{-1}\sigma\gamma)^A$ (in A).

Notice that, for each s, $\lambda(s,1)=\gamma_0^{-1}\sigma\gamma_0(s)$. Then

$$L(s,1)=\begin{cases} \gamma_0((3s+1)/2) & \text{if } 0\le s\le 1/3, \\ \cup\{\gamma_0^{-1}\sigma\gamma_0(u): 3s-1\le u\le q(s,1)\} & \\ \quad\text{where } \mu(L(s,1))=\mu(\beta(0))=T & \text{if } 1/3\le s\le 2/3, \\ \gamma_0((3(1-s)+1)/2) & \text{if } 2/3\le s\le 1. \end{cases}$$

$$=(\gamma^*(\gamma_0^{-1}\sigma\gamma_0)^{\mathcal{C}}\gamma^{*-1})(s).$$

Therefore, $(\gamma^{-1}\sigma\gamma)^{\mathcal{A}}\simeq\gamma^*(\gamma_0^{-1}\sigma\gamma_0)^{\mathcal{C}}\gamma^{*-1}\simeq\alpha$ (in $\mathcal{B}$). Thus $\alpha\simeq(\gamma^{-1}\sigma\gamma)^{\mathcal{A}}$ (in $\mathcal{B}$). □

2. Theorem A.

Construction 2.1. *Let $\mathcal{A}=\mu^{-1}(T)$ be a Whitney level for $C(X)$. Let $\mathcal{C}$ be a pathwise connected subset of $\mathcal{A}^-$. Choose elements $A_0\in\mathcal{A}$ and $C_0\in\mathcal{C}$ such that $C_0\subset A_0$. Let γ be an order arc from C_0 to A_0. Define $\phi_0:\pi_1(\mathcal{C},C_0)\to\pi_1(\mathcal{A},A_0)$ by $\phi_0([\alpha])=[(\gamma^{-1}\alpha\gamma)^{\mathcal{A}}]$. Let $\mathcal{B}=\mu^{-1}(R,S)$ be a Whitney block for $C(X)$ such that $R<T<S$. Then we consider the function $\phi:\pi_1(\mathcal{C},C_0)\to\pi_1(\mathcal{B},A_0)$ defined by $\phi([\alpha])=[(\gamma^{-1}\alpha\gamma)^{\mathcal{A}}]$.*

Lemma 2.2. *ϕ_0 and ϕ are well defined homomorphisms.*

Proof. It follows from Theorem 1.2. □

Theorem 2.3. *Suppose that X is a Peano continuum. If $\mathcal{C}$ contains a Whitney level $\nu^{-1}(Q)$ and C_0 is a fixed element in $\nu^{-1}(Q)$, then $\phi:\pi_1(\mathcal{C},C_0)\to\pi_1(\mathcal{B},A_0)$ is surjective.*

Proof. Let $[\alpha] \in \pi_1(\mathcal{B}, A_0)$. Fix a point $a \in C_0$ and let γ_1 be an order arc from $\{a\}$ to C_0. From Theorem 1.4, there exists a map $\sigma : I \to X$ such that $\sigma(0) = \sigma(1) = a$ and $\alpha \simeq ((\gamma_1\gamma)^{-1}\sigma\gamma_1\gamma)^{\mathcal{A}}$ (in $\mathcal{B}$). Let $D = \nu^{-1}(Q) \subset \mathcal{C}$. Define $\beta = \gamma_1^{-1}\sigma\gamma_1$, then $\beta(0) = C_0 = \beta(1)$ and $\operatorname{Im}\beta \subset D^-$. We will prove that $\phi([\beta^D]) = [\alpha]$. It is enough to prove that $(\gamma^{-1}\beta^D\gamma)^A(s) = (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$ for every s.

We will assume that $\gamma^{-1}\beta^D\gamma$ and $\gamma^{-1}\beta\gamma$ are parametrized in thirds, that is,

$$(\gamma^{-1}\beta^D\gamma)(s) = \begin{cases} \gamma^{-1}(3s) & \text{if } 0 \le s \le 1/3, \\ \beta^D(3s-1) & \text{if } 1/3 \le s \le 2/3, \\ \gamma(3s-2) & \text{if } 2/3 \le s \le 1. \end{cases}$$

$$(\gamma^{-1}\beta\gamma)(s) = \begin{cases} \gamma^{-1}(3s) & \text{if } 0 \le s \le 1/3, \\ \beta(3s-1) & \text{if } 1/3 \le s \le 2/3, \\ \gamma(3s-2) & \text{if } 2/3 \le s \le 1. \end{cases}$$

Let $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) = \cup\{\gamma^{-1}\beta\gamma(r) : s \le r \le q_1(s)\}$, $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) = \cup\{\gamma^{-1}\beta^D\gamma(r) : s \le r \le q_2(s)\}$, and $\beta^D(s) = \cup\{\beta(r) : s \le r \le q_3(s)\}$.

If $q_1(s) \le q_2(s)$, since $\beta(r) \subset \beta^D(r)$ for every r, then $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$, thus $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$. If $q_2(s) \le 1/3$, then $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) = \cup\{\gamma^{-1}(3r) : s \le r \le q_2(s)\}$ is contained or contains to $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$, so $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$. The case $2/3 \le s$ is similar. Then we may assume that $1/3 < q_2(s) < q_1(s)$ and $s < 2/3$.

If $2/3 \le q_1(s)$, $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) \subset \cup\{\gamma^{-1}\beta^D\gamma(r) : s \le r \le q_1(s)\} = \cup\{\gamma^{-1}\beta\gamma(r) : s \le r \le q_1(s)\} = (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$. Thus, $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$. Hence, we may assume that $q_1(s) < 2/3$.

Define $J = [1/3, 2/3] \cap [s, q_2(s)] \neq \varnothing$. Let $q_4 = \sup\{q_3(3r-1) : r \in J\}$. If $(q_4+1)/3 \le q_1(s)$, let $r \in J \subset [1/3, 2/3]$, so $\gamma^{-1}\beta^D\gamma(r) = \beta^D(3r-1) \subset \cup\{\beta(t) : 3r-1 \le t \le q_4\} \subset \cup\{\beta(t) : 3r-1 \le t \le 3q_1(s) - 1\} = \cup\{\gamma^{-1}\beta\gamma(t) : r \le t \le q_1(s)\} \subset (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$. It follows that $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) \subset (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$. Thus $(\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s)$.

Finally, if $q_1(s) < (q_4+1)/3$, then $3q_1(s) - 1 < q_4$. Thus there exists $r_0 \in J$ such that $3q_1(s) - 1 < q_3(3r_0 - 1)$. Let $r \in [s, q_1(s)]$ be such that $1/3 \le r$. If $r_0 \le r$, then $3r_0 - 1 \le 3r - 1 < q_3(3r_0 - 1)$, so $\gamma^{-1}\beta\gamma(r) = \beta(3r-1) \subset \beta^D(3r_0 - 1) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$.

Thus, $\gamma^{-1}\beta\gamma(r) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$. If $r \leq r_0$, then $s \leq r \leq q_2(s)$, so $\gamma^{-1}\beta\gamma(r) = \beta(3r-1) \subset \beta^D(3r-1) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$. This implies that $\gamma^{-1}\beta\gamma(r) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$ for every $r \in [s, q_1(s)]$. Hence $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) \subset (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$. Thus, $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$.

Therefore, for every $s \in I$, $(\gamma^{-1}\beta\gamma)^{\mathcal{A}}(s) = (\gamma^{-1}\beta^D\gamma)^{\mathcal{A}}(s)$.

Therefore, ϕ is surjective. $\square$

Proof of Theorem A. From Theorem 2.3, it is enough to show that $\mu^{-1}(Q, R)$ is pathwise connected. From [**10**, Theorem 14.8], every Whitney level for $C(X)$ is pathwise connected. Taking a fixed Whitney level contained in $\mu^{-1}(Q, R)$ and, using order arcs, it follows that $\mu^{-1}(Q, R)$ is pathwise connected. $\square$

Theorem B.

Lemma 3.1. *Suppose that X is a Peano continuum. Let $\mathcal{U}$ be a finite nonempty family of subsets of X such that each element in $\mathcal{U}$ is open and connected. Let $F = \{P_U \in U : U \in \mathcal{U}\}$ be a chosen set. Then there exists a finite connected graph $G \subset X$ such that $F \subset G$ and if $U, V \in \mathcal{U}$ and $U \cap V \neq \varnothing$, then there exists a path in $G \cap (U \cup V)$ joining P_U and P_V.*

Proof. If X is an arc, the lemma is immediate. So we may suppose that X is not an arc.

First, we will prove that if W is an open subset of X, $\varepsilon > 0$ and H is a finite connected graph contained in X, then only a finite number of components of $H \cap W$ have diameter larger than ε.

Suppose, on the contrary, that there exists a sequence $\{C_1, C_2, \dots\}$ of pairwise different components of $H \cap W$ such that $\dim C_n > \varepsilon$ for all n. Since H has a finite number of vertices, we may assume that there are no vertices in the set $\cup\{C_n : n \geq 1\}$. Thus each C_n is contained in some segment of H. Since H has a finite number of segments, we may also assume that every C_n is contained in a fixed segment J of H. Then each C_n is a subinterval of J of diameter larger than ε. Since it is not possible, then the assertion is proved.

Now we will prove the following assertion:

($*$) If W is an open subset of X, H is a finite connected graph contained in X and $\alpha : I \to W$ is an arc such that $\alpha(0) \in H$, then there exists a finite connected graph H_1 and there exists an arc $\beta : I \to H_1 \cap W$ such that $H \subset H_1 \subset X$, $\alpha(1) \in H_1$ and β joins $\alpha(0)$ and $\alpha(1)$.

Let Z be an open proper subset of X such that $\operatorname{Im} \alpha \subset Z \subset \operatorname{Cl}_X(Z) \subset W$. Let $\varepsilon = \{\min\{\boldsymbol{d}(a, z) : a \in \operatorname{Im} \alpha \text{ and } z \in X - Z\})/2 > 0$. From the assertion proved above, there exists only a finite number of components $C_1, \dots , Cn$ of $H \cap Z$ such that $\dim C_i \geq \varepsilon$. Suppose that $\alpha(0) \in C_1$.

Let C be a component of $H \cap Z$ such that $C \notin \{C_1, \dots , C_n\}$. Since $\alpha(0) \notin C$, then $C \neq H$, so there exists a point $p \in \operatorname{Bd}_H(C)$. Since H is a Peano continuum, C is an open subset of H, then $p \notin C$. Thus $p \notin Z$. This implies that $C \cap \operatorname{Im} \alpha = \varnothing$. Therefore $\operatorname{Im} \alpha \cap H \cap Z \subset C_1 \cup \dots \cup C_n$.

Let $t_1 = \max \alpha^{-1}(\operatorname{Cl}_X(C_1))$. Then $\alpha(t_1) \in C_1$. Since C_1 is pathwise connected, then $0 = t_1$ or there exists an arc $\beta_1 : I \to C_1$ joining $\alpha(0)$ and $\alpha(t_1)$. If $\alpha(t_1, 1] \cap (C_2 \cup \dots \cup C_n) = \varnothing$, then $\alpha(t_1, 1] \cap H = \varnothing$. In this case, define $H_1 = H \cup \alpha[t_1, 1]$ and let β be the product of the paths: $\beta_1(\alpha \mid [t_1, 1])$ with an injective parametrization.

If $\alpha(t_1, 1] \cap (C_2 \cup \dots \cup C_n) \neq \varnothing$, let $t_2 = \min([t_1, 1] \cap \alpha^{-1}(\operatorname{Cl}_X(C_2 \cup \dots \cup C_n)))$. Since $\alpha(t_1) \notin \operatorname{Cl}_X(C_2 \cup \dots \cup C_n)$, then $t_1 < t_2$. Suppose, by example, that $\alpha(t_2) \in \operatorname{Cl}_X(C_2)$. Then $\alpha(t_2) \in \operatorname{Cl}_X(C_2) \cap H \cap Z = C_2$. Let $t_3 = \max([t_2, 1] \cap \alpha^{-1}(\operatorname{Cl}_X(C_2))$. Since C_2 is pathwise connected, then $t_2 = t_3$ or there exists an arc $\beta_2 : I \to C_2$ joining $\alpha(t_2)$ and $\alpha(t_3)$. If $\alpha(t_3, 1] \cap (C_3 \cup \dots \cup C_n) = \varnothing$, then $\alpha(t_3, 1] \cap H = \varnothing$. In this case define $H_1 = H \cup \alpha[t_1, t_2] \cup \alpha[t_3, 1]$ and $\beta = \beta_1(\alpha \mid [t_1, t_2])\beta_2(\alpha \mid [t_3, 1])$.

In the case $\alpha(t_3, 1] \cap (C_3 \cup \dots \cup C_n) \neq \varnothing$, applying repeatedly a similar procedure, we can conclude the existence of H_1 and β.

Now we are ready to prove inductively the lemma. If $\mathcal{U}$ has only one element, $\mathcal{U} = \{U\}$, then define $H = p_U$. Suppose that the lemma has been proved for families with exactly n elements, and suppose that $\mathcal{U} = \{U_1, \dots , U_{n+1}\}$. Let H be the respective graph for $\mathcal{U}_0 = \{U_1, \dots , U_n\}$. Suppose that $U_1, \dots , U_m$, $0 \leq m \leq n$, are the elements in $\mathcal{U}_0$ which intersect U_{n+1}.

If $m = 0$, that is, if $U_{n+1} \cap (U_1 \cap \dots \cap U_n) = \varnothing$, applying ($*$) to H, to a fixed arc $\alpha : I \to W = X$ joining p_{U_1} and $p_{U_{n+1}}$ with $\alpha(0) = p_{U_1}$, then there exists a finite connected graph H_1 such that $H \subset H_1 \subset X$ and $p_{U_{n+1}} \in H_1$. Then $G = H_1$ satisfies the required properties. Then we may assume that $m \geq 1$.

If $p_{U_1} \neq p_{U_{n+1}}$, applying $(*)$ to H and to a fixed arc $\alpha_1 : I \to U_1 \cup U_{n+1}$ joining p_{U_1} and $p_{U_{n+1}}$ with $\alpha_1(0) = p_{U_1}$, we have that there exists a finite connected graph H_1 and there exists an arc $\beta_1 : I \to H_1 \cap (U_1 \cup U_{n+1})$ such that $H \subset H_1 \subset X$, $p_{U_{n+1}} \in H_1$ and β_1 joins p_{U_1} and $p_{U_{n+1}}$. If $p_{U_1} = p_{U_{n+1}}$, define $H_1 = H$ and $\beta_1 : I \to H_1 \cap (U_1 \cup U_{n+1})$ by $\beta_1(t) = p_{U_1}$ for every t.

If $m \geq 2$ and $p_{U_2} \neq p_{U_{n+1}}$, again applying $(*)$ to H_1 and to a fixed arc $\alpha_2 : I \to U_2 \cup U_{n+1}$ joining p_{U_2} and $p_{U_{n+1}}$, with $\alpha_2(0) = p_{U_2}$, we have that there exists a finite connected graph H_2 and there exists an arc $\beta_2 : I \to H_2 \cap (U_2 \cup U_{n+1})$ such that $H_1 \subset H_2 \subset X$ and β_2 joins p_{U_2} and $p_{U_{n+1}}$. If $p_{U_2} = p_{U_{n+1}}$, define $H_2 = H_1$ and define $\beta_2 : I \to H_1 \cap (U_2 \cup U_{n+1})$ by $\beta_2(t) = p_{U_2}$ for every t.

Proceeding in this way, it is possible to construct $H \in H_1 \subset H_2 \subset \cdots \subset H_m$ and paths $\beta_1, \dots, \beta_m$ such that $\beta_i : I \to H_i \cap (U_i \cap U_{n+1})$ and β_i joins p_{U_i} and $p_{U_{n+1}}$. Therefore, $G = H_m$ satisfies the required properties.

This completes the induction and the proof of the lemma. $\square$

Lemma 3.2. *Suppose that X is a Peano continuum. Let $\mathcal{A} = \mu^{-1}(T)$ be a Whitney level with $0 < T$, and let p be a point in X. Then there exists a finite connected graph $G \subset X$ such that $p \in G$ and, for each loop α in the pointed space (X, p), there exists a loop β in the pointed space (G, p) such that $\alpha \simeq \beta$ (in $\mathcal{A}^-$).*

Proof. Suppose that the metric $\boldsymbol{d}$ is a convex metric. Let $\varepsilon > 0$ be such that if $A, B \in C(X)$ and $\mathcal{H}(A, B) < 2\varepsilon$, then $|\mu(A) - \mu(B)| < T$. Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of X such that each U_i is connected and $\operatorname{diam} U_i < \varepsilon$. Suppose that $p \in U_1$. Choose points $p_1 = p \in U_1, p_2 \in U_2, \dots, p_n \in U_n$. Let $G \subset X$ be a finite connected graph as in Lemma 3.1 for the family $\mathcal{U}$ and the set $F = \{p_1, \dots, p_n\}$.

Let α be a loop in (X, p). From the Lebesgue number theorem, there exists a $\delta > 0$ such that if A is a subset of I and $\operatorname{diam} A < \delta$, then $A \subset \alpha^{-1}(U_i)$ for some i. Choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$

of I such that $t_k - t_{k-1} < \delta$ for each $k \in \{1, \dots, m\}$. For each $k \in \{1, \dots, m\}$, choose a number $s_k \in (t_{k-1}, t_k)$ and choose an index $i(k) \in \{1, \dots, n\}$ such that $[t_{k-1}, t_k] \subset \alpha^{-1}(U_{i(k)})$. Since $U_{i(k)}$ is pathwise connected, there exists a path $\gamma_k : I \to U_{i(k)}$ such that $\gamma_k(0) = \alpha(s_k)$ and $\gamma_k(1) = p_{i(k)}$.

If $1 \le k < m$, $\alpha(t_k) \in U_{i(k)} \cap U_{i(k+1)}$, then there exists a path $\beta_k : I \to (U_{i(k)} \cup U_{i(k+1)}) \cap G$ such that $\beta_k(0) = p_{i(k)}$ and $\beta_k(1) = p_{i(k+1)}$. Since $p = \alpha(0) \in U_1 \cap U_{i(1)}$ and $p = \alpha(1) \in U_{i(m)} \cap U_1$, there exist paths $\beta_0 : I \to (U_1 \cup U_{i(1)}) \cap G$ and $\beta_m : I \to (U_{i(m)} \cup U_1) \cap G$ such that $\beta_0 = p$, $\beta_0(1) = p_{i(1)}$, $\beta_m(0) = p_{i(m)}$ and $\beta_m(1) = p$.

Defining $\beta = \beta_0\beta_1\beta_2 \cdots \beta_{m-1}\beta_m$, then β is a loop in (G, p).

If $1 \le k < m$, the set $C_k = \operatorname{Im}\beta_k \cup \operatorname{Im}(\gamma_k^{-1}(\alpha \mid [s_k, s_{k+1}])\gamma_{k+1})$ is contained in $U_{i(k)} \cup U_{i(k+1)}$, then $\dim C_k < 2\varepsilon$, this implies that $C_k \subset \mathcal{A}^-$. Then, from Lemma 1.1, $\beta_k \simeq \gamma_k^{-1}(\alpha \mid [s_k, s_{k+1}])\gamma_{k+1}$ (in $\mathcal{A}^-$). Similarly, $\beta_0 \simeq (\alpha \mid [0, s_1])\gamma_1$ (in $\mathcal{A}^-$) and $\beta_m \simeq \gamma_m^{-1}(\alpha \mid [s_m, 1])$ (in $\mathcal{A}^-$).

Therefore, $\alpha \simeq (\alpha \mid [0, s_1])\gamma_1)(\gamma_1^{-1}(\alpha \mid [s_1, s_2])\gamma_2) \cdots (\gamma_{m-1}^{-1}(\alpha \mid [s_{m-1}, s_m])\gamma_m)(\gamma_m^{-1}(\alpha \mid [s_m, 1])) \simeq \beta$ (in $\mathcal{A}^-$). $\square$

Proof of Theorem B. Fix an element $A_0 \in \mu^{-1}(S, T)$. Let $R = \mu(A_0)$ and $\mathcal{A} = \mu^{-1}(R)$. Let $\mathcal{C} = F_1(X)$ and fix a point $p \in A_0$. Let γ be an order arc from p to A_0. Consider $\phi : \pi_1(\mathcal{C}, \{p\}) \to \pi_1(\mathcal{B}, A_0)$ defined by $\phi([\alpha]) = [(\gamma^{-1}\alpha\gamma)^{\mathcal{A}}]$. From Theorem 2.3, ϕ is surjective.

Let G be as in Lemma 3.2. Consider the natural homomorphism $\psi : \pi_1(G, p) \to \pi_1(\mathcal{C}, p)$, that is, $\psi([\beta]) = [\beta]$. Then $\phi \circ \psi : \pi_1(G, p) \to \pi_1(\mathcal{B}, A_0)$ is a homomorphism. In order to show that $\phi \circ \psi$ is surjective, let $[\sigma] \in \pi_1(\mathcal{B}, A_0)$; then there exists $[\alpha] \in \pi_1(\mathcal{C}, \{p\})$ such that $\sigma \simeq (\gamma^{-1}\alpha\gamma)^{\mathcal{A}}$ (in $\mathcal{B}$). From Lemma 3.2 there exists a loop β in (G, p) such that $\beta \simeq \alpha$ (in $\mathcal{A}^-$). Then $\phi \circ \psi([\beta]) = [(\gamma^{-1}\beta\gamma)^{\mathcal{A}}] = [\sigma]$. Hence $\phi \circ \psi$ is surjective.

Since $\pi_2(G, p)$ is finitely generated (see [**8**, Theorem 5.2]), we conclude that $\pi_1(\mathcal{B}, A_0)$ is finitely generated. $\square$

4. Theorem C.

Proof of Theorem C. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are immediate. In order to prove that (c) $\Rightarrow$ (a), let $T_0 = 1/2$. Fix a point $p_0 \in X$. Let G be a finite connected graph contained in X as in Lemma 3.2 for $\mathcal{A}_0 = \mu^{-1}(T_0)$ and the point p_0. Define $T^* = \max\{\mu(H) : H$ is a proper subgraph of $G\}$. Then $0 \leq T^* < \mu(G) \leq 1$. Then there exists T_1 such that T_0, $T^* < T_1 < 1$. If $G \neq X$, then we may ask that $\mu(G) < T_1$. From the hypothesis, there exists $T_1 < R < S < 1$ such that $\pi_1(\mu^{-1}(R,S))$ is a nontrivial group. Choose $T \in (R,S)$, let $\mathcal{A} = \mu^{-1}(T)$ and $\mathcal{B} = \mu^{-1}(R,S)$.

Suppose that X is not a simple closed curve.

Choose a maximal tree L of G. If $L = G$, choose a vertex $p \in L$. If $L \neq G$, let J be a segment of G such that J is not a segment of L; in this case, choose a vertex p of J, then $p \in L$. Clearly, for each loop α in (X,p), there exists a loop β in (G,p) such that $\alpha \simeq \beta$ (in $\mathcal{A}^-$).

Choose $A \in \mathcal{A}$ such that $p \in A$. Let $\mathcal{C} = F_1(X)$. Let γ be an order arc from $\{p\}$ to A. Consider the homomorphisms $\phi : \pi_1(\mathcal{C},p) \to \pi_1(\mathcal{B},A)$ and $\psi : \pi_1(G,p) \to \pi_1(\mathcal{C},p)$ defined by $\phi([\alpha]) = [(\gamma^{-1}\alpha\gamma)^{\mathcal{A}}]$ and $\psi([\alpha]) = [\alpha]$. Reasoning as in the proof of Theorem B, we have that $\phi \circ \psi$ is surjective. We will obtain a contradiction by proving that $\phi \circ \psi$ is constant.

If $G \neq X$, then for every $[\alpha] \in \pi_1(G,p)$, $\mu(\cup\{\alpha(r) : r \in I\}) \leq \mu(G) < T$. Then, from Lemma 1.1, $\alpha \simeq$ the constant map p. Thus, $(\gamma^{-1}\alpha\gamma)^{\mathcal{A}} \simeq$ the constant map A_0. Hence $\phi \circ \psi$ is constant. So we may assume that $G = X$.

Let M be a segment of G with extremes a and b such that M is not a segment of L. Let α_M be the path $\sigma_1\delta\sigma_2^{-1}$, where σ_1, respectively σ_2, is a parametrization of the unique arc in L joining p and a, respectively p and b, notice that σ_i can be a constant map, and δ is a parametrization of the segment M.

From [**8**, Theorem 5.2], the set $\mathcal{L} = \{\alpha_M : M$ is a segment of G and M is not a segment of $L\}$ generates the group $\pi_1(G,p)$.

Let $\alpha_M \in S$; if $M = J$, we may assume that $a = p$. Then σ_1 is a constant map, and this implies that $\operatorname{Im}\alpha_M$ is a simple closed curve. Since X is not a simple closed curve, $\operatorname{Im}\alpha_M$ is a proper subgraph of G. Hence, $\mu(\operatorname{Im}\alpha_M) < T$. From Lemma 1.1, $\alpha_M \simeq$ the constant map p (in A^-). Therefore, $\phi \circ \psi([\alpha])$ is the unit element in $\pi_1(\mathcal{B},A)$.

If $M \neq J$, then $\operatorname{Im}\alpha_M$ is a subgraph of G and J is not a segment of

$\operatorname{Im}\alpha_M$. Thus $\operatorname{Im}\alpha_M$ is a proper subgraph of G. Reasoning as in the paragraph above, $\phi \circ \psi([\alpha])$ is the unit element in $\pi_1(\mathcal{B}, A)$.

Therefore, $\phi \circ \psi$ is a surjective constant map. This contradiction completes the proof of the theorem. $\square$

5. Examples.

Example 5.1. Theorem A and implication (c) $\Rightarrow$ (a) in Theorem C do not hold without the hypothesis of local connectedness.

Consider X to be the Warzaw circle in the Euclidean plane. It is easy to check that if μ is a Whitney map for $C(X)$, then:

a) If $0 \leq R < S \leq 1$ are small numbers, then the Whitney block $\mu^{-1}(R, S)$ is homeomorphic to the open cylinder $X \times (0, 1)$, so $\pi_1(\mu^{-1}(R, S))$ consists only of the unit element, and

(b) If $0 \leq R < S \leq 1$ are large numbers, then the Whitney block $\mu^{-1}(R, S)$ is homeomorphic to the open cylinder $S^1 \times (0, 1)$, where S^1 is a simple closed curve, so $\pi_1(\mu^{-1}(R, S))$ is isomorphic to the integers.

Example 5.2. Theorem B does not hold without the hypothesis of local connectedness.

If p and q are two points in the Euclidean plane, let pq denote the segment joining them.

Define $X = (0, -1)(0, 1) \cup (\cup\{(1/n, -1)(1/n, 1) : n \geq 1\}) \cup (0, 1)(1, 1) \cup (0, -1)(1, -1)$. It is easy to prove that if $0 \leq R < S \leq 1$ are small numbers then the Whitney block $\mu^{-1}(R, S)$ has the homotopy type of X, and the group $\pi_1(\mu^{-1}(R, S))$ is not finitely generated.

6. Remarks and questions.

Question 6.1. Theorems A, B and C were inspired in Theorems A, B and D in the author's paper [**4**]. In that paper, the following result was also proved:

Let X be a Peano continuum, for a connected space Y, let $\boldsymbol{r}(Y)$ denote the multicoherence degree of Y. If $0 < m \leq r(X)$, then there exists a Whitney map $\boldsymbol{v} : C(x) \to [0,1]$ and there exists $t \in [0,1]$ such that $r(\boldsymbol{v}^{-1}(t)) = m$.

What could be an appropriate version of this result for fundamental group and Whitney blocks?

Question 6.2. Are Theorems A, B and C true for Whitney levels instead of Whitney blocks?

Question 6.3. Is the implication (b) $\Rightarrow$ (a) true for every pathwise connected continuum X (instead of Peano continuum)?

Question 6.4. Is the following greater dimensional version of implication (c) $\Rightarrow$ (a) in Theorem C true:

Let X be a Peano continuum and let μ be a Whitney map for $C(X)$. If for each $R < 1$, there exist $R < S < T \leq 1$ and there exists an integer $n \geq 1$ such that $\pi_n(\mu^{-1}(S,T))$ is a non-trivial group, then X is a finite connected graph?

Question 6.5. Characterize the finite connected graphs satisfying the assertion in Question 6.4 (compare with Question 3.5 in [**5**]).

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