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THE EXTERIOR PROBLEM IN THE PLANE FOR A NON-NEWTONIAN INCOMPRESSIBLE BIPOLAR VISCOUS FLUID

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0. Introduction. The problem of steady flow of an incompressible fluid past a fixed body Ω' in R^2, as modeled by the Navier-Stokes system, has been extensively studied but is still mostly unresolved.

Using the Navier-Stokes equations to model this flow gives rise to the system:

$$\Delta \mathbf{v} - \mathbf{v}\nabla\mathbf{v} - \nabla p = 0 \tag{0.1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{0.2}$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega' \tag{0.3}$$

$$\lim_{|x|\to\infty} \mathbf{v}(x) = \mathbf{v}_\infty \tag{0.4}$$

Léray has proved in 1933 (see [**12**]) the existence of a function which satisfies in a weak sense (0.1), (0.2) and (0.3). However, it is not known yet whether or not the weak solution constructed by Léray satisfies the boundary condition (0.4) (see [**9**] for example). R. Finn and D. Smith in [**3**] have resolved the situation under the additional condition that $|\mathbf{v}_\infty|$ be small enough. However, to this day and in spite of much talent and effort devoted to it, the question of existence of a solution to problem (0.1)–(0.4) is still open. For a more complete bibliography as well as a discussion of the different results, see the review paper by J. Heywood [**9**] and the references therein.

The Navier-Stokes model of fluid flow is based on the Stokes-hypothesis, which simplifies and restricts the relation between the stress tensor and the velocity. By relaxing the constraints of the Stokes hypothesis, the mathematical theory of multipolar viscous fluids generalizes the usual Navier-Stokes model in three important respects: it allows for nonlinear constitutive relations between the viscous part of

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the stress tensor and velocity gradients; it allows for a dependence of the viscous stress on velocity gradients of order two or higher; and it introduces constitutive relations for higher order stress tensors (moments of stress), which must be present in the balance of energy equation as soon as higher order velocity gradients are admitted into the theory.

This constitutive theory was shown to be compatible with all known thermodynamical processes and the frame indifference principle (see [**2, 5, 6, 11, 16, 20**] and the references therein).

In [**2**] we considered the plane Poiseuille flow within the framework of this model, and it was found that the velocity distribution profile predicted by the model was in agreement with the experimental results, see [**18**, p. 599]. One of the directions in which we are continuing the work of [**2**] is to consider some of the classical flow problems within the framework of the multipolar non-Newtonian theory.

Here we consider, within this framework, the classical problem of stationary flow past a body in the plane. We show that this problem has a solution. We also show that this model predicts the existence of a drag on the immersed body and thus does not suffer from the D'Alembert paradox.

We will, in subsequent work, study properties of this solution and investigate the questions of wake and stream lines, etc.

Let Ω' be a simply connected bounded domain of $\mathbf{R}^2$ with smooth boundary Γ. Set $\Omega = \mathbf{R}^2 \backslash \overline{\Omega}'$. We are interested in the flow of a fluid around the body Ω'. Within the framework of bipolar non-Newtonian fluids, the motion is then described by the equations

$$L(\mathbf{v}) \equiv \mu_1 \frac{\partial}{\partial x_j} \Delta e_{ij} - \frac{\partial}{\partial x_j}(\gamma(\mathbf{v}) e_{ij}) + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} \quad \text{in } \Omega \tag{0.5}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \tag{0.6}$$

$$\mathbf{v} = 0 \quad \text{on } \Gamma \tag{0.7}$$

$$\left(\frac{\partial}{\partial x_k} e_{ij}\right) \nu_j \nu_k \tau_i = 0 \quad \text{on } \Gamma \tag{0.8}$$

$$\lim_{|x| \to \infty} \mathbf{v}(x) = \mathbf{v}_\infty \equiv \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \tag{0.9}$$

$$\lim_{|x| \to \infty} D^2 \mathbf{v}(x) = 0 \tag{0.10}$$

where $\mathbf{v}$ is the velocity vector field, p is the pressure, $\boldsymbol{\nu}$ is the interior unit normal vector to Ω', $\boldsymbol{\tau}$ is the unit tangent vector to $\partial\Omega'$ and e_{ij} are the components of the rate of deformation tensor defined by

$$e_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{0.11}$$

Remark 1. We recall that the theory of bipolar fluid flows postulates the existence of a new higher order stress tensor $\tau_{ijk} = \mu_1(\partial e_{ij}/\partial x_k)$, which expresses a momentum of the fluid (for details, see [**5, 6, 16**] and [**2**]). The classical stress tensor is, in our case,

$$\tau_{ij} = -p\delta_{ij} + \gamma(\mathbf{v})e_{ij} - \mu_1 \Delta e_{ij}$$

and we get for $\partial\Omega, \mathbf{u}, \mathbf{v}$ smooth enough, satisfying the nonslip condition (0.7) and the incompressibility condition (0.6):

$$\begin{aligned} \int_\Omega \frac{\partial \tau_{ij}}{\partial x_j} u_i \, dx = -\mu_1 \int_\Omega \frac{e_{ij}(\mathbf{v})}{\partial x_k} \frac{e_{ij}(\mathbf{u})}{\partial x_k} \, dx \\ - \int_\Omega \gamma(\mathbf{v}) e_{ij}(\mathbf{v}) e_{ij}(\mathbf{u}) \, dx \\ + \int_{\partial\Omega} \tau_{ijk}(\mathbf{v}) \nu_j \nu_k \frac{\partial u_i}{\partial \boldsymbol{\nu}} \, dS. \end{aligned}$$

The boundary condition (0.8) follows from the assumption that the above surface integral vanishes for all $\mathbf{u}$ satisfying (0.6) and (0.7).

The function γ is taken to be

$$\gamma(\mathbf{v}) = \mu_0 (e_{ij}(\mathbf{v}) \cdot e_{ij}(\mathbf{v}))^{-\alpha/2} \tag{0.12}$$

sand μ_0, μ_1, λ are positive constants and $\alpha \in (0,1)$.

It should be noted that if we set $\alpha = \mu_1 = 0$ in (0.5) and $\mu_0 = 2$ in (0.12), then (0.5) reduces to the usual Navier-Stokes equations (0.1).

We will use the notations

$$\begin{aligned} ||D^2\mathbf{u}||_{L^2}^2 \equiv \sum_{i,j,k} \iint \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)^2 dx \\ ||D\mathbf{u}||_{L^s}^s \equiv \sum_{i,j} \iint \left|\frac{\partial u_i}{\partial x_j}\right|^s dx. \end{aligned} \tag{0.13}$$

Elementary algebraic estimates yield that there exists a constant c such that for all $\mathbf{u}$

$$\frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}) \geq c \sum_{i,j,k} \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} \right)^2, \tag{0.14}$$

$$\gamma(\mathbf{u}) e_{ij}(\mathbf{u}) e_{ij}(\mathbf{u}) \geq c\mu_0 \sum_{i,j} |e_{ij}(\mathbf{u})|^{2-\alpha}. \tag{0.15}$$

Next we introduce some functional spaces

$$\begin{aligned} H_\alpha(\Omega) &= \{\mathbf{u} : \mathbf{u} = 0 \text{ on } \Gamma, \operatorname{div} \mathbf{u} = 0, \nabla \mathbf{u} \in L^{2-\alpha}(\Omega)\} \\ V &= \{\mathbf{u}; \mathbf{u} \in H_\alpha; D^2\mathbf{u} \in L^2(\Omega)\} \\ V_b &= \{\mathbf{u} \in V;\ \text{such that supp } \mathbf{u} \text{ is bounded}\}. \end{aligned}$$

We will need the following Korn inequality.

Lemma 1.1. *There exists $c > 0$ such that for all $\mathbf{u} \in V_b$,*

$$\iint \sum_{i,j} |e_{ij}(\mathbf{u})|^{2-\alpha}\, dx \geq c||D\mathbf{u}||^{2-\alpha}_{L^{2-\alpha}}. \tag{0.16}$$

For a proof see [**10**], for example.

We wish to transform the problem (0.5)–(0.10) into an equivalent problem with homogeneous boundary conditions. For this purpose we recall (see [**8**], for example) that there exists a function $\mathbf{w}(x)$ defined in Ω and such that

$$\mathbf{w}(x) \equiv \begin{pmatrix} -\lambda \\ 0 \end{pmatrix} \quad \text{for } |x| < M_2, \tag{0.17}$$

$$\mathbf{w}(x) \equiv 0 \quad \text{for } |x| > M_1, \tag{0.18}$$

$$\operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega \tag{0.19}$$

and $\mathbf{w} \in C^\infty(\Omega)$.

Furthermore (see [**13**, p. 103], for example), for all $\varepsilon > 0$ we can choose $\mathbf{w}$ such that for all $\mathbf{u} \in V$

$$\sum_{i,j} \iint_{\Omega_{M_1}} (w_j, u_i)^2\, dx \leq \varepsilon ||D\mathbf{u}||^2_{L^2(\Omega_{M_1})}, \tag{0.20}$$

where $\Omega_{M_1} = \Omega \cap B(0, M_1)$.

Setting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty - \mathbf{w}$ we then have that $\mathbf{u}$ satisfies:

$$L(\mathbf{u} + \mathbf{v}_\infty + \mathbf{w}) = -\nabla p \quad \text{in } \Omega \tag{0.21}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma \tag{0.22}$$

$$\left(\frac{\partial}{\partial x_k} e_{ij}\right) \nu_j \nu_k \tau_i = 0 \quad \text{on } \Gamma \tag{0.23}$$

$$\lim_{|x|\to\infty} \mathbf{u} = 0 \tag{0.24}$$

$$\lim_{|x|\to\infty} D^2 \mathbf{u} = 0 \tag{0.25}$$

$$\operatorname{div} u = 0. \tag{0.26}$$

The precise meaning of the equalities above will be given later.

Definition 1.1. A function $\mathbf{u}$ is said to be a *weak solution* of (0.21)–(0.25) if $\mathbf{u} \in V$, $\mathbf{u}$ satisfies (0.23)–(0.25) and for all $\mathbf{v} \in V_b$

$$\begin{aligned} \iint_\Omega \gamma(\mathbf{u} + \mathbf{w}) e_{ij}(\mathbf{u} + \mathbf{w}) e_{ij}(\mathbf{v})\, dx \\ + \mu_1 \iint_\Omega \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u} + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v})\, dx \\ + \iint_\Omega (\mathbf{u} + \mathbf{v}_\infty + \mathbf{w})_j \frac{\partial (\mathbf{u} + \mathbf{w})_i}{\partial x_j} \cdot v_i \, dx = 0 \end{aligned} \tag{0.27}$$

In order to prove the existence of a solution, we will first show the existence of a solution $\mathbf{u}^N$ in a truncated domain $\Omega_N = \Omega \cap B(0, N)$, then let N go to infinity and prove that the limiting function $\mathbf{u}$ is a weak solution.

1. Solution in the truncated domain. Let $B(0, N) = \{x \in \mathbf{R}^2; |x| < N\}$. We will assume that N is large enough so that $N > M_1 > M_2 >$ diameter (Ω'), and set $\Omega_N = \Omega \cap B(0, N)$. We intend to show the existence of a function $\mathbf{u}^N$ which satisfies

$$\mathbf{u}^N \in W^{2,2}(\Omega_N) \cap W_0^{1,2}(\Omega_N); \quad \operatorname{div} \mathbf{u}^N = 0 \tag{1.1}$$

$$\left(\frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N)\right) \nu_j \nu_k \tau_i = 0 \quad \text{on } \partial\Omega_N \tag{1.2}$$

and such that for all $\mathbf{v} \in W^{2,2}(\Omega_N) \cap W_0^{1,2}(\Omega_N)$ $\operatorname{div} \mathbf{v} = 0$;

$$\begin{aligned}(1.3)\quad &\iint_{\Omega_N} \gamma(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty) e_{ij}(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty) e_{ij}(\mathbf{v})\, dx \\ &\quad + \mu_1 \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v})\, dx \\ &\quad + \iint_{\Omega_N} (\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty)_j \frac{\partial (\mathbf{u}^N + \mathbf{w})_i}{\partial x_j} \cdot v_i\, dx = 0.\end{aligned}$$

We start by establishing an apriori estimate.

Theorem 1.2. *There exists a constant C independent of N such that any solution $\mathbf{u}^N$ of* (1.3) *satisfies*:

$$(1.4)\qquad \iint_{\Omega_N} (D^2 \mathbf{u}^N)^2\, dx + \iint_{\Omega_N} (D\mathbf{u}^N)^{2-\alpha}\, dx \le C.$$

Proof. Setting $\mathbf{v} = \mathbf{u}^N$ in (1.3), it follows that

$$\begin{aligned}(1.5)\quad &\iint \gamma(\mathbf{u}^N + \mathbf{w}) e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot e_{ij}(\mathbf{u}^N + \mathbf{w})\, dx \\ &\quad + \mu_1 \iint \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} (e_{ij}(\mathbf{u}^N + \mathbf{w}))\, dx \\ &\le \left| \iint \gamma(\mathbf{u}^N + \mathbf{w}) e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot e_{ij}(\mathbf{w})\, dx \right| \\ &\quad + \mu_1 \left| \iint \frac{\partial}{\partial x_k} e_k(\mathbf{u}^N + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{w})\, dx \right| \\ &\quad + \left| \iint (\mathbf{u}^N + \mathbf{v}_\infty + \mathbf{w})_j \frac{\partial (\mathbf{u}^N + \mathbf{w})_i}{\partial x_j} u_i^N\, dx \right|.\end{aligned}$$

Since (while the summation convention over repeated indices is used throughout the paper, it is not used in the next expression)

$$\begin{aligned}\left| \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{w}) \right| &\le \frac{\delta}{2} \left(\frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w}) \right)^2 \\ &\quad + \frac{1}{2\delta} \left(\frac{\partial}{\partial x_k} e_{ij}(\mathbf{w}) \right)^2,\end{aligned}$$

and

$$\left| \iint \gamma(\mathbf{u}^N + \mathbf{w}) e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot \varepsilon_{ij}(\mathbf{w})\, dx \right| \\ \leq \sum_{ij} \left(c_1 \delta ||e_{ij}(\mathbf{u}^N + \mathbf{w})||^{2-\alpha}_{L^{2-\alpha}} + c_1 \delta^{\alpha-1} ||e_{ij}(\mathbf{w})||^{2-\alpha}_{L^{2-\alpha}} \right)$$

we deduce from (1.5), (0.14), (0.15) and Lemma 1.1 that, for some positive constants c_1 and c_2,

$$\begin{aligned} c_2 \iint_{\Omega_N} (D^2(\mathbf{u}^N + \mathbf{w}))^2\, dx &+ c_2 \iint_{\Omega_N} (D(\mathbf{u}^N + \mathbf{w}))^{2-\alpha}\, dx \\ &\leq (1 - \delta c_1) \iint \gamma(\mathbf{u}^N + \mathbf{w}) e_{ij}(\mathbf{u}^N + \mathbf{w}) e_{ij}(\mathbf{u}^N + \mathbf{w})\, dx \\ &\quad + \mu_1 \left(1 - \frac{\delta}{2}\right) \iint \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N + \mathbf{w})\, dx \\ &\leq \mu_1 \frac{1}{2\delta} \sum_{ij} \left\| \frac{\partial}{\partial x_k} e_{ij}(\mathbf{w}) \right\|^2_{L^2} + c_1 \delta^{\alpha-1} \sum_{ij} ||e_{ij}(\mathbf{w})||^{2-\alpha}_{L^{2-\alpha}} \\ &\quad + |b(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty, \mathbf{u}^N + \mathbf{w}, \mathbf{u}^N)| \end{aligned}$$

where

$$b(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \iint f_j \frac{\partial g_i}{\partial x_j} h_i\, dx.$$

Integration by parts yields that $b(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty, \mathbf{u}^N, \mathbf{u}^N) = 0$, hence

$$b(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty, \mathbf{u}^N + \mathbf{w}, \mathbf{u}^N) \\ = b(\mathbf{w} + \mathbf{v}_\infty, \mathbf{w}, \mathbf{u}^N) + b(\mathbf{u}^N, \mathbf{w}, \mathbf{u}^N).$$

Since $\partial w_i / \partial x_j$ has a fixed support (independent of N) we have that:

$$\begin{aligned} (1.6) \quad |b(\mathbf{w} + \mathbf{v}_\infty, \mathbf{w}, \mathbf{u}^N)| &\leq \delta ||\mathbf{u}^N||^2_{L^\infty(\Omega_{M_2})} + \frac{1}{\delta} ||\mathbf{w} + \mathbf{v}_\infty||^2_{L^\infty} \left\| \frac{\partial w_i}{\partial x_j} \right\|^2_{L^2} \\ &\leq c_3 \delta ||D^2 \mathbf{u}^N||^2_{L^2(\Omega_{M_2})} + \frac{1}{\delta} ||\mathbf{w} + \mathbf{v}_\infty||^2_{L^\infty} \left\| \frac{\partial w_i}{\partial x_j} \right\|^2_{L^2}, \end{aligned}$$

where c_3 depends on Γ, M_2 but not on N.

From (0.20), we deduce that

$$|b(\mathbf{u}^N, \mathbf{w}, \mathbf{u}^N)| \leq \varepsilon ||D\mathbf{u}^N||^2_{L^2(\Omega_{M_2})} \leq \varepsilon c_7 ||D^2\mathbf{u}^N||^2_{L^2(\Omega_{M_2})} \tag{1.7}$$

where c_7 is independent of N. Choosing ε and δ small enough, it then follows from (1.5) that there exists $C(\varepsilon, \delta, \mu_0, \mu_1)$, independent of $N, \mathbf{u}^N$ such that

$$||D^2\mathbf{u}^N||^2_{L^2(\Omega_N)} + ||D\mathbf{u}^N||^{2-\alpha}_{L^{2-\alpha}(\Omega_N)} \leq c. \qquad \square$$

We will use the Galerkin method to prove the existence of a solution u^N. In order to introduce the needed basis we define the space

$$\mathcal{H} = \{\mathbf{u} : \mathbf{u} \in W^{2,2}(\Omega_N) \cap W_0^{1,2}(\Omega_N); \operatorname{div} \mathbf{u} = 0\}. \tag{1.8}$$

In $\mathcal{H}$ the scalar product is taken to be

$$((w, \boldsymbol{\psi})) = \int_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(w) \cdot \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{\psi})\, dx.$$

We will denote by $(w, \boldsymbol{\psi})$ the usual L^2 scalar product.

Lemma 1.3. *The eigenvalue problem*

$$((w, \boldsymbol{\psi})) = \lambda(w, \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in \mathcal{H} \tag{1.9}$$

has a sequence of solutions $\mathcal{W}^l \in \mathcal{H} \cap C^\infty(\Omega_N)$ *corresponding to a sequence of positive eigenvalues* λ_l. *Furthermore,*

1. $((\partial/\partial x_k) e_{ij}(\mathcal{W}^l))\nu_j \nu_k \tau_i = 0$ *for all* l *on* $\partial\Omega_N$;
2. *The sequence* $\mathcal{W}^l$ *is a basis for the closure of* $\mathcal{H}$ *under the* L^2 *norm*;
3. *The sequence* $\mathcal{W}^l$ *is a basis of* $\mathcal{H}$;
4. $(\mathcal{W}^l, \mathcal{W}^k) = \delta_{lk}$.

Proof. This is a standard consequence of the estimate (0.14). $\square$

Now, for K fixed, let $u^{N,K} \in E_K = \mathrm{Span}\,\{\mathcal{W}^1 \cdots \mathcal{W}^K\}$

$$u^{N,K}(x) = \sum_{l=1}^{K} A_l \mathcal{W}^l(x)$$

be the solution of

$$\begin{aligned}(1.10)\qquad &\iint_{\Omega_N} \gamma(\mathbf{u}^{N,K} + \mathbf{w} + \mathbf{v}_\infty) e_{ij}(\mathbf{u}^{N,K} + \mathbf{w} + \mathbf{v}_\infty) e_{ij}(\mathbf{v})\, dx \\ &+ \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^{N,K} + \mathbf{w} + \mathbf{v}_\infty) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v})\, dx \\ &+ \iint_{\Omega_N} (\mathbf{u}^{N,K} + \mathbf{w} + \mathbf{v}_\infty)_j \frac{\partial(\mathbf{u}^{N,K} + \mathbf{w})_i}{\partial x_j} \cdot v_i\, dx = 0\end{aligned}$$

for all $\mathbf{w} \in E_K$.

Lemma 1.4. *The problem* (1.10) *has a solution* $\mathbf{u}^{N,K}$.

Proof. The existence of the solution $\mathbf{u}^{N,K}$ can be established using Lemma 1.4, page 164 of [**19**], which we recall here for convenience:

Lemma 1.5 (see [**19**]). *Let X be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$, and let P be a continuous mapping from X into itself such that*

$$(1.11)\qquad [P(\xi), \xi] > 0 \quad \textit{for } [\xi] = A > 0.$$

Then there exists $\xi \in X$, $[\xi] \le A$, such that $P(\xi) = 0$.

We set $X = E_K$ and define P by

$$\begin{aligned}[P(\mathbf{u}), \mathbf{v}] = \mu_1 &\iint \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u} + \mathbf{w}) \cdot \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v})\, dx \\ &+ \iint \gamma(\mathbf{u} + \mathbf{w}) e_{ij}(\mathbf{u} + \mathbf{w}) e_{ij}(\mathbf{v})\, dx \\ &+ b(\mathbf{u} + \mathbf{w} + \mathbf{v}_\infty, \mathbf{u} + \mathbf{w}, \mathbf{v})\end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in E_K$. The continuity of P is clear; also, proceeding as in the proof of the a priori estimate, it can be shown that there exists c (independent of K and N) such that if $[P(\mathbf{u}), \mathbf{u}] \le 0$, then $||\mathbf{u}|| \le c$. Therefore, if $A > c$, condition (1.11) of the lemma is satisfied. Hence, there exists a solution $\mathbf{u}^{N,K}$ to the problem (1.10). $\square$

Since (1.10) holds for $\mathbf{v} = \mathbf{u}^{N,K}$ the solution $\mathbf{u}^{N,K}$ satisfies the apriori estimate of Theorem 1.2. In particular, there exists c independent of K such that

$$||D^2\mathbf{u}^{N,K}||_{L^2(\Omega_N)} \le c.$$

Hence for fixed N the sequence $\mathbf{u}^{N,K}$ is weakly compact in $W^{2,2}(\Omega_N) \cap W_0^{1,2}(\Omega_N)$, from which the following existence theorem can be deduced by letting $K \to \infty$ in (1.10) and the use of standard arguments.

Proposition 1.1. *The problem* (1.3) *has a weak solution* $\mathbf{u}^N \in \mathcal{H}$.

Next we state and prove a regularity result for the solution $\mathbf{u}^N$.

Proposition 1.2. *Let* $\mathbf{u}^N \in \mathcal{H}$ *be a weak solution of* (1.3). *Then* $\mathbf{u}^N \in W^{3,2}(\Omega_N)$ *and satisfies the boundary condition*

$$(\partial/\partial x_k)e_{ij}(\mathbf{u}^N)\nu_j\nu_k\tau_i = 0.$$

Moreover, there exists $p^N \in L^2(\Omega_N)$ *such that* $\mathbf{u}^N$ *satisfies*

$$L(\mathbf{u}^N + \mathbf{v}_\infty + \mathbf{w}) = -\nabla p^N \quad \text{in } \Omega_N. \tag{1.12}$$

Proof. That $\mathbf{u}^N$ satisfies (1.12) is an immediate consequence of (1.3) and the De Rham theorem (see [**4, 17**], for example).

Since $\mathbf{u}^N \in W_0^{1,2}(\Omega_N) \cap W^{2,2}(\Omega_N)$ and is divergence free, there exists a unique function $\Psi^N(x, y) \in W^{3,2}(\Omega_N)$ such that

$$\mathbf{u}^N = (-\Psi_y^N, \Psi_x^N)$$

$$\Psi^N\Big|_\Gamma = c^N; \qquad \Psi^N\Big|_{|x|=N} = 0 \tag{1.13}$$

$$\frac{\partial \Psi^N}{\partial n}\Big|_{\partial\Omega_N} = 0. \tag{1.14}$$

Substituting $(-\Psi_y^N, \Psi_x^N)$ for $\mathbf{u}^N$ in the partial differential equation and taking the curl, we find that

$$\begin{aligned} -\mu_1 \Delta^3 \Psi^N = {} & \frac{\partial^2}{\partial x_1 \partial x_j}(\gamma(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty) e_{2j}(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty)) \\ & - \frac{\partial^2}{\partial x_2 \partial x_j}(\gamma(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty) e_{1j}(\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty)) \\ & + \frac{\partial}{\partial x_1}\left((\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty)_j \frac{\partial(\mathbf{u}^N + \mathbf{w})_2}{\partial x_j}\right) \\ & - \frac{\partial}{\partial x_2}\left((\mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty)_j \frac{\partial(\mathbf{u}^N + \mathbf{w})_1}{\partial x_j}\right) \\ & + \mu_1 \frac{\partial^2}{\partial x_1 \partial x_j} \Delta e_{2j}(\mathbf{w}) - \mu_1 \frac{\partial^2}{\partial x_2 \partial x_j} \Delta e_{1j}(\mathbf{w}). \end{aligned} \tag{1.15}$$

From the regularity of $\mathbf{u}^N$, $\mathbf{w}$ the righthand side is in $W^{-2,2}(\Omega_N)$, whence $\Delta^3 \Psi^N \in W^{-2,3}(\Omega_N)$.

Using that $(\Psi^N, \Delta^3 \Psi^N) \in W^{3,2}(\Omega_N) \times W^{-2,3}(\Omega_N)$ we deduce via duality, in the usual fashion, that we can define $\partial^2 \Psi^N / \partial \boldsymbol{\nu}^3 \in W^{-1/2,2}(\partial \Omega_N)$. Therefore, the traces of all third order derivatives of Ψ^N are defined, and the traces of all second order derivatives of $\mathbf{u}^N$ are defined. From (1.12) and (1.3), it then follows that

$$\int_{\partial \Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N) \nu_j \nu_k e_{ij}(\mathbf{v})\, d\sigma = 0, \quad \forall\, \mathbf{v} \in \mathcal{H}.$$

By Theorem 3.2 of [**7**] it follows that the tangential component of the vector $(\partial/\partial x_k) e_{ij}(\mathbf{u}^N) \nu_j \nu_k$ is 0, which yields the boundary condition

$$\left. \frac{\partial}{\partial x_k} e_{ij}(\mathbf{u}^N) \nu_j \nu_k \tau_i \right|_{\partial \Omega_S} = 0. \tag{1.16}$$

It follows now that Ψ^N satisfies

$$\left. \frac{\partial}{\partial \nu} \Delta \Psi^N \right|_{\partial \Omega_N} = 0.$$

Now from the regularity theory of elliptic partial differential equations (see [**14, 15**], for example), we have that $\Psi^N \in W^{4,2}(\Omega_N)$, which then yields that $\mathbf{u}^N \in W^{3,2}(\Omega_N)$ and $p^N \in L^2(\Omega_N)$. □

2. Existence of a solution for the unbounded domain case.

Proposition 1.3. *There exists a constant $c > 0$ independent of N such that*

$$\|\mathbf{u}^N\|_{L^{4/\alpha-2}(\Omega_N)} \le c. \tag{2.1}$$

Proof. This is a direct consequence of estimate (1.4) and the estimate

$$\|\mathbf{u}\|_{L^q(\Omega)} \le c\|\nabla \mathbf{u}\|_{L^p(\Omega)}$$

with $1/p - 1/2 = 1/q$. (See [**19**, p. 158], for example.) □

From (2.1) it follows that the sequence $\mathbf{u}^N$ has a subsequence, denoted also by $\mathbf{u}^N$, which is weakly convergent in $L^{4/\alpha-2}(\Omega)$ to a function $\mathbf{u}$. Using the estimate (1.4) and a diagonal process, there exists a subsequence, which will again be denoted by $\mathbf{u}^N$, such that $\mathbf{u}^N$ converges strongly to $\mathbf{u}$ in $W^{1,2-\alpha}_{\text{loc}}(\Omega)$ and weakly in $W^{2,2}_{\text{loc}}(\Omega)$. From (1.4), it follows that

$$\iint_\Omega (D^2\mathbf{u})^2\,dx + \iint_\Omega |D\mathbf{u}|^{2-\alpha}\,dx \le c \tag{2.2}$$

and, from (2.1), that

$$\iint_\Omega |\mathbf{u}|^{4/\alpha-2}\,dx \le c. \tag{2.3}$$

Going to the limit in (1.3), we then find that

$$\begin{aligned}\iint_\Omega &\gamma(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)e_{ij}(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)\cdot e_{ij}(\mathbf{v})\,dx \\ &+ \mu_1\iint_\Omega \frac{\partial}{\partial x_k}e_{ij}(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)\cdot\frac{\partial e_{ij}}{\partial x_k}(\mathbf{v})\,dx \\ &+ \iint_\Omega (\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)_j\frac{(\partial(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)_i}{\partial x_j}\cdot v_i\,dx = 0\end{aligned} \tag{2.4}$$

for all $\mathbf{v} \in V_b$. It now follows from (2.4) that $\mathbf{u}$ satisfies

$$\begin{aligned}(2.5)\quad \mu_1 \frac{\partial}{\partial x_j}\Delta e_{ij}(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty) - \frac{\partial}{\partial x_j}(\gamma(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)\cdot e_{ij}(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty))\\ + (\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)_j \cdot \frac{\partial}{\partial x_j}(\mathbf{u}+\mathbf{w}+\mathbf{v}_\infty)_i = -\frac{\partial p}{\partial x_i} \quad \text{in } \Omega,\end{aligned}$$

where $p \in W^{-1,2}(\Omega)$.

Since the righthand side of (1.15) is uniformly bounded in $W^{-2,2}(\Omega)$, it follows from local regularity results for elliptic equations that $\Psi^N - c^N$ is uniformly bounded in $W^{4,2}_{\text{loc}}(\Omega)$, hence $\mathbf{u}^N$ converges to $\mathbf{u}$ in $W^{3,2}_{\text{loc}}(\Omega)$. It then follows from (1.16) that $\mathbf{u}$ satisfies

$$\frac{\partial}{\partial k} e_{ij}(\mathbf{u}) \cdot \nu_j \nu_k \tau_i \Big|_{\Gamma} = 0.$$

Lemma 1.6. 1. $\mathbf{u} \in L^\infty(\Omega)$.

2. $\lim_{x\to\infty} \mathbf{u}(x) = 0$.

Proof.

Part 1. Since $\mathbf{u} = 0$ on Γ, it follows from (2.2) that $\mathbf{u} \in L^\infty_{\text{loc}}(\Omega)$. Assume that $\mathbf{u} \notin L^\infty(\Omega)$. Then there exists an $x_n \to \infty$ such that $|\mathbf{u}(x_n)| \to \infty$. But, since

$$||D^2\mathbf{u}||_{L^2(B(x_n,1))} \le c, \qquad ||\mathbf{u}||_{L^{4/\alpha-2}(B(x_n,1))} \le c,$$

(c independent of n) it follows from the estimate

$$(2.6)\qquad ||\mathbf{u}||_{L^\infty(B(x_n,1))} \le c(||D^2\mathbf{u}||_{L^2(B(x_n,1))} + ||\mathbf{u}||_{L^{4/\alpha-2}(B(x_n,1))})$$

that

$$||\mathbf{u}||_{L^\infty(B(x_n,1))} \le c$$

where c is independent of n. □

Part 2. From (2.2) and (2.3) we have that the righthand side of (2.6) goes to 0 as $n \to \infty$. □

We have thus proved the following existence theorem.

Theorem 1.7. *The problem* (0.21)–(0.25) *has a weak solution in the sense of Definition* 1.1.

Remark 2. Condition (0.24) is satisfied in the sense of pointwise limit, while condition (0.25) is satisfied in the sense that $D^2\mathbf{u} \in L^2(\Omega)$.

3. Properties of the solution. We will show that the force exerted on the immersed body actually produces a drag. We denote by $\mathcal{F}(\mathbf{v}_\infty)$ the force due to the motion on the body with

$$\mathcal{F}(\mathbf{v}_\infty) = \int_{\partial\Omega'} \tau_{ij} n_j \, d\sigma$$

where $\mathbf{n}$ is the unit normal vector to $\partial\Omega'$ and τ_{ij} is the stress tensor which in the framework of bipolar fluids is given by

$$\tau_{ij} = -p\delta_{ij} + \gamma(\mathbf{v})e_{ij} - 2\mu_1 \Delta e_{ij}.$$

Theorem 1.8. *The solution exhibits a drag force in the direction of* $\mathbf{v}_\infty$, *i.e.*,

$$\mathcal{F}(\mathbf{v}_\infty) \cdot \mathbf{v}_\infty > 0.$$

Proof.

$$\mathcal{F}(\mathbf{v}_\infty) \cdot \mathbf{v}_\infty = \lambda \int_{\partial\Omega'} \tau_{1,j}(\mathbf{v}) n_j \, d\sigma.$$

Set $\mathbf{v}^N = \mathbf{u}^N + \mathbf{w} + \mathbf{v}_\infty$. Multiplying the equation (1.12) by $(\mathbf{v}^N - \mathbf{v}_\infty)$, integrating by parts, and using that $v_j^N(\mathbf{v}^N - \mathbf{v}_\infty)_i = 0$ on $\partial\Omega_l$, we find that

$$\lambda \int_{\partial\Omega'} \tau_{1,j}(\mathbf{v}^N) n_j \, d\sigma = \iint_{\Omega_l} \gamma(\mathbf{v}^N) e_{ij}(\mathbf{v}^N) e_{ij}(\mathbf{v}^N) \, dx + \mu_1 \iint_{\Omega_l} \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v}^N) \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v}^N) \, dx > 0.$$

Letting $N \to \infty$ yields

$$\lambda \int_{\partial\Omega'} \tau_{1,j}(\mathbf{v}) n_j \, d\sigma = \iint_{\Omega_l} \gamma(\mathbf{v}) e_{ij}(\mathbf{v}) e_{ij}(\mathbf{v}) + \iint_{\Omega_l} \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v}) \frac{\partial}{\partial x_k} e_{ij}(\mathbf{v}) \, dx > 0. \qquad \square$$

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