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ON BOUNDED VECTOR FIELDS

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ABSTRACT. We introduce the notion of a strongly bounded vector field, which is closely related to the usual notion of a bounded vector field, and we prove that any $\mathcal{C}^1$ strongly bounded vector field in $\mathbf{r}^n$ with finitely many critical points satisfies that the sum of the indices of the vector field at these critical points is equal to $(-1)^n$. In the planar case we improve this result since we prove it for $\mathcal{C}^1$ bounded vector fields. Moreover, when $n \geq 3$, we present examples of $\mathcal{C}^\infty$ bounded vector fields in $\mathbf{r}^n$, being obviously not strongly bounded, not satisfying that the sum of the indices at the critical points is $(-1)^n$.

1. Introduction. Let $X : U \to \mathbf{R}^n$ be a $\mathcal{C}^1$ vector field where U is an open set of $\mathbf{R}^n$, and let $\dot{x} = X(x)$ be the differential system associated to X. Consider $\varphi(x,t)$, the integral curve of X such that $\varphi(x,0) = x$, and let I_x be its maximal interval of definition. We say that X is a *bounded vector field* if for each $x \in U$ there exists some compact set $K \subset U$ such that $\varphi(x,t) \in K$ for all $t \in I_x \cap (0,+\infty)$.

It is a well-known fact, see [**10**], that if X is a bounded vector field, then the integral curve of any point is defined for all positive time and that the ω-limit of any point x, $\omega(x)$, is not empty and compact.

Bounded vector fields are interesting from the theoretical and practical point of view. Thus we can mention, for instance, previous approaches in [**2, 3**] and [**7**]. Given a vector field, it may be very difficult to know if it is bounded or not. In this setting it is interesting to give necessary and sufficient conditions in order to assure that a vector field is bounded. The goal of this paper is to generalize, as far as possible, a property which is satisfied by certain families of bounded vector fields. This property takes into account the index of the vector field at its critical points.

In what follows we will say that X is a *strongly bounded vector field*

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if it is a bounded vector field and the set

$$\omega(\mathbf{R}^n) = \bigcup_{x \in \mathbf{R}^n} \omega(x)$$

is bounded. Considering, for example, a planar global center, notice that a bounded vector field may not be strongly bounded.

Let $q \in \mathbf{R}^n$ be an isolated critical point of X. Then the function $X(u)/\|X(u)\|$ maps a small sphere centered at q into the unit $(n-1)$-sphere. The topological degree of this mapping is called the *index of X at q* and it is denoted by $i_X(q)$.

Our main results, proved in Sections 2 and 3, respectively, are the following theorems:

Theorem A. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ strongly bounded vector field with finitely many critical points $p_1, p_2, \dots, p_k$. Then*

$$\sum_{j=1}^{k} i_X(p_j) = (-1)^n.$$

Theorem B. *Let $X : \mathbf{R}^2 \to \mathbf{R}^2$ be a $\mathcal{C}^1$ bounded vector field having finitely many critical points $p_1, p_2, \dots, p_k$. Then*

$$\sum_{j=1}^{k} i_X(p_j) = 1.$$

The conclusion of Theorem A is proved in [**2**] for a family of bounded polynomial vector fields satisfying an algebraic condition, see Theorem 11 in Section 4 for more details. In [**2**] the authors use completely different tools than the ones here since they compactify the polynomial vector field, and this is not possible in the $\mathcal{C}^1$ case. In Section 4 we give a dynamical consequence of this required condition and a different proof of their result. In the same paper they also prove Theorem B for bounded polynomial vector fields.

Lastly we show that the condition in Theorem A that requires X to be a strongly bounded vector field cannot be relaxed when $n \geq 3$. To this end, we give in Section 5 an example of a bounded vector field, being obviously not strongly bounded, that does not satisfy the conclusion of Theorem A. In brief, we construct a $\mathcal{C}^\infty$ bounded vector field X in $\mathbf{R}^3$ that has finitely many critical points and that satisfies that the sum of the indices of X at all its critical points is equal to zero. In fact, by using an idea of Erle, see [**4**], given any integer k it is possible to construct a $\mathcal{C}^\infty$ bounded vector field in $\mathbf{R}^n$, $n \geq 4$, in such a way that the sum of the indices of its critical points is equal to k. When $n = 3$, there are only examples with $k \geq -1$.

In what follows, the closure, interior, boundary and complement of $S \subset \mathbf{R}^n$ are written $\overline{S}$, $\operatorname{Int} S$, ∂S and S^c, respectively. For any $r > 0$ and any $q \in \mathbf{R}^n$, the set $\{x \in \mathbf{R}^n : \|x - q\| < r\}$ will be denoted by $B_r(q)$.

2. Strongly bounded vector fields. In order to prove Theorem A we need some previous results. The first one is obvious but it will be used frequently.

Lemma 1. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ strongly bounded vector field. Let $r > 0$ be such that $B_r(0) \supset \omega(\mathbf{R}^n)$. Then, for each $x \in \mathbf{R}^n$, there exists $\tau(x) \geq 0$ such that $\varphi(x, \tau(x)) \in B_r(0)$.*

Proof. Assume it is false for $\bar{r} > 0$. Then there is an $\bar{x} \in \mathbf{R}^n$ such that $\varphi(\bar{x}, t) \notin B_{\bar{r}}(0)$ for all $t \geq 0$. Since X is a bounded vector field, there exists a compact set K such that $\varphi(\bar{x}, t) \in K$ for all $t > 0$. Then, since $K \backslash B_{\bar{r}}(0)$ is a compact set and $\varphi(\bar{x}, t) \in K \backslash B_{\bar{r}}(0)$ for all $t \geq 0$, the ω-limit set of $\bar{x}$, $\omega(\bar{x})$, is not in $B_{\bar{r}}(0)$. But this is a contradiction because $\omega(\mathbf{R}^n) \subset B_{\bar{r}}(0)$. $\square$

The following result and Lemma 1 show that, for a strongly bounded vector field, the positive trajectory of any bounded set is a bounded set.

Lemma 2. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ vector field. Assume that $r > 0$ is such that, for all $x \in \mathbf{R}^n$, there exists $\tau(x) \geq 0$ such that*

$\varphi(x, \tau(x)) \in B_r(0)$. Then there exists $R \geq r$ such that, if $x \in \overline{B_r(0)}$, then $\varphi(x,t) \in B_R(0)$ for all $t > 0$.

Proof. It is clear that it is enough to prove it for all $x \in \partial B_r(0)$. For each $x \in \partial B_r(0)$, define

$$T(x) = \inf\{t \geq 0 : \varphi(x,t) \in B_r(0)\}.$$

Then $T(x) \geq 0$, $\varphi(x, T(x)) \in \partial B_r(0)$ and $\varphi(x,t) \notin B_r(0)$ for all $t \in [0, T(x)]$. Define $d(x) = \sup\{\|\varphi(x,t)\| : 0 \leq t \leq T(x)\}$. Then $d(x) = \|\varphi(x, \bar{t}(x))\|$ with $\bar{t}(x) \in [0, T(x)]$ and $r \leq d(x) < \infty$ for all $x \in \partial B_r(0)$. Define $R = \sup\{d(x) : x \in \partial B_r(0)\}$.

In order to finish the proof we have to show that $R < \infty$. Suppose that $R = \infty$. Then there exists a sequence $(x_n)_{n \in \mathbf{N}}$, with $x_n \in \partial B_r(0)$, such that $d(x_n) \to \infty$ as $n \to \infty$. Since $\partial B_r(0)$ is a compact set, there exists (x_{n_k}) and $y_0 \in \partial B_r(0)$ such that $x_{n_k} \to y_0$ as $k \to \infty$ (and also $d(x_{n_k}) \to \infty$ as $k \to \infty$). Notice that $d(y_0) < \infty$. From (1), there exists $\tilde{t} > T(y_0)$ such that $\varphi(y_0, t) \in B_r(0)$ for all $t \in (T(y_0), \tilde{t}]$. Now let $\varepsilon > 0$ be such that $B_\varepsilon(\varphi(y_0, \tilde{t})) \subset B_r(0)$. Due to continuity with respect to initial conditions, there exists $\delta > 0$ such that if $x \in B_\delta(y_0)$, then $\|\varphi(x,t) - \varphi(y_0,t)\| < \varepsilon$ for all $t \in [0, \tilde{t}]$. It follows that if $x \in B_\delta(y_0)$, then $\varphi(x, \tilde{t}) \in B_r(0)$. So if $x \in (\partial B_r(0)) \cap B_\delta(y_0)$, then $T(x) < \tilde{t}$. Therefore, if $d(x) = \|\varphi(x, \bar{t}(x))\|$, then $\bar{t}(x) < \tilde{t}$. It follows that, if $x \in (\partial B_r(0)) \cap B_\delta(y_0)$, then $d(x) < d(y_0) + \varepsilon$. Due to $x_{n_k} \to y_0$ as $k \to \infty$, there exists k_0 such that $x_{n_k} \in B_\delta(y_0)$ for all $k \geq k_0$. Then $d(x_{n_k}) < d(y_0) + \varepsilon$ for all $k \geq k_0$. But this is a contradiction to the fact that $d(x_{n_k}) \to \infty$ as $k \to \infty$. □

Proposition 3. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ strongly bounded vector field. Then, for each $r > 0$ such that $B_r(0) \supset \omega(\mathbf{R}^n)$, there exists a bounded positively invariant set M with $B_r(0) \subset M$.*

Proof. Let $r > 0$ be such that $B_r(0)$ contains $\omega(\mathbf{R}^n)$. We define

$$M = \bigcup_{x \in B_r(0)} \{\varphi(x,t), t \geq 0\}.$$

Then $B_r(0) \subset M$ by construction and, due to Lemmas 1 and 2, M is bounded. On the other hand, that M is a positively invariant set is obvious because it is the union of positive semi-paths. □

When $n = 2$, the set M can be constructed in such a way that it is possible to describe it completely. In fact, it is not difficult to prove the following result:

Let X be a $\mathcal{C}^1$ bounded planar vector field having all its critical points in a compact set. Then, for any $r > 0$ such that $B_r(0)$ contains all the critical points of X, there exists a $\mathcal{C}^\infty$ Jordan curve γ, with $B_r(0) \subset Int\,\gamma$, such that $M = \overline{Int\,\gamma}$ is positively invariant.

Then Theorem B would easily follow from this result, see for instance [**1**], but, since the construction of such a Jordan curve is quite tedious, we will prove Theorem B using different techniques in Section 3.

Proposition 4. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ strongly bounded vector field. Then there exists an $r > 0$ with $\omega(\mathbf{R}^n) \subset B_r(0)$ such that, if $R > r$, then there exists $T = T(R) > 0$ such that $\varphi(x, T) \in B_r(0)$ for all $x \in \partial B_R(0)$.*

Proof. Let $\bar{r} > 0$ be such that $B_{\bar{r}}(0) \supset \omega(\mathbf{R}^n)$. By Proposition 3, let M be a bounded positively invariant set with $B_{\bar{r}}(0) \subset M$. Choose any $r > 0$ such that $M \subset B_r(0)$ and any $R > r$. Given $x \in \partial B_R(0)$, by Lemma 1 there exists $\tau(x) > 0$ such that $\varphi(x, \tau(x)) \in B_{\bar{r}}(0)$. Let $\varepsilon_x > 0$ be such that $B_{\varepsilon_x}(\varphi(x, \tau(x))) \subset B_{\bar{r}}(0)$, and let $\delta_x > 0$ be such that, if $y \in B_{\delta_x}(x)$, then $\varphi(y, \tau(x)) \in B_{\varepsilon_x}(\varphi(x, \tau(x)))$. Since $\partial B_R(0)$ is compact, there exist $x_1, x_2, \dots, x_k \in \partial B_R(0)$ such that

$$\partial B_R(0) \subset \bigcup_{i=1}^{k} B_{\delta_{x_i}}(x_i).$$

Define $T = \max\{\tau(x_i); i = 1, 2, \dots, k\}$. Since $B_{\bar{r}}(0) \subset M \subset B_r(0)$ and M is positively invariant, it follows that $\varphi(x, T) \in B_r(0)$ for all $x \in \partial B_R(0)$. Hence, Proposition 4 is proved. $\square$

In order to prove Theorem A we need another definition and two technical results:

Let M be a compact manifold with boundary ∂M, and let X be a continuous tangent vector field on M vanishing nowhere on ∂M. The *topological index* of X, see [**5**] or [**6**], $I(X)$, is an integer such that if X

has finitely many critical points $p_1, p_2, \dots, p_k$, then

$$I(X) = \sum_{j=1}^{k} i_X(p_j).$$

In this context we can state the following well-known result, see [**6**] or [**8**].

Lemma 5. *Let X and Y be continuous tangent vector fields on a compact manifold M vanishing nowhere on ∂M. Assume that X and Y are homotopic on the boundary ∂M, that is, there is an H_s with $s \in [0,1]$, a continuous family of tangent vector fields on M, such that $H_0 = X$, $H_1 = Y$ and $H_s(x) \neq 0$ for all $x \in \partial M$ and for all $s \in [0,1]$. Then $I(X) = I(Y)$.*

Lemma 6. *Let X be a continuous vector field on $\overline{B_r(0)} \subset \mathbf{R}^n$ vanishing nowhere on $\partial B_r(0)$ such that there does not exist $x \in \partial B_r(0)$ with $X(x)$ pointing in the outward normal direction to $\partial B_r(0)$. Then $I(X) = (-1)^n$.*

Proof. We define $H_s(x) = (1-s)X(x) - sx$, $s \in [0,1]$, a family of continuous vector fields on $\overline{B_r(0)}$. Since there does not exist any $x \in \partial B_r(0)$ such that $X(x)$ points in the outward normal direction to $\partial B_r(0)$ it is easy to see that H_s is a continuous homotopy on $\partial B_r(0)$ between $H_0 = X$ and $H_1 = -Id$. Then $I(X) = I(-Id)$ from Lemma 5. Since $-Id$ has $x = 0$ as a unique critical point and $i_{-Id}(0) = (-1)^n$, the result follows. $\square$

Proof of Theorem A. Let $r > 0$ with $\omega(\mathbf{R}^n) \subset B_r(0)$ be defined in Proposition 4. Let $R > r$ and $T > 0$ be such that $\varphi(x,T) \in B_r(0)$ for all $x \in \partial B_R(0)$. Define H_s with $s \in [0,T]$, a family of vector fields on $\overline{B_R(0)}$ to be

$$H_s(x) = \lim_{t \to s} \frac{\varphi(x,t) - x}{t}.$$

Then

$$H_0(x) = \lim_{t\to 0} \frac{\varphi(x,t) - x}{t} = \frac{d}{dt}(\varphi(x,t))\Big|_{t=0}$$
$$= X(\varphi(x,t))\Big|_{t=0} = X(x).$$

We claim that $0 \notin H_s(\partial B_R(0))$ for all $s \in [0,T]$. This is obvious when $s = 0$ since $H_0(x) = X(x)$ and X has no critical points on $\partial B_R(0)$. Assume now that there exists some $\bar{x} \in \partial B_R(0)$ and $\bar{s} \in (0,T]$ such that $H_{\bar{s}}(\bar{x}) = 0$. It follows then that $\varphi(\bar{x},\bar{s}) = \bar{x}$ for $\bar{s} \neq 0$. Hence, $\bar{x}$ belongs to a periodic orbit. But this is impossible since $\omega(\mathbf{R}^n) \subset B_r(0)$ and $R > r$. Thus the claim is proved.

Therefore we have shown that $H_0 = X|_{\overline{B_R(0)}}$ and H_T are homotopic on $\partial B_R(0)$.

Now consider H_T. Since $\varphi(x,T) \in B_r(0)$ for all $x \in \partial B_R(0)$ and $r < R$, it follows that

$$H_T(x) = \frac{\varphi(x,T) - x}{T}$$

never points in the outward normal direction to $\partial B_R(0)$ for any $x \in \partial B_R(0)$. Then, by Lemmas 5 and 6,

$$\sum_{j=1}^{k} i_X(p_j) = I(X|_{\overline{B_R(0)}}) = I(H_T) = (-1)^n,$$

and Theorem A follows. $\square$

As in Section 1, we can define a concept of boundedness taking into account now the negative semi-orbits of the trajectories defined by the vector field X. Thus, denoting by I_x the maximal interval of definition of the integral curve of X with $\varphi(x,0) = x$, we say that X is a *negatively bounded vector field* if, for each $x \in \mathbf{R}^n$ there exists some compact set K such that $\varphi(x,t) \in K$ for all $t \in I_x \cap (-\infty, 0)$. Then it is obvious that the α-limit of any point x, $\alpha(x)$, is not empty and compact, see [**10**]. Clearly we have in this case an analogous result to Theorem A. That is, if X is a $\mathcal{C}^1$ negatively bounded vector field in $\mathbf{R}^n$ having finitely many critical points and satisfying that

$$\alpha(\mathbf{R}^n) = \bigcup_{x\in\mathbf{R}^n} \alpha(x)$$

is a bounded set, then the sum of the indices of X at all its critical points is equal to 1. To see this, notice that in this situation $-X$ is a strongly bounded vector field and that if $x_0 \in \mathbf{R}^n$ is a critical point of X, then $i_X(x_0) = (-1)^n i_{-X}(x_0)$.

We note that there exist vector fields being simultaneously bounded and negatively bounded. The simplest example of this behavior is a planar global center. In even dimensions we have the obvious generalization of this situation. In these examples the whole $\mathbf{R}^n$ is foliated by periodic orbits. In Section 5 we present examples of vector fields in dimension greater than two being simultaneously bounded and negatively bounded and having more complicated dynamics.

The next proposition shows that a strongly bounded vector field cannot be negatively bounded. In other words, if X is a $\mathcal{C}^1$ strongly bounded vector field, then there exists at least one point $x_0 \in \mathbf{R}^n$ such that, for each compact set K there is a $\tau < 0$ with $\varphi(x_0, \tau) \notin K$.

Proposition 7. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ vector field. If X is a bounded and negatively bounded vector field, then neither $\omega(\mathbf{R}^n)$ nor $\alpha(\mathbf{R}^n)$ are bounded sets.*

Proof. We will prove by contradiction that $\omega(\mathbf{R}^n)$ is an unbounded set, the same fact can be proved for $\alpha(\mathbf{R}^n)$ similarly. So assume that $\omega(\mathbf{R}^n)$ is a bounded set. Hence X is a $\mathcal{C}^1$ strongly bounded vector field. Taking any $r > 0$ such that $\omega(\mathbf{R}^n) \subset B_r(0)$, by Lemmas 1 and 2, there exists $R \geq r$ such that, if $x \in \overline{B_r(0)}$ then $\varphi(x,t) \in B_R(0)$ for all $t \geq 0$. Consider any $x_0 \notin B_R(0)$. Since X is a negatively bounded vector field, $\alpha(x_0) \neq \varnothing$. Consider $y_0 \in \alpha(x_0)$. On account of $\omega(\mathbf{R}^n) \subset B_r(0)$, there exists a $t_1 \geq 0$ such that $\varphi(y_0, t_1) \in B_r(0)$. Take $\varepsilon > 0$ small enough such that $B_\varepsilon(\varphi(y_0, t_1)) \subset B_r(0)$. Due to continuity with respect to initial conditions, take $\delta > 0$ such that, if $x \in B_\delta(y_0)$, then $\varphi(x, t_1) \in B_\varepsilon(\varphi(y_0, t_1))$. Now consider $t_2 \geq t_1$ such that $\varphi(x_0, -t_2) \in B_\delta(y_0)$, and set $z_0 = \varphi(\varphi(x_0, -t_2), t_1)$. Then

$z_0 \in B_\varepsilon(\varphi(y_0, t_1)) \subset B_r(0)$, but $\varphi(z_0, t_2 - t_1) = x_0 \notin B_R(0)$ with $t_2 - t_1 \geq 0$ contradicts the choice of R. □

Finally we prove a technical result that will be used in Section 4. Lemma 8 gives a sufficient condition for a bounded vector field to be strongly bounded.

Lemma 8. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a $\mathcal{C}^1$ bounded vector field. Denoting the scalar product by $\langle\ ,\ \rangle$, if the set $\mathcal{T} = \{x \in \mathbf{R}^n : \langle X(x), x\rangle = 0\}$ is bounded, then X is a strongly bounded vector field.*

Proof. We will show that if X is not strongly bounded then $\mathcal{T}$ is not bounded. So assume that $\omega(\mathbf{R}^n)$ is not bounded. Then there exists $(x_k)_{k\in\mathbf{N}}$ such that the set

$$\bigcup_{k\in\mathbf{N}} \omega(x_k)$$

is not bounded. Since X is a bounded vector field, the ω-limit of any point is a compact set and hence there exists $(y_k)_{k\in\mathbf{N}}$ such that $\|y_k\| = \sup\{\|y\| : y \in \omega(x_k)\}$. It is obvious that $\|y_k\| \to +\infty$ as $k \to +\infty$. We claim that $\langle X(y_k), y_k\rangle = 0$ for all $k \in \mathbf{N}$. It is clear that, if the claim is true, then Lemma 8 will follow.

We will prove the claim by contradiction. So assume that there exists some $\tilde{k} \in \mathbf{N}$ such that $\langle X(y_{\tilde{k}}), y_{\tilde{k}}\rangle \neq 0$. Then $y_{\tilde{k}}$ is not a critical point, and by the tubular flow theorem, see, for instance, [**10**], there exists $t_{\tilde{k}} \neq 0$ small enough such that

$$\varphi(y_{\tilde{k}}, t_{\tilde{k}}) \notin \overline{B_{\|y_{\tilde{k}}\|}(0)}.$$

But, since $y_{\tilde{k}} \in \omega(x_{\tilde{k}})$ and the ω-limits are invariant by the flow, $\varphi(y_{\tilde{k}}, t_{\tilde{k}}) \in \omega(x_{\tilde{k}})$. Then $\|\varphi(y_{\tilde{k}}, t_{\tilde{k}})\| > \|y_{\tilde{k}}\|$ contradicts $\|y_{\tilde{k}}\| = \sup\{\|y\| : y \in \omega(x_{\tilde{k}})\}$, and the claim is proved. □

3. Bounded planar vector fields.

Lemma 9. *Let $X : \mathbf{R}^2 \to \mathbf{R}^2$ be a $\mathcal{C}^1$ bounded vector field with all its critical points in a compact set. Then, for each $r > 0$ such that $B_r(0)$*

contains all the critical points of X, either there exists a periodic orbit γ with $B_r(0) \subset Int\gamma$ or for each $x \in \mathbf{R}^2$ there exists $\tau(x) \geq 0$ such that $\varphi(x, \tau(x)) \in B_r(0)$.

Proof. Consider $r > 0$ such that $B_r(0)$ contains all the critical points of X and assume that there is no periodic orbit γ with $B_r(0) \subset \operatorname{Int}\gamma$.

By contradiction, suppose that there is some $\bar{x} \in \mathbf{R}^2$ such that $\varphi(\bar{x}, t) \notin B_r(0)$ for all $t \geq 0$. From the hypothesis, X is a bounded vector field, so there exists a compact set K such that $\varphi(\bar{x}, t) \in K$ for all $t \geq 0$. Let Q be defined by $K \backslash B_r(0)$. Then Q is a compact set and $\varphi(\bar{x}, t) \in Q$ for all $t \geq 0$. It follows that the ω-limit of $\bar{x}$, $\omega(\bar{x})$, satisfies $\omega(\bar{x}) \neq \varnothing$ and $\omega(\bar{x}) \subset Q$. Since there are no critical points of X in Q, by the Poincaré-Bendixson theorem, see [**10**], $\omega(\bar{x})$ is a periodic orbit, say γ, contained in Q. Since every periodic orbit in the plane has a critical point in its interior, then it follows that $B_r(0) \subset \operatorname{Int}\gamma$, otherwise there would be at least one critical point not in $B_r(0)$. But this contradicts the assumption. Hence, Lemma 9 is proved. □

Theorem 10. *Let $X : \mathbf{R}^2 \to \mathbf{R}^2$ be a $\mathcal{C}^1$ bounded vector field with all its critical points in a compact set. Then either X is strongly bounded or, for each $r > 0$ such that $B_r(0)$ contains all the critical points of X, there exists a periodic orbit γ with $B_r(0) \subset Int\gamma$.*

Proof. By contradiction, assume that X is not strongly bounded and that there exists some $\bar{r} > 0$ with $B_{\bar{r}}(0)$ containing all the critical points such that there does not exist any periodic orbit γ with $B_{\bar{r}}(0) \subset \operatorname{Int}\gamma$. Then, due to Lemma 9, we can apply Lemma 2 and assert that there exists $R > \bar{r}$ such that, if $x \in \overline{B_{\bar{r}}(0)}$, then $\varphi(x, t) \in B_R(0)$ for all $t > 0$.

Since X is not strongly bounded, there exists an $\bar{x} \notin B_R(0)$ with $\bar{x} \in \omega(\bar{y})$ for some $\bar{y} \in \mathbf{R}^2$. Since $\bar{x}$ is not a critical point and does not belong to a periodic orbit, it must belong to a cycle of separatrices, see, for instance, [**10**]. Then its α-limit is a critical point. Since all the critical points of X are in $B_{\bar{r}}(0)$, there exists a $\bar{t} < 0$ such that $\varphi(\bar{x}, \bar{t}) \in B_{\bar{r}}(0)$. But this contradicts Lemma 2 since $\bar{x} \notin B_R(0)$. □

Proof of Theorem B. If X is strongly bounded it follows from Theorem A. Otherwise, applying Theorem 10, there is a periodic orbit γ with $p_j \in \operatorname{Int}\gamma$ for all $j = 1, 2, \dots, k$. Then, using that the sum of the indices at the critical points contained in the interior of any periodic orbit is equal to one, see, for instance, [**1**], Theorem B follows again. □

As has been noted, in [**2**] Cima and Llibre proved Theorem B for bounded polynomial vector fields with finitely many critical points. Using the index formula of Bendixson they showed that the infinite critical point of the stereographic compactification of the vector field has index equal to one. Then the result follows from the Poincaré-Hopf theorem.

4. Bounded polynomial vector fields. We shall say that $X = (P^1, \dots, P^n)$ belongs to Δ_m if X is a polynomial vector field with degree $(P^i) = m$ for all $i = 1, 2, \dots, n$. For $i = 1, 2, \dots, n$, we denote the homogeneous part of P^i of degree k by P_k^i, $k = 0, 1, \dots, m$. Finally we will say that $X \in \Delta_m$ satisfies condition $*$ if the system $P_m^i(x_1, \dots, x_n) = 0$ for $i = 1, 2, \dots, n$, has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. In this sense we can state Theorem D of [**2**] as follows:

Theorem 11. *Let $X \in \Delta_m$ be a bounded polynomial vector field in $\mathbf{R}^n$ satisfying condition $*$ and having finitely many critical points $p_1, p_2, \dots, p_k$. If the Poincaré compactification of X has all the critical points at infinity isolated, then*

$$\sum_{j=1}^{k} i_X(p_j) = (-1)^n.$$

In the first part of this section we give a proof of Theorem 11 different to the one in [**2**], but first we need the following result that gives a dynamical consequence of condition $*$.

Lemma 12. *If $X \in \Delta_m$ is a bounded polynomial vector field* *satisfying condition* $*$, *then*

$$\mathcal{N} = \{x \in \mathbf{R}^n : X(x) \textit{ points in the outward normal direction to } \partial B_{\|x\|}(0)\}$$

is a bounded set.

Proof. It is clear that $x \in \mathcal{N}$ if and only if $X(x) = \langle X(x), s\rangle s$ with $\langle X(x), s\rangle > 0$, where $\langle\ ,\ \rangle$ denotes the scalar product and $s = x/\|x\|$.

We will prove that if $\mathcal{N}$ is not bounded, then X is not a bounded vector field. So assume that there exists $(x_k)_{k\in\mathbf{N}}$, with $x_k \in \mathcal{N}$ for all k, such that $\|x_k\| \to +\infty$ as $k \to +\infty$. Define $r_k = \|x_k\|$ and $s_k \in S^{n-1} = \{x \in \mathbf{R}^n : \|x\| = 1\}$ to be such that $x_k = r_k s_k$. Since S^{n-1} is compact we can assume without loss of generality that $s_k \to \tilde{s}$ as $k \to \infty$ for some $\tilde{s} \in S^{n-1}$.

For $k = 0, 1, \dots, m$, define $X_k = (P_k^1, \dots, P_k^n)$. Then $X = X_0 + X_1 + \cdots + X_m$.

For any $x \in \mathbf{R}^n\backslash\{0\}$, where $r = \|x\|$ and $s = x/\|x\|$, it follows that

$$X(x) = X_0 + rX_1(s) + \cdots + r^{m-1}X_{m-1}(s) + r^m X_m(s). \tag{2}$$

Applying (2) and the bilinearity of the scalar product, we have

$$\begin{aligned}\langle X(x), s\rangle = \langle X_0, s\rangle + r\langle X_1(s), s\rangle + \cdots + r^{m-1}\langle X_{m-1}(s), s\rangle \\ + r^m\langle X_m(s), s\rangle.\end{aligned} \tag{3}$$

Since for $x \in \mathcal{N}$ with $x = \|x\|s$ we have that $X(x) = \langle X(x), s\rangle s$, applying (2) and (3) we can assert that, for any $x \in \mathcal{N}$ with $x = rs$, where $r = \|x\|$, the following holds

$$\begin{aligned}X_m(s) = &\frac{\langle X_0, s\rangle s + r\langle X_1(s), s\rangle s + \cdots + r^{m-1}\langle X_{m-1}(s), s\rangle s}{r^m} \\ &+ \frac{r^m\langle X_m(s), s\rangle s}{r^m} - \frac{X_0 + rX_1(s) + \cdots + r^{m-1}X_{m-1}(s)}{r^m}.\end{aligned}$$

Evaluating $(x_k)_{k\in\mathbf{N}}$ in the above expression and making $k \to \infty$ afterward, we get

$$X_m(\tilde{s}) = \langle X_m(\tilde{s}), \tilde{s}\rangle\tilde{s}.$$

Since $\langle X(x_k), s_k\rangle > 0$ for all $k \in \mathbf{N}$, we have $\langle X_m(\tilde{s}), \tilde{s}\rangle \geq 0$. Then, due to condition $*$ and noting that $\tilde{s} \in S^{n-1}$, we finally get $\langle X_m(\tilde{s}), \tilde{s}\rangle > 0$.

Thus we have proved that if $\mathcal{N}$ is not bounded, then there exists $\tilde{s} \in S^{n-1}$ such that $X_m(\tilde{s}) = \lambda\tilde{s}$ with $\lambda > 0$.

Using $\mathcal{P}(X)$, the Poincaré compactification of X, see [**2**] for more details, it follows that $\tilde{s}$ is an infinite critical point and that the linear part of $\mathcal{P}(X)$ at $\tilde{s}$ has $-\lambda$ as eigenvalue in the finite direction. Then there exists some orbit of $\mathcal{P}(X)$ lying in the finite part of $\mathbf{R}^n$ whose ω-limit is $\tilde{s}$. Hence, X is not a bounded vector field and Lemma 12 follows. $\square$

Proof of Theorem 11. Due to Lemma 12 we can assert that $\mathcal{N}$ is a bounded set. Then there exists $R > 0$ large enough such that $B_R(0)$ contains all the critical points of X and such that, for all $x \in \partial B_R(0)$, $X(x)$ is not pointing in the outward normal direction to $\partial B_R(0)$. Then, by Lemma 6, it follows that

$$\sum_{j=1}^{k} i_X(p_j) = I(X|_{\overline{B_R(0)}}) = (-1)^n,$$

and Theorem 11 is proved. $\square$

Finally we give an algebraic condition that is sufficient for a bounded polynomial vector field to be strongly bounded:

Proposition 13. *Let $X = (P^1, \ldots, P^n) \in \Delta_m$ be a bounded polynomial vector field. Denote $X_m = (P^1_m, \ldots, P^n_m)$. If the system $\langle X_m(x), x\rangle = 0$ has only the trivial solution $x = 0$, then X is a strongly bounded vector field.*

Proof. By Lemma 8 we must only show that the set $\mathcal{T} = \{x \in \mathbf{R}^n : \langle X(x), x\rangle = 0\}$ is bounded. We will prove that, if $\mathcal{T}$ is not a bounded set, then there exists $\tilde{s} \neq 0$ such that $\langle X_m(\tilde{s}), \tilde{s}\rangle = 0$. So assume that there is an $(x_k)_{k\in\mathbf{N}}$, with $\|x_k\| \to +\infty$ as $k \to +\infty$, such that $x_k \in \mathcal{T}$ for all $k \in \mathbf{N}$.

Define $r_k = \|x_k\|$ and $s_k \in S^{n-1} = \{x \in \mathbf{R}^n : \|x\| = 1\}$ to be such that $x_k = r_k s_k$. Since S^{n-1} is compact, we can assume without loss of generality that $s_k \to \tilde{s}$ as $k \to +\infty$ for some $\tilde{s} \in S^{n-1}$. For $k = 0, 1, \dots, m$, denote $X_k = (P_k^1, \dots, P_k^n)$. Then, since $X = X_0 + X_1 + \cdots + X_m$, we get

$$\begin{aligned}\langle X(x_k), x_k\rangle = r_k\langle X_0, s_k\rangle + \cdots + r_k^m \langle X_{m-1}(s_k), s_k\rangle \\ + r_k^{m+1}\langle X_m(s_k), s_k\rangle.\end{aligned}$$

Hence it follows that

$$\frac{\langle X(x_k), x_k\rangle}{r_k^{m+1}} \longrightarrow \langle X_m(\tilde{s}), \tilde{s}\rangle \quad \text{as} \quad k \longrightarrow +\infty.$$

Since $x_k \in \mathcal{T}$ for all $k \in \mathbf{N}$, we get $\langle X_m(\tilde{s}), \tilde{s}\rangle = 0$ with $\tilde{s} \in S^{n-1}$, and Proposition 13 follows. $\square$

5. Examples of bounded vector fields not satisfying the conclusion of Theorem A. By using an idea of Erle, see [**4**], we construct a $\mathcal{C}^\infty$ bounded vector field X in $\mathbf{R}^3$ satisfying that the sum of the indices of X at all its critical points is equal to zero. This vector field will also be negatively bounded.

Let T be a three-dimensional solid torus, and let $r > 0$ be such that $T \subset B_r(0) = \{x \in \mathbf{R}^3 : \|x\| < r\}$. Let G be a $\mathcal{C}^\infty$ vector field defined in a neighborhood of $B_r(0)$, tangent to ∂T and with finitely many critical points, all of them contained in $B_r(0)\backslash\partial T$. Let $p_1, p_2, \dots, p_k$ (respectively, $q_1, q_2, \dots, q_{k'}$) be the critical points of G inside of $\operatorname{Int} T$ (respectively, $B_r(0)\backslash T$). Let D_1 and D_2 be two sets diffeomorphic to closed balls satisfying:

(1) $D_1 \subset \operatorname{Int} T$ and $p_1, p_2, \dots, p_k \in \operatorname{Int} D_1$.

(2) $T \subset \operatorname{Int} D_2$, $D_2 \subset B_r(0)$ and $q_1, q_2, \dots, q_{k'} \notin D_2$.

Let φ be a diffeomorphism that carries D_2 to $\overline{B_2(0)}$ and D_1 to $\overline{B_1(0)}$. For any point $p \in \overline{B_2(0)}$, define $F(p)$ as

$$(D\varphi \circ G \circ \varphi^{-1})(p).$$

Then F is a $\mathcal{C}^\infty$ vector field on $\overline{B_2(0)}$ that is tangent to $\varphi(\partial T)$, and it is clear that it has finitely many critical points, all contained in $B_1(0)$.

Let T_0 be the three-dimensional solid torus with $\partial T_0 = \varphi(\partial T)$. It is clear that $B_1(0) \subset \operatorname{Int} T_0$ and that $\partial T_0 \subset B_2(0)\backslash\overline{B_1(0)}$.

Take $\varepsilon > 0$ small enough such that $\partial T_0 \subset B_{2-\varepsilon}(0)\backslash\overline{B_{1+\varepsilon}(0)}$. Consider a $\mathcal{C}^\infty$ map $\rho_1 : [1,2] \to [1,2]$ satisfying $\rho_1(r) = r$ for all $r \in [1, 2-\varepsilon]$ and $\rho_1(r) = 2$ for all $r \in [2-\varepsilon/2, 2]$. Consider a $\mathcal{C}^\infty$ map $\rho_2 : [1,2] \to [1,2]$ satisfying $\rho_2(r) = r$ for all $r \in [1+\varepsilon, 2-\varepsilon]$ and $\rho_2(r) = 2$ for all $r \in [1, 1+\varepsilon/2] \cup [2-\varepsilon/2, 2]$. For a point p with $1 \leq \|p\| \leq 2$, define

$$F_1(p) = F\left(\frac{\rho_1(\|p\|)}{\|p\|}p\right) \quad \text{and} \quad F_2(p) = F\left(\frac{\rho_2(\|p\|)}{\|p\|}p\right).$$

Then F_1 and F_2 are $\mathcal{C}^\infty$ nonvanishing vector fields on $\overline{B_2(0)}\backslash B_1(0)$ satisfying

(a) For all p with $\|p\| \in [1, 1+\varepsilon/2] \cup [2-\varepsilon/2, 2]$ the value of F_2 at p depends only on $p/\|p\|$.

(b) For all p with $\|p\| \in [2-\varepsilon/2, 2]$ the value of F_1 at p depends only on $p/\|p\|$ and it is equal to $F_2(p)$.

(c) $F_2(p) = F(p)$ for all p with $\|p\| \in [1+\varepsilon, 2-\varepsilon]$.

(d) $F_1(p) = F(p)$ for all p with $\|p\| \in [1, 2-\varepsilon]$.

Notice that, due to (c) and (d), F_1 and F_2 are tangent to ∂T_0.

We use F_1 and F_2 to define a vector field X on $\mathbf{R}^3$ as follows:

$$X(p) = \begin{cases} F(p) & \text{if } \|p\| < 1, \\ F_1(p) & \text{if } 1 \leq \|p\| < 2, \\ F_2(2^{-n}p) & \text{if } 2^n \leq \|p\| < 2^{n+1},\ n \in \mathbf{N}. \end{cases}$$

It is clear that X is well defined and that it has finitely many critical points, all contained in $B_1(0)$. Moreover, by using properties (a), (b) and (d), notice that X is a $\mathcal{C}^\infty$ vector field.

For $n \in \mathbf{N}$, define T_n to be the image of T_0 under the map $p \to 2^n p$. It is clear that T_n is a three-dimensional solid torus, and it follows from property (c) that X is tangent to ∂T_n for all $n \in \mathbf{N}$. Hence, for all $n \in \mathbf{N}$, T_n is invariant under the flow defined by X. This implies that X is a bounded vector field since $B_{2^n}(0) \subset T_n$ for all $n \in \mathbf{N}$. Notice that in fact X is also a negatively bounded vector field.

On the other hand, X has finitely many critical points, all of them contained in T_0. Since X is tangent to ∂T_0, by the Poincaré-Hopf theorem, see [**9**] for instance, we can assert that the sum of their indices is equal to zero, the Euler characteristic of T_0.

Notice that X is obviously not strongly bounded. For all $n \in \mathbf{N}$, $x \in \partial T_n$ implies $\omega(x) \subset \partial T_n$ and by construction $\operatorname{dist}(\partial T_n, 0) \to +\infty$ as $n \to +\infty$.

Remarks. (a) Choosing the initial vector field G vanishing nowhere in the solid torus T, it is possible to construct a $\mathcal{C}^\infty$ bounded vector field in $\mathbf{R}^3$ without critical points.

(b) As has been noted in Section 1, given any integer k, respectively $k \geq -1$, when $n \geq 4$, respectively $n = 3$, it is possible to construct similarly a $\mathcal{C}^\infty$ bounded vector field in $\mathbf{R}^n$ satisfying the sum of the indices at all its critical points is equal to k. It is enough to take initially, instead of the three-dimensional solid torus, an n-dimensional compact differentiable manifold M with connected boundary ∂M, satisfying that its Euler characteristic is $(-1)^n k$, when $n = 3$ there are only examples of such manifolds with $k \geq -1$. Furthermore the original vector field G defined in a neighborhood of M must point inward at all the boundary points.

(c) In order to get examples of vector fields being simultaneously bounded and negatively bounded, we must take the original vector field G nonvanishing and tangent to ∂M. To this end, ∂M must have Euler characteristic equal to zero.

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