ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1999

DISCONJUGACY OF SYMPLECTIC SYSTEMS AND POSITIVE DEFINITENESS OF BLOCK TRIDIAGONAL MATRICES

ROMAN HILSCHER

ABSTRACT. In this paper we discuss disconjugacy of symplectic difference systems in the relation with positive definiteness of a certain associated block tridiagonal matrix. Analogous results have been recently proved for a special form of symplectic systems–linear Hamiltonian difference systems and Sturm-Liouville difference equations. Finally, reciprocal systems are also discussed.

1. Introduction. The principal aim of this paper is to study the relationship between disconjugacy of symplectic systems

$$z_{k+1} = S_k z_k, \quad 0 \le k \le N \tag{S}$$

and positive definiteness of a certain symmetric block tridiagonal matrix associated with (S). Symplectic systems cover two important objects as its special cases: linear Hamiltonian difference systems (LHdS, see below) and Sturm-Liouville difference equations (SLdE, the special case of LHdS). Lately a considerable effort has been made to define disconjugacy for LHdS, and hence for SLdE, and to prove the so-called Reid roundabout theorem for such systems, see [**3**]. Recently the above-mentioned results have been extended by Bohner and Došlý also to symplectic systems, see [**4**]. The approach used in the above references is based on the *discrete Picone's identity*, which is not needed in [**5**], and in the present work, too.

Consider an LHdS

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \qquad 0 \le k \le N, \tag{H}$$

Received by the editors on May 5, 1997, and in revised form on December 10, 1997.
1991 AMS *Mathematics Subject Classification.* 39A10, 39A12, 15A09, 15A63.
Key words and phrases. Symplectic system, linear Hamiltonian difference system, disconjugacy, principal solution, Sturm-Liouville difference equation.
Supported by grants 201/96/0410 and 201/98/0677 of the Czech Grant Agency.

where A, B, C are sequences of $n \times n$-matrices, B and C symmetric, $I - A$ nonsingular. We denote $\tilde{A} := (I - A)^{-1}$. The associated discrete quadratic functional takes the form

$$\mathcal{F}_H(x, u) = \sum_{k=1}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}.$$

System (H) can be rewritten into (S) with the matrix

$$S = \begin{pmatrix} \tilde{A} & \tilde{A}B \\ C\tilde{A} & C\tilde{A}B + \tilde{A}^{T-1} \end{pmatrix}.$$

Define the $n \times n$-matrices $\bar{T}_k$, $\bar{S}_k$ and $kn \times kn$-matrix $\mathcal{L}_k$ in the following way:

$$\begin{aligned} \bar{T}_k &:= C_k + (I - A_k^T) B_k^\dagger (I - A_k) + B_{k+1}^\dagger, \\ \bar{S}_k &:= -B_k^\dagger (I - A_k), \quad 0 \le k \le N - 1, \end{aligned} \tag{1}$$

$$\mathcal{L}_k = \begin{pmatrix} \bar{T}_0 & \bar{S}_1 & & \\ \bar{S}_1^T & \bar{T}_1 & \ddots & \\ & \ddots & \ddots & \bar{S}_{k-1} \\ & & \bar{S}_{k-1}^T & \bar{T}_{k-1} \end{pmatrix}, \quad 1 \le k \le N.$$

In [**5**] it is shown that disconjugacy of the system (H), and hence positive definiteness of the quadratic functional $\mathcal{F}_H$, is equivalent to positive definiteness of the matrix $\mathcal{L} := \mathcal{L}_N$ on some appropriate subspace of "N-vectors" (here every item of such an "N-vector" is an n-vector itself). For more details on disconjugacy of LHdS, (S) and positive definiteness of $\mathcal{F}$, see the work of Bohner and Došlý [**3, 5**], and the references given therein. A comprehensive treatment of difference equations and LHdS is contained in the recent monograph of Ahlbrandt and Peterson [**1**].

The subject of this paper is to extend the above-mentioned results to symplectic systems.

2. Preliminary results. Let $n, N \in \mathbf{N}$, $J := [0, N] \cap \mathbf{Z}$, $J^* := [0, N + 1] \cap \mathbf{Z}$. By $M^\dagger$ we denote the Moore-Penrose generalized inverse of a matrix M, i.e., the unique matrix satisfying $MM^\dagger M = M$, $M^\dagger M M^\dagger = M^\dagger$ such that $MM^\dagger$ and $M^\dagger M$ are symmetric. For a symmetric matrix M, we write $M > 0$ if M is positive definite and $M \geq 0$ if M is positive semi-definite. By $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{rank} M$, M^T, M^{-1}, $\det M$, we denote the kernel, image, rank, transpose, inverse and determinant of the matrix M, respectively. By Δ we denote the usual forward difference operator. We denote by I the $n \times n$-identity matrix and define $2n \times 2n$-matrices $\mathcal{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and $\mathcal{K} = \left(\begin{smallmatrix} 0 & 0 \\ I & 0 \end{smallmatrix}\right)$. A real $2n \times 2n$-matrix S is called *symplectic* if $S^T \mathcal{J} S = \mathcal{J}$ holds.

Lemma 1 (Properties of symplectic matrices). *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbf{R}^{n\times n}$ and $S := \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right) \in \mathbf{R}^{2n\times 2n}$ be matrices. Then S is symplectic if and only if*

$$(2) \qquad \begin{cases} \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = \mathcal{A}\mathcal{D}^T - \mathcal{B}\mathcal{C}^T = I, & \text{and} \\ \mathcal{A}\mathcal{B}^T, \mathcal{C}\mathcal{D}^T, \mathcal{C}^T\mathcal{A}, \mathcal{D}^T\mathcal{B} & \text{symmetric.} \end{cases}$$

In this case S is nonsingular, $S^{-1} = \mathcal{J}^T S^T \mathcal{J} = \left(\begin{smallmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{smallmatrix}\right)$ and both S^{-1} and S^T are symplectic as well. Consequently, the set of all (*real*) *symplectic $2n \times 2n$-matrices form a group with respect to the matrix multiplication.*

Proof. Rewriting the definition of a symplectic matrix, we get formulae (2). From $\mathcal{J}^{-1} = \mathcal{J}^T$ and $1 = \det \mathcal{J} = \det(S^T \mathcal{J} S) = (\det S)^2$, the rest follows. □

Consider a *symplectic system*

$$(\mathrm{S}) \qquad z_{k+1} = S_k z_k, \quad k \in J,$$

where z_k is a sequence of $2n$-vectors defined on J^* and S_k is a sequence of $2n \times 2n$-matrices defined on J. The matrices S_k are supposed to be symplectic. Simultaneously with the system (S) we consider its matrix analogy $Z_{k+1} = S_k Z_k$, $k \in J$, where Z_k is a sequence of $2n \times n$-matrices defined on J^*. When referring to solutions of (S), we use a usual agreement that the vector-valued solutions of (S) are denoted by small letters and the matrix-valued solutions by capital ones.

In the sequel we use the following notation:

$$S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ u \end{pmatrix}, \qquad Z = \begin{pmatrix} X \\ U \end{pmatrix}$$

with $x, u : J^* \to \mathbf{R}^n$, $X, U : J^* \to \mathbf{R}^{n\times n}$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : J \to \mathbf{R}^{n\times n}$. The system (S) can then be rewritten into the form

$$\text{(S)} \qquad x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \qquad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in J.$$

By Lemma 1, the time-reversed system $z_k = S_k^{-1} z_{k+1}$ reads as

$$\text{(3)} \quad x_k = \mathcal{D}_k^T x_{k+1} - \mathcal{B}_k^T u_{k+1}, \qquad u_k = -\mathcal{C}_k^T x_{k+1} + \mathcal{A}_k^T u_{k+1}, \quad k \in J.$$

Simultaneously with (S) consider the discrete quadratic functional

$$\mathcal{F}(z) = \sum_{k=0}^{N} z_k^T \left\{ S_k^T \mathcal{K} S_k - \mathcal{K} \right\} z_k,$$

which can be rewritten into the form

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \left\{ x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{B}_k^T \mathcal{C}_k x_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \right\}.$$

For the LHdS the above quadratic functional $\mathcal{F}$ reduces to $\mathcal{F}_H$.

According to [**4**], we say that

(a) z satisfies the *boundary conditions* if $\mathcal{K} z_0 = 0 = \mathcal{K} z_{N+1}$, i.e., if $x_0 = 0 = x_{N+1}$;

(b) z is *admissible* if $\mathcal{K} z_{k+1} = \mathcal{K} S_k z_k$ on J, i.e., if $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ on J;

(c) x is *admissible* if there exists u such that $z = \binom{x}{u}$ is admissible;

(d) the discrete quadratic functional $\mathcal{F}$ is *positive definite* ($\mathcal{F} > 0$) if $\mathcal{F}(z) > 0$ for all nontrivial admissible z satisfying the boundary conditions, i.e., if $\mathcal{F}(x,u) > 0$ for all nontrivial admissible x with $x_0 = 0 = x_{N+1}$;

(e) a solution Z of (S) is a *conjoined basis* of (S) if $\operatorname{rank} Z = n$ and $Z^T \mathcal{J} Z = 0$ hold on J^*, i.e., if $\operatorname{rank}(X^T U^T) = n$ and $X^T U = U^T X$ on J^*;

(f) the solution Z of (S) is *principal* at $m \in J$ if $Z_m = \binom{0}{I}$, i.e., if $X_m = 0$ and $U_m = I$;

(g) a conjoined basis (X, U) has a *focal point* in the interval $(k, k+1]$, $k \in J$, if

$$\text{Ker}\, X_{k+1} \subseteq \text{Ker}\, X_k \quad \text{and} \quad P_k := X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0 \tag{4}$$

does not hold;

(h) the solution (x, u) of (S) has a *generalized zero* in $(k, k+1]$, $k \in J$, if

$$x_k \neq 0, \quad x_{k+1} \in \text{Im}\, \mathcal{B}_k \quad \text{and} \quad x_k^T \mathcal{B}_k^\dagger x_{k+1} \leq 0;$$

(i) the system (S) is *disconjugate* on J if no solution of (S) has more than one and no solution (x, u) of (S) with $x_0 = 0$ has any generalized zeros on J.

Lemma 2. *For any two matrices V and W we have*

$$\text{Ker}\, V \subseteq \text{Ker}\, W \quad \textit{iff} \quad W = W V^\dagger V \quad \textit{iff} \quad W^\dagger = V^\dagger V W^\dagger.$$

Proof. See [**2**] or [**3**, Remark 2(iii)]. □

Lemma 3. *Let $P_k = X_k X_{k+1}^\dagger \mathcal{B}_k$ for $k \in J$. If* $\text{Ker}\, X_{k+1} \subseteq \text{Ker}\, X_k$, *then*

$$P_k \quad \textit{is symmetric and} \quad \text{Ker}\, X_{k+1}^T \subseteq \text{Ker}\, \mathcal{B}_k^T.$$

Proof. See [**4**] for details. □

Remark 1. By Lemma 2 we have that if $\text{Ker}\, X_{k+1} \subseteq \text{Ker}\, X_k$, then $\mathcal{B}_k = X_{k+1} X_{k+1}^\dagger \mathcal{B}_k$ and $\mathcal{B}_k^\dagger = \mathcal{B}_k^\dagger X_{k+1} X_{k+1}^\dagger$.

The following Reid roundabout theorem for symplectic systems has been proved in [**4**].

Theorem 1. *The following statements are equivalent.*

(i) $\mathcal{F} > 0$;

(ii) *the system* (S) *is disconjugate on* J;

(iii) *the principal solution* $Z = (X, U)$ *of* (S) *has no focal points in* $(0, N+1]$.

The goal of this paper is to relate the condition (4) to a condition on a certain block tridiagonal $(N+1)n \times (N+1)n$-matrix. Namely, our Corollary 2 explains why matrices P_k appear in the definition of focal points for a conjoined basis of (S).

3. Main results. We proceed similarly as in [**5**]. In this section we always assume that (X, U) is the principal solution of (S) at 0, i.e., $X_0 = 0$ and $U_0 = I$.

For $m \in J$ we define $(m+1)n \times (m+1)n$-matrices $\mathcal{U}_m$ by $\mathcal{U}_0 := \mathcal{T}_0$ and for $1 \le m \le N$,

$$\mathcal{U}_m = \begin{pmatrix} \mathcal{T}_0 & \mathcal{S}_0 & & \\ \mathcal{S}_0^T & \mathcal{T}_1 & \ddots & \\ & \ddots & \ddots & \mathcal{S}_{m-1} \\ & & \mathcal{S}_{m-1}^T & \mathcal{T}_m \end{pmatrix},$$

where

$$\mathcal{T}_k = \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{A}_k^T \mathcal{C}_k + \mathcal{E}_{k-1} \quad \text{and} \quad \mathcal{S}_k = \mathcal{C}_k^T - \mathcal{A}_k^T \mathcal{E}_k, \quad k \in J, \tag{5}$$

with $\mathcal{E}_{-1} := 0$; the matrix $\mathcal{E}$ is any symmetric $n \times n$-matrix for which $\mathcal{B}^T \mathcal{E} \mathcal{B} = \mathcal{D}^T \mathcal{B}$ holds on J, for example, $\mathcal{B}\mathcal{B}^\dagger \mathcal{D} \mathcal{B}^\dagger$, $\mathcal{D}(\mathcal{D}^T \mathcal{B})^\dagger \mathcal{D}^T$, $(\mathcal{D}\mathcal{B}^\dagger/2) + ((\mathcal{D}\mathcal{B}^\dagger)^T/2)$ or any other. Note that $\mathcal{T}$, and hence $\mathcal{U}$, are symmetric.

Note also that, in contrast to [**5**], we employ $(N+1)n \times (N+1)n$-matrices $\mathcal{U}_N, \mathcal{M}_N$, cf., $Nn \times Nn$-matrices $\mathcal{L}, \mathcal{M}$ of [**5**], the space $\mathcal{V}$ of $(N+1)n$-vectors (cf. the space $\mathcal{A}$ of [**5**] consisting of Nn-vectors). The reason is that we include x_0 as the first entry of the elements of $\mathcal{V}$. Then the computations are, we believe, smoother.

Theorem 2. *Let (x,u) be admissible on J with $x_0 = 0 = x_{N+1}$. Then*

$$\mathcal{F}(x,u) = \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}^T \mathcal{U}_N \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}.$$

Proof. Let (x,u) be admissible, and let $x_0 = 0 = x_{N+1}$. Then $\mathcal{B}_k u_k = x_{k+1} - \mathcal{A}_k x_k$ holds on J, and so

$$\begin{aligned}
\mathcal{F}(x,u) &= \sum_{k=0}^{N} \big\{ x_k^T \mathcal{C}_k^T (\mathcal{A}_k x_k + \mathcal{B}_k u_k) \\
&\qquad + u_k^T \mathcal{B}_k^T \mathcal{C}_k x_k + u_k^T \mathcal{B}_k^T \mathcal{E}_k \mathcal{B}_k u_k \big\} \\
&= \sum_{k=0}^{N} \big\{ x_k^T \mathcal{C}_k^T x_{k+1} + (x_{k+1}^T - x_k^T \mathcal{A}_k^T) \mathcal{C}_k x_k \\
&\qquad + (x_{k+1}^T - x_k^T \mathcal{A}_k^T) \mathcal{E}_k (x_{k+1} - \mathcal{A}_k x_k) \big\} \\
&= \sum_{k=0}^{N} \big\{ x_k^T \mathcal{T}_k x_k + x_k^T \mathcal{S}_k x_{k+1} + x_{k+1}^T \mathcal{S}_k^T x_k \big\} \\
&\quad + x_{N+1}^T \mathcal{E}_N x_{N+1} \\
&= \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}^T \mathcal{U}_N \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}. \qquad \square
\end{aligned}$$

Let us introduce the space $\mathcal{V}$ of $(N+1)n$-vectors

$$\mathcal{V} := \left\{ \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} \text{ such that } x = \{x_k\}_{k=0}^{N+1} \text{ is admissible on } J \right.$$
$$\left. \text{with } x_0 = 0 = x_{N+1} \right\}.$$

Then we easily formulate a consequence of Theorem 2.

Corollary 1. $\mathcal{F} > 0$ *if and only if* $\mathcal{U}_N > 0$ *on* $\mathcal{V}$.

Remark 2. Note that $\mathcal{U}_N > 0$ on $\mathcal{V}$, i.e., $\chi^T \mathcal{U}_N \chi > 0$ for all $\chi \in \mathcal{V} \backslash \{0\}$, if and only if

$$\mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N \geq 0 \quad \text{and} \quad \operatorname{Ker} \mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N \subseteq \operatorname{Ker} \mathcal{M}_N \tag{6}$$

whenever $\mathcal{M}_N$ is a matrix with $\operatorname{Im} \mathcal{M}_N = \mathcal{V}$. We will construct such matrix $\mathcal{M}_N$ and show that (6) is equivalent to the condition given in (4).

Lemma 4. *Let* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *on* J. *Then, for all* $k \in J$, *we have*

$$\begin{aligned} X_{k+1}^T \mathcal{T}_{k+1} X_{k+1} = \Delta \left\{ X_k^T U_k + X_k^T [\mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{C}_k^T \mathcal{A}_k] X_k \right\} \\ - X_k^T \mathcal{S}_k X_{k+1} - X_{k+1}^T \mathcal{S}_k^T X_k. \end{aligned}$$

Proof. Let $k \in J$. Then

$$\begin{aligned} & X_{k+1}^T \mathcal{T}_{k+1} X_{k+1} + X_k^T \mathcal{S}_k X_{k+1} + X_{k+1}^T \mathcal{S}_k^T X_k \\ &= X_{k+1}^T \mathcal{A}_{k+1}^T \mathcal{E}_{k+1} \mathcal{A}_{k+1} X_{k+1} - X_{k+1}^T \mathcal{C}_{k+1}^T \mathcal{A}_{k+1} X_{k+1} \\ &\quad + X_{k+1}^T \mathcal{E}_k X_{k+1} + X_k^T \mathcal{C}_k^T X_{k+1} - X_k^T \mathcal{A}_k^T \mathcal{E}_k X_{k+1} \\ &\quad + X_{k+1}^T \mathcal{C}_k X_k - X_{k+1}^T \mathcal{E}_k \mathcal{A}_k X_k \\ &= X_{k+1}^T \mathcal{A}_{k+1}^T \mathcal{E}_{k+1} \mathcal{A}_{k+1} X_{k+1} - X_{k+1}^T \mathcal{C}_{k+1}^T \mathcal{A}_{k+1} X_{k+1} \\ &\quad + X_{k+1}^T \mathcal{E}_k X_{k+1} + X_k^T (\mathcal{A}_k^T U_{k+1} - U_k) - X_k^T \mathcal{A}_k^T \mathcal{E}_k (\mathcal{A}_k X_k + \mathcal{B}_k U_k) \\ &\quad + X_{k+1}^T (U_{k+1} - \mathcal{D}_k U_k) - X_{k+1}^T \mathcal{E}_k (X_{k+1} - \mathcal{B}_k U_k) \\ &= \Delta \left\{ X_k^T U_k + X_k^T \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k X_k \right\} + (X_{k+1} - \mathcal{A}_k X_k)^T \mathcal{E}_k \mathcal{B}_k U_k \\ &\quad - X_{k+1}^T \mathcal{C}_{k+1}^T \mathcal{A}_{k+1} X_{k+1} + X_k^T \mathcal{A}_k^T (\mathcal{C}_k X_k + \mathcal{D}_k U_k) - X_{k+1}^T \mathcal{D}_k U_k \\ &= \Delta \left\{ X_k^T U_k + X_k^T \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k X_k - X_k^T \mathcal{C}_k^T \mathcal{A}_k X_k \right\} \\ &\quad + U_k^T \mathcal{B}_k^T \mathcal{E}_k \mathcal{B}_k U_k + (\mathcal{A}_k X_k - X_{k+1})^T \mathcal{D}_k U_k \\ &= \Delta \left\{ X_k^T U_k + X_k^T [\mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{C}_k^T \mathcal{A}_k] X_k \right\}. \qquad \square \end{aligned}$$

Let us define matrices P_{ij} for $0 \leq i \leq j \leq N$ by

$$P_{ij} = X_i X_j^{\dagger} P_j,$$

where $P_j = X_j X_{j+1}^\dagger \mathcal{B}_j$. Then we have $P_{0j} = X_0 X_j^\dagger P_j = 0$, $P_{ii} = X_i X_i^\dagger P_i = X_i X_{i+1}^\dagger \mathcal{B}_i = P_i$ and if $\operatorname{Ker} X_j \subseteq \operatorname{Ker} X_i$, i.e., if $X_i = X_i X_j^\dagger X_j$, then

$$P_{ij} = X_i X_j^\dagger P_j = X_i X_j^\dagger X_j X_{j+1}^\dagger \mathcal{B}_j = X_i X_{j+1}^\dagger \mathcal{B}_j.$$

For $m \in J$ we define $(m+1)n \times (m+1)n$-matrices $\mathcal{M}_m$ by

$$\mathcal{M}_m = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ 0 & P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{mm} \end{pmatrix}.$$

Proposition 1 (Characterization of $\operatorname{Im} \mathcal{M}_N$). *If* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *holds on* J, *then* $\operatorname{Im} \mathcal{M}_N = \mathcal{V}$.

Proof. Let $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ on J. Let $\begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} \in \mathcal{V}$, i.e., $x_0 = 0 = x_{N+1}$. We put

$$c_0 := 0, \qquad c_{k+1} := c_k - X_{k+1}^\dagger \mathcal{B}_k (U_k c_k - u_k) \quad \text{for } k \in J,$$

where $u = \{u_k\}_{k \in J}$ is such that (x, u) is admissible on J. We will prove that there exists $\begin{pmatrix} d_0 \\ \vdots \\ d_N \end{pmatrix} \in \mathbf{R}^{(N+1)n}$ such that $\begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} = \mathcal{M}_N \begin{pmatrix} d_0 \\ \vdots \\ d_N \end{pmatrix}$, i.e., $\begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} \in \operatorname{Im} \mathcal{M}_N$.

We define $d_k := U_k c_k - u_k$ for $k \in J$. Then we have $X_0 c_0 = 0 = x_0$. Hence, by induction,

$$\begin{aligned} X_{k+1} c_{k+1} &= X_{k+1} c_k - X_{k+1} X_{k+1}^\dagger \mathcal{B}_k (U_k c_k - u_k) \\ &= (\mathcal{A}_k X_k + \mathcal{B}_k U_k) c_k - \mathcal{B}_k (U_k c_k - u_k) = \mathcal{A}_k X_k c_k + \mathcal{B}_k u_k \\ &= \mathcal{A}_k x_k + \mathcal{B}_k u_k = x_{k+1}. \end{aligned}$$

Thus $X_k c_k = x_k$ for all $k \in J^*$. Next we have for $j \in J$

$$P_j d_j = X_j X_{j+1}^\dagger \mathcal{B}_j (U_j c_j - u_j) = X_j (c_j - c_{j+1}) = -X_j \Delta c_j$$

and

$$P_{ij}d_j = X_iX_j^\dagger P_jd_j = -X_iX_j^\dagger X_j\Delta c_j \quad \text{for } 0 \le i \le j \le N.$$

Therefore, for $i \in J$,

$$\begin{aligned}\sum_{j=i}^{N} P_{ij}d_j &= \sum_{j=i}^{N}(-X_i\Delta c_j) = -X_i\sum_{j=i}^{N}\Delta c_j \\ &= -X_i(c_{N+1} - c_i) \\ &= -X_iX_{N+1}^\dagger X_{N+1}c_{N+1} + X_ic_i \\ &= -X_iX_{N+1}^\dagger x_{N+1} + x_i = x_i.\end{aligned}$$

Thus,

$$\mathcal{M}_N\begin{pmatrix} d_0 \\ \vdots \\ d_N\end{pmatrix} = \begin{pmatrix} P_{00} & \cdots & P_{0N} \\ & \ddots & \vdots \\ & & P_{NN}\end{pmatrix}\begin{pmatrix} d_0 \\ \vdots \\ d_N\end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_N\end{pmatrix}. \tag{7}$$

Conversely, let $\begin{pmatrix} x_0 \\ \vdots \\ x_N\end{pmatrix} \in \operatorname{Im}\mathcal{M}_N$ and put $x_{N+1} = 0$. There are $d_0, \dots, d_N \in \mathbf{R}^n$ satisfying (7), i.e., $x_i = \sum_{j=i}^{N} P_{ij}d_j$ for $i \in J$. Then $x_0 = 0$ and for $k \in J\backslash\{N\}$, we have

$$\begin{aligned}x_{k+1} - \mathcal{A}_kx_k &= \sum_{j=k+1}^{N} X_{k+1}X_j^\dagger P_jd_j - \mathcal{A}_k\sum_{j=k}^{N} X_kX_j^\dagger P_jd_j \\ &= (X_{k+1} - \mathcal{A}_kX_k)\sum_{j=k+1}^{N} X_j^\dagger P_jd_j - \mathcal{A}_kX_kX_k^\dagger P_kd_k \\ &= \mathcal{B}_kU_k\sum_{j=k+1}^{N} X_j^\dagger P_jd_j - \mathcal{A}_k\mathcal{B}_k^TX_{k+1}^{\dagger T}X_k^Td_k \\ &= \mathcal{B}_k\left[U_k\sum_{j=k+1}^{N} X_j^\dagger P_jd_j - \mathcal{A}_k^TX_{k+1}^{\dagger T}X_k^Td_k\right] \in \operatorname{Im}\mathcal{B}_k,\end{aligned}$$

and

$$x_{N+1} - \mathcal{A}_Nx_N = -\mathcal{A}_NP_{NN}d_N = -\mathcal{B}_N\mathcal{A}_N^TX_{N+1}^{\dagger T}X_N^Td_N \in \operatorname{Im}\mathcal{B}_N.$$

Thus x is admissible and $\begin{pmatrix} x_0 \\ \cdots \\ x_N \end{pmatrix} \in \mathcal{V}$. $\square$

For $m \in J$, define $(m+1)n\times n$-matrices $\mathcal{Q}_m, \mathcal{R}_m, \Omega_m$ and $n\times n$-matrix Λ_m by

$$\mathcal{Q}_m = \begin{pmatrix} X_0 \\ \vdots \\ X_m \end{pmatrix}, \qquad \mathcal{R}_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathcal{S}_m \end{pmatrix},$$

$$\Omega_m = \mathcal{M}_m^T \mathcal{U}_m \mathcal{Q}_m + \mathcal{M}_m^T \mathcal{R}_m X_{m+1},$$

$$\Lambda_m = \mathcal{Q}_m^T \mathcal{U}_m \mathcal{Q}_m + \mathcal{Q}_m^T \mathcal{R}_m X_{m+1} + X_{m+1}^T \mathcal{R}_m^T \mathcal{Q}_m + X_{m+1}^T \mathcal{T}_{m+1} X_{m+1}.$$

Lemma 5. *Let* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *on* J. *Then*

(i) $\Lambda_m = X_{m+1}^T\{U_{m+1} + [\mathcal{A}_{m+1}^T \mathcal{E}_{m+1} \mathcal{A}_{m+1} - \mathcal{C}_{m+1}^T \mathcal{A}_{m+1}] X_{m+1}\}$ *for* $m \in J$;

(ii) $P_{m+1} X_{m+1}^{\dagger T} \Lambda_m X_{m+1}^{\dagger} P_{m+1} = P_{m+1}$ *for* $m \in J\backslash\{N\}$;

(iii) $\Omega_m = 0$ *for* $m \in J\backslash\{N\}$.

Proof. See Appendix A. $\square$

The following statement is the key to our main result, Theorem 3.

Proposition 2. *Let* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *on* J. *Then, for any* $m \in J\backslash\{N\}$, *we have*

$$\mathcal{M}_{m+1}^T \mathcal{U}_{m+1} \mathcal{M}_{m+1} = \begin{pmatrix} \mathcal{M}_m^T \mathcal{U}_m \mathcal{M}_m & 0 \\ 0 & P_{m+1} \end{pmatrix}.$$

Proof. Let $m \in J$. Then we have

$$\mathcal{M}_{m+1} = \begin{pmatrix} P_{00} & \cdots & P_{0m} & P_{0m+1} \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & P_{mm} & P_{mm+1} \\ 0 & \cdots & 0 & P_{m+1m+1} \end{pmatrix} = \begin{pmatrix} \mathcal{M}_m & \mathcal{Q}_m X_{m+1}^{\dagger} P_{m+1} \\ 0 & X_{m+1} X_{m+1}^{\dagger} P_{m+1} \end{pmatrix}$$

and

$$\mathcal{U}_{m+1} = \begin{pmatrix} \mathcal{T}_0 & \mathcal{S}_0 & & & \\ \mathcal{S}_0^T & \mathcal{T}_1 & \ddots & & \\ & \ddots & \ddots & \mathcal{S}_{m-1} & \\ & & \mathcal{S}_{m-1}^T & \mathcal{T}_m & \mathcal{S}_m \\ & & & \mathcal{S}_m^T & \mathcal{T}_{m+1} \end{pmatrix} = \begin{pmatrix} \mathcal{U}_m & \mathcal{R}_m \\ \mathcal{R}_m^T & \mathcal{T}_{m+1} \end{pmatrix}.$$

Hence, by putting $\tilde{\mathcal{M}}_m = \begin{pmatrix} \mathcal{M}_m \\ 0 \end{pmatrix}$, we have

$$\mathcal{M}_{m+1} = \begin{pmatrix} \tilde{\mathcal{M}}_m & \mathcal{Q}_{m+1} X_{m+1}^{\dagger} P_{m+1} \end{pmatrix}.$$

Moreover,

$$\tilde{\mathcal{M}}_m^T \mathcal{U}_{m+1} \tilde{\mathcal{M}}_m = \begin{pmatrix} \mathcal{M}_m \\ 0 \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_m & \mathcal{R}_m \\ \mathcal{R}_m^T & \mathcal{T}_{m+1} \end{pmatrix} \begin{pmatrix} \mathcal{M}_m \\ 0 \end{pmatrix} = \mathcal{M}_m^T \mathcal{U}_m \mathcal{M}_m.$$

Therefore,

$$\begin{aligned} &\mathcal{M}_{m+1}^T \mathcal{U}_{m+1} \mathcal{M}_{m+1} \\ &= \begin{pmatrix} \tilde{\mathcal{M}}_m^T \\ P_{m+1} X_{m+1}^{\dagger T} \mathcal{Q}_{m+1}^T \end{pmatrix} \mathcal{U}_{m+1} \begin{pmatrix} \tilde{\mathcal{M}}_m & \mathcal{Q}_{m+1} X_{m+1}^{\dagger} P_{m+1} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\mathcal{M}}_m^T \mathcal{U}_{m+1} \tilde{\mathcal{M}}_m & \tilde{\mathcal{M}}_m^T \mathcal{U}_{m+1} \mathcal{Q}_{m+1} X_{m+1}^{\dagger} P_{m+1} \\ P_{m+1} X_{m+1}^{\dagger T} \mathcal{Q}_{m+1}^T \mathcal{U}_{m+1} & P_{m+1} X_{m+1}^{\dagger T} \mathcal{Q}_{m+1}^T \mathcal{U}_{m+1} \mathcal{Q}_{m+1} X_{m+1}^{\dagger} P_{m+1} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}_m^T \mathcal{U}_m \mathcal{M}_m & \Omega_m X_{m+1}^{\dagger} P_{m+1} \\ P_{m+1} X_{m+1}^{\dagger T} \Omega_m^T & P_{m+1} X_{m+1}^{\dagger T} \Lambda_m X_{m+1}^{\dagger} P_{m+1} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Omega_m &= \tilde{\mathcal{M}}_m^T \mathcal{U}_{m+1} \mathcal{Q}_{m+1} = \begin{pmatrix} \mathcal{M}_m^T & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U}_m & \mathcal{R}_m \\ \mathcal{R}_m^T & \mathcal{T}_{m+1} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_m \\ X_{m+1} \end{pmatrix} \\ &= \mathcal{M}_m^T \mathcal{U}_m \mathcal{Q}_m + \mathcal{M}_m^T \mathcal{R}_m X_{m+1}, \\ \Lambda_m &= \mathcal{Q}_{m+1}^T \mathcal{U}_{m+1} \mathcal{Q}_{m+1} = \begin{pmatrix} Q_m^T & X_{m+1}^T \end{pmatrix} \begin{pmatrix} \mathcal{U}_m & \mathcal{R}_m \\ \mathcal{R}_m^T & \mathcal{T}_{m+1} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_m \\ X_{m+1} \end{pmatrix} \\ &= \mathcal{Q}_m^T \mathcal{U}_m \mathcal{Q}_m + \mathcal{Q}_m^T \mathcal{R}_m X_{m+1} + X_{m+1}^T \mathcal{R}_m^T \mathcal{Q}_m + X_{m+1}^T \mathcal{T}_{m+1} X_{m+1}. \end{aligned}$$

Thus we are done if we prove that

$$\Omega_m = 0 \quad \text{and} \quad P_{m+1} X_{m+1}^{\dagger T} \Lambda_m X_{m+1}^{\dagger} P_{m+1} = P_{m+1},$$

which is a content of Lemma 5. $\square$

Corollary 2. *Let* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *on* J. *Then*

$$\mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N = \operatorname{diag}\{P_0, P_1, \dots, P_N\}.$$

Proof. By applying Proposition 2 for $m = N-1, \dots, 0$, we have

$$\mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N = \operatorname{diag}\{\mathcal{M}_0^T \mathcal{U}_0 \mathcal{M}_0, P_1, \dots, P_N\}.$$

However, $\mathcal{M}_0^T \mathcal{U}_0 \mathcal{M}_0 = P_0 \mathcal{T}_0 P_0 = 0 = P_0$, so the result follows. $\square$

Now we may state the main result of this paper, the theorem relating positivity of the discrete quadratic functional $\mathcal{F}$ to (among others) condition (4) but without using the discrete Picone's identity. We remind the reader that (X, U) is the principal solution of (S) at zero.

Theorem 3. *The following are equivalent.*

(i) $\mathcal{F} > 0$;

(ii) $\mathcal{U}_N > 0$ *on* $\mathcal{V}$;

(iii) $\operatorname{Ker} \mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N \subseteq \operatorname{Ker} \mathcal{M}_N$ *and* $\mathcal{M}_N^T \mathcal{U}_N \mathcal{M}_N \geq 0$;

(iv) $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *and* $P_k \geq 0$ *on* J, *i.e.,* (X, U) *has no focal points in* $(0, N+1]$.

Proof. It is a direct consequence of Corollary 1, Remark 2 and Corollary 2. $\square$

Remark 3. If $\mathcal{B}$ is nonsingular and if the system (S) is rewritten system (H), i.e., if both the matrix $\mathcal{A}$ of (S) and B of the Hamiltonian system are nonsingular, the so-called "regular case," then the above procedure gives the results of [**5**] noted in Section 1. However, for a general matrix $\mathcal{B}$ this cannot be expected since, in the case of LHdS, the matrix $\mathcal{A} = \tilde{A}$ is nonsingular and this fact is essential for the construction of the matrices $\bar{T}_k$ and $\bar{S}_k$ for system (H). It means that one cannot obtain $\bar{T}_k, \bar{S}_k$ as special cases of our $\mathcal{T}_k, \mathcal{S}_k$ in spite of the fact that the procedure for deriving them is in both cases the same. For, if $\mathcal{A}$ is nonsingular, then $\mathcal{D} = \mathcal{C}\mathcal{A}^{-1}\mathcal{B} + \mathcal{A}^{T-1}$, and the matrices $\mathcal{C}\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}\mathcal{B}$ are symmetric and (x, u) is admissible if and only if $x_k + \mathcal{A}_k^{-1}\mathcal{B}_k u_k = \mathcal{A}_k^{-1} x_{k+1}$. Then the quadratic functional $\mathcal{F}$ can be brought into the form from Theorem 2 with

$$\begin{aligned}
\bar{\mathcal{T}}_0 &= (\mathcal{A}_0^{-1}\mathcal{B}_0)^\dagger, \\
\bar{\mathcal{T}}_k &= \mathcal{C}_{k-1}\mathcal{A}_{k-1}^{-1} + \mathcal{A}_{k-1}^{T-1}(\mathcal{A}_{k-1}^{-1}\mathcal{B}_{k-1})^\dagger \mathcal{A}_{k-1}^{-1} \\
&\quad + (\mathcal{A}_k^{-1}\mathcal{B}_k)^\dagger \quad \text{for } 1 \le k \le N, \\
\bar{\mathcal{S}}_k &= -(\mathcal{A}_k^{-1}\mathcal{B}_k)^\dagger \mathcal{A}_k^{-1} \quad \text{for } k \in J,
\end{aligned}$$

which reduce to $\bar{T}_k$ and $\bar{S}_k$ when substituting $\mathcal{A} = \tilde{A}$, $\mathcal{B} = \tilde{A}B$ and $\mathcal{C} = C\tilde{A}$, cf. (1), although $\bar{\mathcal{T}}_k \neq \mathcal{T}_k$ and $\bar{\mathcal{S}}_k \neq \mathcal{S}_k$.

4. Reciprocal symplectic systems. The following transformation lemma is an easy consequence of Lemma 1.

Lemma 6. *Let R_k be a sequence of symplectic $n \times n$-matrices. Then the transformation $z = R\tilde{z}$ takes the symplectic system $z_{k+1} = S_k z_k$ into another symplectic system $\tilde{z}_{k+1} = \tilde{S}_k \tilde{z}_k$. Particularly, $\tilde{S}_k = R_{k+1}^{-1} S_k R_k$.*

The *reciprocal symplectic system* is the symplectic system

$$z_{k+1}^* = S_k^* z_k^* \tag{S*}$$

arising from (S) upon the transformation $z = \mathcal{J} z^*$, i.e., we have $S^* = \mathcal{J}^T S \mathcal{J} = S^{T-1}$. Thus,

$$z^* = \begin{pmatrix} -u \\ x \end{pmatrix}, \qquad S^* = \begin{pmatrix} \mathcal{D} & -\mathcal{C} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix}.$$

The corresponding quadratic functional $\mathcal{F}^*$ takes the form

$$\begin{aligned}\mathcal{F}^*(z^*) &= \sum_{k=0}^{N} z_k^{*T}\{S_k^{*T}\mathcal{K}S_k^* - \mathcal{K}\}z_k^* \\ &= \sum_{k=0}^{N} z_k^T \mathcal{J}\{S_k^{-1}\mathcal{K}S_k^{T-1} - \mathcal{K}\}\mathcal{J}^T z_k \\ &= \sum_{k=0}^{N} z_k^T\{\mathcal{K}^T - S_k^T\mathcal{K}^T S_k\}z_k \\ &= -\mathcal{F}(z).\end{aligned}$$

Reformulating the definitions from page 4, we get

(a) z^* satisfies the *boundary conditions* if $\mathcal{K}^T z_0 = 0 = \mathcal{K}^T z_{N+1}$, i.e., if $u_0 = 0 = u_{N+1}$;

(b) z^* is *admissible* if $\mathcal{K}^T z_{k+1} = \mathcal{K}^T S_k z_k$ on J, i.e., if $u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k$ on J;

(c) u is *admissible* if there exists an x such that $z^* = \begin{pmatrix} -u \\ x \end{pmatrix}$ is admissible;

(d) the solution Z^* of (S*) is *principal* at $m \in J$ if the solution $Z = (X, U)$ of (S) satisfies $Z_m = \begin{pmatrix} I \\ 0 \end{pmatrix}$, i.e., if $X_m = I$ and $U_m = 0$.

Let us define the matrices $\mathcal{T}_k^*$ and $\mathcal{S}_k^*$ by

$$\mathcal{T}_k^* := \mathcal{D}_k^T\mathcal{E}_k^*\mathcal{D}_k + \mathcal{D}_k^T\mathcal{B}_k + \mathcal{E}_{k-1}^* \quad \text{and} \quad \mathcal{S}_k^* := -\mathcal{B}_k^T - \mathcal{D}_k^T\mathcal{E}_k^*, \quad k \in J,$$

with $\mathcal{E}_{-1}^* := 0$; the matrix $\mathcal{E}^*$ is any symmetric $n \times n$-matrix satisfying $\mathcal{C}^T\mathcal{E}^*\mathcal{C} = -\mathcal{A}^T\mathcal{C}$, for example, $-\mathcal{C}\mathcal{C}^\dagger\mathcal{A}\mathcal{C}^\dagger$, $-\mathcal{A}(\mathcal{A}^T\mathcal{C})^\dagger\mathcal{A}^T$, $-(\mathcal{A}\mathcal{C}^\dagger/2) - ((\mathcal{A}\mathcal{C}^\dagger)^T/2)$.

In this section we always assume that (X, U) is the solution of (S) satisfying $X_0 = I$, $U_0 = 0$. Define the matrices P_k^* and P_{ij}^* by

$$P_k^* = U_k U_{k+1}^\dagger \mathcal{C}_k \quad \text{and} \quad P_{ij}^* = U_i U_j^\dagger P_j^*.$$

Define the matrices $\mathcal{U}^*$ and $\mathcal{M}^*$ in the analogous way as for the system

(S), i.e., all their entries with the superscript '$*$.' Let

$$\mathcal{V}^* := \left\{ \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix} \text{ such that } u = \{u_k\}_{k=0}^{N+1} \text{ is admissible on } J \right.$$
$$\left. \text{with } u_0 = 0 = u_{N+1} \right\}.$$

Then Theorem 3 for reciprocal symplectic systems reads as:

Theorem 4. *The following are equivalent.*

(i) $\mathcal{F}(x,u) < 0$ *for all* (x,u) *satisfying*

$$u \not\equiv 0, \qquad u_0 = 0 = u_{N+1} \quad \text{and} \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k;$$

(ii) $\mathcal{U}_N^* > 0$ *on* $\mathcal{V}^*$;

(iii) $\operatorname{Ker} \mathcal{M}_N^{*T} \mathcal{U}_N^* \mathcal{M}_N^* \subseteq \operatorname{Ker} \mathcal{M}_N^*$ *and* $\mathcal{M}_N^{*T} \mathcal{U}_N^* \mathcal{M}_N^* \geq 0$;

(iv) *The solution* (X,U) *of* (S) *with* $X_0 = I$, $U_0 = 0$, *satisfies*

$$\operatorname{Ker} U_{k+1} \subseteq \operatorname{Ker} U_k \quad \textit{and} \quad P_k^* = U_k U_{k+1}^{\dagger} \mathcal{C}_k \leq 0 \quad \textit{on } J.$$

Remark 4. The equivalence (i) $\Leftrightarrow$ (iv) is a part of the Reid roundabout theorem for reciprocal symplectic systems of [**4**].

Appendix

A. Proof of Lemma 5.

Proof. (i) First we have, by Lemma 4,

$$\Lambda_0 = \mathcal{Q}_1^T \mathcal{U}_1 \mathcal{Q}_1 = X_1^T \mathcal{T}_1 X_1 = X_1^T \left\{ U_1 + \left[\mathcal{A}_1^T \mathcal{E}_1 \mathcal{A}_1 - \mathcal{C}_1^T \mathcal{A}_1 \right] X_1 \right\}.$$

Hence, by induction, if (i) holds for $0 \leq k < m$, i.e., $1 \leq m \leq N$, then

$$\begin{aligned}
\Lambda_m &= \Lambda_{m-1} + X_{m+1}^T \mathcal{T}_{m+1} X_{m+1} + \mathcal{Q}_m^T \mathcal{R}_m X_{m+1} + X_{m+1}^T \mathcal{R}_m^T \mathcal{Q}_m \\
&= \Lambda_{m-1} + X_{m+1}^T \mathcal{T}_{m+1} X_{m+1} + X_m^T \mathcal{S}_m X_{m+1} + X_{m+1}^T \mathcal{S}_m^T X_m \\
&\overset{\text{Lemma 4}}{=} X_m^T \left\{ U_m + \left[\mathcal{A}_m^T \mathcal{E}_m \mathcal{A}_m - \mathcal{C}_m^T \mathcal{A}_m \right] X_m \right\} \\
&\quad + \Delta \left\{ X_m^T U_m + X_m^T \left[\mathcal{A}_m^T \mathcal{E}_m \mathcal{A}_m - \mathcal{C}_m^T \mathcal{A}_m \right] X_m \right\} \\
&= X_{m+1}^T \left\{ U_{m+1} + \left[\mathcal{A}_{m+1}^T \mathcal{E}_{m+1} \mathcal{A}_{m+1} - \mathcal{C}_{m+1}^T \mathcal{A}_{m+1} \right] X_{m+1} \right\};
\end{aligned}$$

(ii) Let $m \in J \backslash \{N\}$. In the first step we show that

$$\begin{aligned}
&P_{m+1} \left[\mathcal{A}_{m+1}^T \mathcal{E}_{m+1} \mathcal{A}_{m+1} - \mathcal{C}_{m+1}^T \mathcal{A}_{m+1} \right] P_{m+1} \\
&\qquad\qquad = X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} X_{m+1} X_{m+1}^\dagger P_{m+1}.
\end{aligned}$$

We have

$$\begin{aligned}
&P_{m+1} \left[\mathcal{A}_{m+1}^T \mathcal{E}_{m+1} \mathcal{A}_{m+1} - \mathcal{C}_{m+1}^T \mathcal{A}_{m+1} \right] P_{m+1} \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{B}_{m+1} \mathcal{A}_{m+1}^T \mathcal{E}_{m+1} \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T X_{m+2}^{\dagger T} X_{m+1}^T \\
&\quad - P_{m+1} \mathcal{C}_{m+1}^T \mathcal{A}_{m+1} P_{m+1} \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T \mathcal{E}_{m+1} \mathcal{B}_{m+1} \mathcal{A}_{m+1}^T X_{m+2}^{\dagger T} X_{m+1}^T \\
&\quad - X_{m+1} X_{m+2}^\dagger \mathcal{B}_{m+1} \mathcal{A}_{m+1}^T \mathcal{C}_{m+1} P_{m+1} \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T \mathcal{D}_{m+1} \mathcal{A}_{m+1}^T X_{m+2}^{\dagger T} X_{m+1}^T \\
&\quad - X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T \mathcal{C}_{m+1} \mathcal{B}_{m+1}^T X_{m+2}^{\dagger T} X_{m+1}^T \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T \left(\mathcal{D}_{m+1} \mathcal{A}_{m+1}^T - \mathcal{C}_{m+1} \mathcal{B}_{m+1}^T \right) X_{m+2}^{\dagger T} X_{m+1}^T \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} \mathcal{B}_{m+1}^T X_{m+2}^{\dagger T} X_{m+1}^T = X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} P_{m+1} \\
&= X_{m+1} X_{m+2}^\dagger \mathcal{A}_{m+1} X_{m+1} X_{m+1}^\dagger P_{m+1}.
\end{aligned}$$

Thus, by (i) and by the first step

$$
\begin{aligned}
P_{m+1}X_{m+1}^{\dagger T}\Lambda_m X_{m+1}^{\dagger}P_{m+1}& \\
&= P_{m+1}X_{m+1}^{\dagger T}X_{m+1}^{T}U_{m+1}X_{m+1}^{\dagger}P_{m+1} + P_{m+1}X_{m+1}^{\dagger T}X_{m+1}^{T} \\
&\quad \cdot \left[\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}\mathcal{A}_{m+1} - \mathcal{C}_{m+1}^{T}\mathcal{A}_{m+1}\right] X_{m+1}X_{m+1}^{\dagger}P_{m+1} \\
&= P_{m+1}U_{m+1}X_{m+1}^{\dagger}P_{m+1} + P_{m+1} \\
&\quad \cdot \left[\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}\mathcal{A}_{m+1} - \mathcal{C}_{m+1}^{T}\mathcal{A}_{m+1}\right] P_{m+1} \\
&= X_{m+1}X_{m+2}^{\dagger}\mathcal{B}_{m+1}U_{m+1}X_{m+1}^{\dagger}P_{m+1} \\
&\quad + X_{m+1}X_{m+2}^{\dagger}\mathcal{A}_{m+1}X_{m+1}X_{m+1}^{\dagger}P_{m+1} \\
&= X_{m+1}X_{m+2}^{\dagger}(\mathcal{B}_{m+1}U_{m+1} + \mathcal{A}_{m+1}X_{m+1})X_{m+1}^{\dagger}P_{m+1} \\
&= X_{m+1}X_{m+2}^{\dagger}X_{m+2}X_{m+1}^{\dagger}P_{m+1} \\
&= X_{m+1}X_{m+1}^{\dagger}P_{m+1} = P_{m+1}.
\end{aligned}
$$

(iii) First we have

$$\Omega_0 = \mathcal{M}_0^{T}\mathcal{U}_0\mathcal{Q}_0 + \mathcal{M}_0^{T}\mathcal{R}_0X_1 = P_{00}\mathcal{T}_0X_0 + P_{00}\mathcal{S}_0X_1 = 0.$$

Hence, by induction, if $\Omega_m = 0$ for some $0 \le m \le N-1$, then

$$
\begin{aligned}
&\Omega_{m+1} \\
&= \tilde{\mathcal{M}}_{m+1}^{T}\mathcal{U}_{m+2}\mathcal{Q}_{m+2} \\
&= \begin{pmatrix} \mathcal{M}_{m+1} \\ 0 \end{pmatrix}^{T} \begin{pmatrix} \mathcal{U}_{m+1} & \mathcal{R}_{m+1} \\ \mathcal{R}_{m+1}^{T} & \mathcal{T}_{m+1} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_{m+1} \\ X_{m+2} \end{pmatrix} \\
&= \mathcal{M}_{m+1}^{T}(\mathcal{U}_{m+1}\mathcal{Q}_{m+1} + \mathcal{R}_{m+1}X_{m+2}) \\
&= \begin{pmatrix} \mathcal{M}_m & \mathcal{Q}_m X_{m+1}^{\dagger}P_{m+1} \\ 0 & X_{m+1}X_{m+1}^{\dagger}P_{m+1} \end{pmatrix}^{T} (\mathcal{U}_{m+1}\mathcal{Q}_{m+1} + \mathcal{R}_{m+1}X_{m+2}) \\
&= \begin{pmatrix} \tilde{\mathcal{M}}_m^{T} \\ P_{m+1}X_{m+1}^{\dagger T}\mathcal{Q}_{m+1}^{T} \end{pmatrix} (\mathcal{U}_{m+1}\mathcal{Q}_{m+1} + \mathcal{R}_{m+1}X_{m+2}) \\
&= \begin{pmatrix} \tilde{\mathcal{M}}_m^{T}\mathcal{U}_{m+1}\mathcal{Q}_{m+1} + \tilde{\mathcal{M}}_m^{T}\mathcal{R}_{m+1}X_{m+2} \\ P_{m+1}X_{m+1}^{\dagger T}\mathcal{Q}_{m+1}^{T}\mathcal{U}_{m+1}\mathcal{Q}_{m+1} + P_{m+1}X_{m+1}^{\dagger T}\mathcal{Q}_{m+1}^{T}\mathcal{R}_{m+1}X_{m+2} \end{pmatrix} \\
&= \begin{pmatrix} \Omega_m + 0 \\ P_{m+1}X_{m+1}^{\dagger T}\Lambda_m + P_{m+1}X_{m+1}^{\dagger T}X_{m+1}^{T}\mathcal{S}_{m+1}X_{m+2} \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} 0 \\ P_{m+1}X_{m+1}^{\dagger T}\Lambda_m + P_{m+1}\mathcal{S}_{m+1}X_{m+2} \end{pmatrix}.$$

Thus we are done if we prove that $P_{m+1}X_{m+1}^{\dagger T}\Lambda_m + P_{m+1}\mathcal{S}_{m+1}X_{m+2} = 0$. By part (i) we have

$$\begin{aligned}
&P_{m+1}X_{m+1}^{\dagger T}\Lambda_m + P_{m+1}\mathcal{S}_{m+1}X_{m+2} \\
&= P_{m+1}X_{m+1}^{\dagger T}X_{m+1}^{T}\{U_{m+1} + [\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}\mathcal{A}_{m+1} - \mathcal{C}_{m+1}^{T}\mathcal{A}_{m+1}]X_{m+1}\} \\
&\quad + P_{m+1}[\mathcal{C}_{m+1}^{T} - \mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}]X_{m+2} \\
&= P_{m+1}U_{m+1} + P_{m+1}\mathcal{A}_{m}^{T}\mathcal{E}_{m+1}\mathcal{A}_{m+1}X_{m+1} \\
&\quad - P_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{C}_{m+1}X_{m+1} \\
&\quad + P_{m+1}\mathcal{C}_{m+1}^{T}X_{m+2} - P_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}X_{m+2} \\
&= P_{m+1}U_{m+1} + P_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}(\mathcal{A}_{m+1}X_{m+1} - X_{m+2}) \\
&\quad - P_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{C}_{m+1}X_{m+1} + P_{m+1}(\mathcal{A}_{m+1}^{T}U_{m+2} - U_{m+1}) \\
&= -X_{m+1}X_{m+2}^{\dagger}\mathcal{B}_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{E}_{m+1}\mathcal{B}_{m+1}U_{m+1} \\
&\quad + P_{m+1}\mathcal{A}_{m+1}^{T}(U_{m+2} - \mathcal{C}_{m+1}X_{m+1}) \\
&= -X_{m+1}X_{m+2}^{\dagger}\mathcal{A}_{m+1}\mathcal{B}_{m+1}^{T}\mathcal{D}_{m+1}U_{m+1} \\
&\quad + X_{m+1}X_{m+2}^{\dagger}\mathcal{B}_{m+1}\mathcal{A}_{m+1}^{T}\mathcal{D}_{m+1}U_{m+1} = 0.
\end{aligned}$$

The proof is now complete. □

REFERENCES

1. C.D. Ahlbrandt and A.C. Peterson, *Discrete Hamiltonian systems*: *Difference equations, continued fractions and Riccati equations*, Kluwer Academic Publishers, Boston, 1996.

2. A. Ben-Israel and T.N.E. Greville, *Generalized inverses*: *Theory and applications*, John Wiley & Sons, Inc., New York, 1974.

3. M. Bohner, *Linear Hamiltonian difference systems*: *Disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl. **199** (1996), 804–826.

4. M. Bohner and O. Došlý, *Disconjugacy and transformations for symplectic systems*, Rocky Mountain J. Math. **27** (1997), 707–743.

5. ———, *Positivity of block tridiagonal matrices*, SIAM J. Matrix Anal. **20** (1998), 182–195.

Department of Mathematics, Faculty of Science, Masaryk University Brno, Janáčkovo nám. 2A, CZ-66295 Brno, Czech Republic
E-mail address: houska@math.muni.cz