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ON *-BANDS AND THEIR VARIETIES

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ABSTRACT. A *-band is a semigroup with a unary operation * obeying the axioms $(xy)^* = y^*x^*$, $x^{**} = x$, $x = xx^*x$, $x^2 = x$. On a free involutorial semigroup F on a nonempty set X, we define a family of operators δ_{t_n} and prove that each of them is a *-homomorphism of F onto its image with a suitable multiplication and the *-operation of F. We then investigate the interplay of this operator with several others occurring in the literature as well as the relationship of the equivalence relations they induce on F or on X^+. In particular, we obtain the structural description of all relatively free *-bands. We conclude with a brief consideration of the problem of converting *-identities to equivalent star-free identities.

1. Introduction and summary. A *-band is a semigroup S together with a unary operation * satisfying the axioms:

$$(1)\qquad (xy)^* = y^*x^*, \qquad x^{**} = x, \qquad x = xx^*x, \qquad x^2 = x.$$

By the first two axioms, * is an involution, the third axiom makes it "regular," and by the fourth, S is a band (idempotent semigroup). The class of all *-bands thus forms a variety of algebras whose members are pairs $(S, *)$ where S and * satisfy the above axioms.

Adair [**1**] determined the lattice of all $*$-band varieties and provided bases for identities for each variety. We characterized in [**10**] relatively free *-bands, namely free objects in each *-band variety. This was achieved by using the result in [**7**] that *-band varieties admit as bases for their identities the system devised in [**5**] for join irreducible band varieties. To this end, an operator γ_{t_n} on the free involutorial semigroup was defined which induces a congruence that solves the word problem for the relevant free object.

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This provides rudimentary information concerning *-band varieties but much remains to be done. We defined in [**8**] and [**9**] operators τ_{pq} and θ_{pq} which were used for solving the word problem for relatively free bands. An analogue of the latter was used in [**10**] for solving the word problem for relatively free *-bands. Hence there remains an analogue of the former operator and the accompanying deliberations. The interesting interplay among all the operators devised in [**8**], [**9**] and [**10**] also remains to be investigated.

In Section 2 we provide a few concepts and symbols needed throughout. Section 3 consists of auxiliary results needed later. We consider in Section 4 an operator on the free involutorial semigroup on a nonempty set which produces a copy of a free object in most *-band varieties. The free objects constructed in Section 4 and in [**10**] mentioned above are faithfully represented in Section 5 by a construction of *-bands from a collection of rectangular bands indexed by a semilattice. These results are unified in Section 6. Section 7 consists of a study of the mutual relationship of the operators t_n and γ_{t_n} introduced in [**10**]. This study was initiated in Section 5 and is continued here and in Section 8 in the context of congruences they induce on the free involutorial semigroup. Section 9 contains considerations which are aimed at conversion of an identity with stars to an equivalent one which is star-free.

This paper completes the cycle of articles [**8**], [**9**], [**10**] concerning relatively free (*)-bands.

2. Notation and terminology. For these, we follow the standard texts in semigroups and [**5**] with the following supplements.

If Y is a set, $|Y|$ stands for the cardinality of Y. We fix a nonempty set X and consider a bijection $x \to x^*$ of X onto a disjoint copy X^* of X. Let $I = X \cup X^*$ and F be the free semigroup on I which consists of all nonempty words over the alphabet I. We may view F as an involutory semigroup by defining

$$(x^*)^* = x, \qquad (y_1 \dots y_n)^* = y_n^* \dots y_1^*$$

for all $x \in X$, $n \geq 2$ and $y_1, \dots, y_n \in I$. We denote by F^1 the free monoid on I obtained by adjoining the empty word 1 to F.

We refer to a homomorphism of *-bands as a *-homomorphism and the induced congruence as a *-congruence. In this terminology, the terms homomorphism and congruence refer to multiplication alone. We denote by

End(F) – the set of all *-endomorphisms of F,

$\hat{\varphi}$ – the equivalence relation induced by a function φ,

ρ^c – the *-congruence generated by a relation ρ.

For a *-band identity $u = v$, we denote by $[u = v]$ the *-band variety determined by $u = v$. We omit the covering identities (1).

Now let $w \in F$; following [**10**], we define:

$c(w)$ – the set of all letters $x \in X$ such that either x or x^* occurs in w, $c(1) = \varnothing$ (in [**3**] the notation $c_X(w)$ is used),

$\sharp(w) = |c(w)|$,

u – a *prefix* of w if $w = uv$ for some $v \in F^1$,

$\bar{w}$ – the word obtained from w by reversing the order of letters, that is, if $w = x_1x_2 \ldots x_n$, then $\bar{w} = x_n \ldots x_2x_1$, $\bar{1} = 1$,

$s(w)$ and $\sigma(w)$ – there is a unique factorization $w = uyv$ with $u, v \in F^1$, $y \in I$ and $c(u) \subset c(uy) = c(w)$; we write $s(w) = u$ and $\sigma(w) = y$ (in [**3**] the notation $s_X(w)$ and $\sigma_X(w)$ is used),

$e(w)$ – the left-right dual of $s(w)$,

$\varepsilon(w)$ – the left-right dual of $\sigma(w)$.

For any operator t on F, we set $t(1) = 1$ thereby extending it to F^1, and define operators $\bar{t}$ and t^* on F by

$$\bar{t}(w) = \overline{t(\bar{w})}, \qquad t^*(w) = (t(w^*))^*.$$

For any $w \in F$, we have $\bar{\bar{w}} = w^{**} = w$ and thus $\bar{\bar{t}} = t^{**} = t$.

Let $w \in F$. If $w = yz$ with $y \in I$ and $z \in F^1$, we write $h_2(w) = y$. The operator i_2 is defined on F inductively on $\sharp(w)$ by the formula

$$i_2(w) = i_2 s(w)\sigma(w).$$

Hence $i_2(w)$ is the word obtained from w by retaining only the first occurrence of each letter regarding x and x^* as the same letter.

The next set of operators is also defined on F inductively: for $t \in \{h, i\}$ and $n > 2$, let

$$t_n(w) = t_n s(w)\sigma(w)\overline{t_{n-1}}(w). \tag{2}$$

It is important to note that this formula harbors two inductions: one is on n and the other is on $\sharp(w)$. For $\sharp s(w) = \sharp(w) - 1$ unless $w = 1$. The proofs will generally be by (primary) induction on n and occasionally, for the first and/or the inductive step, also by (secondary) induction on $\sharp(w)$.

In several proofs by induction, the following notation will come in handy:

$$i_1(w) = 1, \quad w \in F; \qquad \chi_n = h_{n+1}, \quad n \geq 1. \tag{3}$$

This device will make it possible to start the induction process at $n = 1$. The case $\chi_1 = h_2$ is generally easy to check while the instance i_1 usually holds trivially. Observe that the inductive formula (2) remains valid for operators χ_n and i_n for $n > 1$.

For $t \in \{\chi, i\}$ and $n \geq 3$, in [**10**, Section 4] we defined an operator γ_{t_n} on F by

$$\gamma_{t_n}(w) = t_{n-1}s(w)\sigma(w)\varepsilon(w)\overline{t_{n-1}}e(w).$$

Under the same circumstances, we now define an operator δ_{t_n} by

$$\delta_{t_n}(w) = t_{n-1}(w)\overline{t_{n-1}}(w).$$

These operators are akin, respectively, to θ_{pq} and τ_{pq} defined in [**9**, Section 3] and [**8**, Section 5], but note the technical difference that both γ_{t_n} and δ_{t_n} are defined by means of t_{n-1} and not of t_n. Also let

$$\Gamma = \{\gamma_{t_n} \mid t \in \{\chi, i\}, n \geq 3\}, \qquad \Delta = \{\delta_{t_n} \mid t \in \{\chi, i\}, n \geq 3\}.$$

The following system of words was introduced in [**5**]:

$$G_2 = x_2x_1, \qquad H_2 = x_2, \qquad I_2 = x_2x_1x_2$$

and, for $n > 2$, defined inductively

$$G_n = x_n\overline{G_{n-1}}, \qquad T_n = G_nx_n\overline{T_{n-1}}, \quad T \in \{H, I\}.$$

We assume that $t = h$ if and only if $T = H$ and $t = i$ if and only if $T = I$.

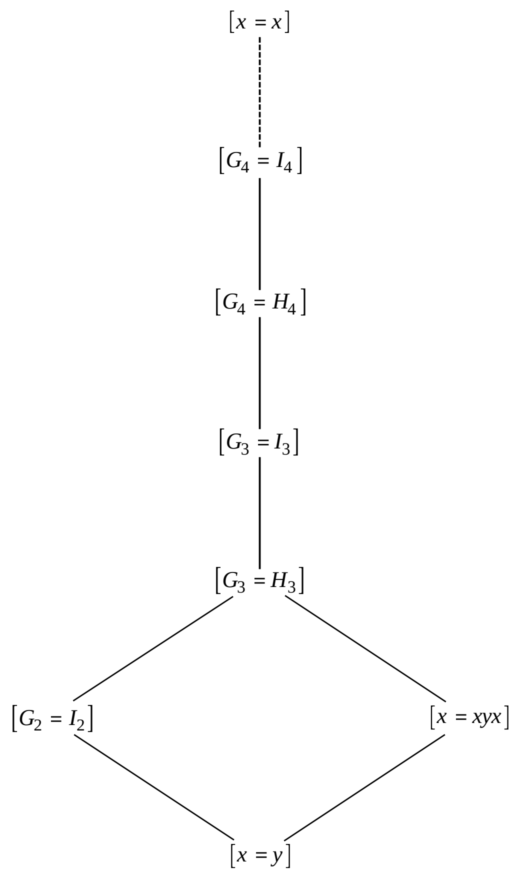


Diagram 1.

The lattice of all *-band varieties, depicted in Diagram 1, was determined in [**1**]. That the bases for these varieties are as shown in the diagram was proved in [**7**].

The bottom five varieties are usually named as follows.

- $[G_3 = I_3]$ – *regular* *-bands,
- $[G_3 = H_3]$ – *normal* *-bands,
- $[G_2 = I_2]$ – *-*semilattices*,

- $[x = xyx]$ – *rectangular* *-bands,
- $[x = y]$ – *trivial* *-bands.

3. Preliminaries. We start with the following simple statement.

Lemma 3.1. *Let f and g be operators on F. Then $\bar{f}\bar{g} = \overline{fg}$ and $f^*g^* = (fg)^*$.*

Proof. For any $w \in F$, we have

$$\overline{fg}(w) = \overline{fg(\bar{w})} = \overline{f(\overline{\bar{g}(w)})} = \bar{f}\bar{g}(w).$$

The same argument is valid for the star. □

We now state several auxiliary results taken from paper [**10**] except the last one which is taken from [**3**]. With the exception of Lemma 3.12 and Theorem 3.14, in the remainder of this section we assume that $t \in \{\chi, i\}$.

Lemma 3.2. *We have $\bar{s} = s^* = e$ and $\bar{\sigma} = \sigma^* = \varepsilon$.*

Lemma 3.3. *For $n \geq 2$ and $w \in F$, we have*

$$\overline{t_n}(w) = t_{n-1}(w)\varepsilon(w)\overline{t_n}e(w).$$

Lemma 3.4. *For $n \geq 2$, we have $ct_n = c$, $st_n = t_n s$, $\sigma t_n = \sigma$.*

Lemma 3.5. *For $n \geq 2$ and $w = uyv$ where $u, v \in F^1$, $y \in I$ and $c(y) \cap c(u) = \varnothing$, we have $t_n(w) \in t_n(u)yF^1$.*

Lemma 3.6. *For $n \geq 1$, we have $\overline{t_n} = t_n^*$.*

We thus could either use $\overline{t_n}$ or t_n^* throughout; our choice is $\bar{t}$ for typographical reasons.

Lemma 3.7. *For $m \geq n \geq 1$ and $u, v, w \in F^1$, we have*

$$t_n(u\,t_m(v)w) = t_n(u\,\overline{t_{m+1}}(v)w) = t_n(uvw).$$

Lemma 3.8. *If $\varphi \in \mathrm{End}\,(F)$, then $\bar{\varphi} \in \mathrm{End}\,(F)$.*

Lemma 3.9. *Let $n \geq 1$, $u, v, w, z \in F^1$ and $\varphi \in \mathrm{End}\,(F)$ be such that $t_n(u) = t_n(v)$. Then $t_n(w\,\varphi(u)z) = t_n(w\,\varphi(v)z)$.*

Lemma 3.10. *For $\gamma \in \Gamma$, we have $\gamma = \bar{\gamma} = \gamma^* = \gamma^2$.*

Lemma 3.11. *For $1 \leq n < m \geq 3$ and $u, v, w \in F^1$, we have*

$$t_n(u\,\gamma_{t_m}(v)w) = t_n(uvw).$$

Lemma 3.12. *For $t \in \{h, i\}$ and $n \geq 2$, we have*

$$t_n(G_n) = t_n(T_n), \qquad t_n(\overline{G_{n+1}}) = t_n(\overline{T_{n+1}}).$$

Lemma 3.13. *For $n \geq 3$, we have*

$$\gamma_{\chi_n}(\overline{G_n}) \neq \gamma_{\chi_n}(\overline{I_n}), \qquad \gamma_{i_n}(\overline{G_n}) \neq \gamma_{i_n}(\overline{H_n}).$$

Theorem 3.14. *Let X be a nonempty set and let $\gamma \in \Gamma$, say $\gamma = \gamma_{t_n}$, $t \in \{h, i\}$. Let $\mathcal{V} = [G_n = T_n]$. On the set $\gamma(F)$ define a multiplication by $u \star v = \gamma(uv)$ and consider the unary operation on F restricted to $\gamma(F)$. Then γ is a *-homomorphism of F onto $\gamma(F)$ which induces the least $\mathcal{V}$-congruence on F. Therefore $\gamma(F)$ is a $\mathcal{V}$-free *-band on X.*

Define an operator b on F inductively on $\sharp(w)$ by

$$b(w) = bs(w)\sigma(w)[bs(w^*)\sigma(w^*)]^* \tag{4}$$

(this is taken from [**3**, Section 4], where b, s and σ are denoted by b^*, s_X and σ_X, respectively). Let $\mathcal{B}$ be the variety of *-bands. On the set $b(F)$ define a multiplication by $u \star v = b(uv)$ and consider the unary operation on F restricted to $b(F)$.

Theorem 3.15. *Let X be a nonempty set. Then b is a *-homomorphism of F onto $b(F)$ which induces the least $\mathcal{B}$-congruence on F. Therefore $b(F)$ is a free *-band on X.*

4. Operators δ_{t_n}. We shall prove that δ_{t_n}, defined in Section 2, is a *-homomorphism of F onto its range on which a suitable multiplication is defined. This is preceded by a sequence of lemmas which may be compared to some concerning $\gamma \in \Gamma$. The first lemma is an analogue of a part of Lemma 3.10.

Lemma 4.1. *For $\delta \in \Delta$, we have $\delta = \bar{\bar{\delta}} = \delta^*$.*

Proof. Let $\delta = \delta_{t_n}$ and $w \in F$. Then

$$\begin{aligned}\bar{\bar{\delta}}(w) &= \overline{\bar{\delta}(\bar{w})} = \overline{t_{n-1}(\bar{w})\,\overline{t_{n-1}}(\bar{w})} = \overline{\overline{t_{n-1}}(\bar{w})}\,\overline{t_{n-1}(\bar{w})} \\ &= t_{n-1}(w)\,\overline{t_{n-1}}(w) = \delta(w)\end{aligned}$$

so that $\bar{\bar{\delta}} = \delta$; the argument for the star is the same. $\square$

The first part of the second lemma is an analogue of Lemma 3.11.

Lemma 4.2. *For $t \in \{\chi, i\}$, $1 \leq n < m \geq 3$ and $u, v, w \in F^1$, we have*

$$t_n(u\,\delta_{t_m}(v)w) = t_n(uvw), \qquad \overline{t_n}(u\,\delta_{t_m}(v)w) = \overline{t_n}(uvw).$$

Proof. First we note that

$$\begin{aligned} t_n(u\,\delta_{t_m}(v)w) = t_n(uvw) \quad &\Longleftrightarrow \quad \overline{t_n(u\,\delta t_m(v)w)} \;=\; \overline{t_n(uvw)} \\ &\Longleftrightarrow \quad \overline{t_n}\,\overline{(u\,\delta_{t_m}(v)w)} \;=\; \overline{t_n}\,(\overline{uvw}) \\ &\Longleftrightarrow \quad \overline{t_n}\,(\bar{w}\,\overline{\delta_{t_m}}(\bar{v})\bar{u}) \;=\; \overline{t_n}\,(\bar{w}\bar{v}\bar{u}) \\ &\Longleftrightarrow \quad \overline{t_n}\,(\bar{w}\,\delta_{t_m}(\bar{v})\bar{u}) \;=\; \overline{t_n}\,(\bar{w}\bar{v}\bar{u})\end{aligned}$$

by Lemma 4.1 and thus

$$(5) \quad t_n(u\,\delta_{t_m}(v)w) \;=\; t_n(uvw) \quad\Longleftrightarrow\quad \overline{t_n}(\bar{w}\,\delta_{t_m}(\bar{v})\bar{u}) \;=\; \overline{t_n}(\bar{w}\bar{v}\bar{u}).$$

The case $n = 1$ is trivial for $t = i$ and follows easily from Lemma 3.7 for $t = \chi$, hence we may assume that $n > 1$. By (5), we only need to show that $t_n(u\,\delta_{t_m}(v)w) = t_n(uvw)$. Consider first the case $m > n+1$. We wish to prove that $t_n(u\,t_{m-1}(v)\overline{t_{m-1}}(v)w) = t_n(uvw)$. Double application of Lemma 3.7 yields the desired equality.

Hence it remains to show that

$$(6) \qquad t_n(u\,\delta_{t_{n+1}}(v)w) = t_n(uvw).$$

By (5) and Lemma 3.7, we have

$$(7) \qquad \overline{t_{n-1}}\,(u\,\delta_{t_{n+1}}(v)w) = \overline{t_{n-1}}\,(uvw).$$

Next we use induction on $d = |c(w) \setminus c(uv)|$.

Let $d = 0$, that is, $c(w) \subseteq c(uv)$. In view of (7), we only have to show that $p(u\delta_{t_{n+1}}(v)w) = p(uvw)$ for $p \in \{t_n s, \sigma\}$. Note that $t_n^2 = t_n$ by Lemma 3.7 and so $pt_n = p$ by Lemma 3.4. Thus,

$$\begin{aligned} p(u\,\delta_{t_{n+1}}(v)w) &= p(u\,t_n(v)\overline{t_n}(v)w) = p(u\,t_n(v)) && \text{since } d = 0 \\ &= pt_n(u\,t_n(v)) = pt_n(uv) && \text{by Lemma 3.7} \\ &= p(uv) = p(uvw) && \text{since } d = 0. \end{aligned}$$

Now let $d > 0$ and assume that (6) holds for all values smaller than d. We can write $w = zxr$ with $s(uvw) = uvz$ and $\sigma(uvw) = x$. We have

$$\begin{aligned} t_n s(u\,\delta_{t_{n+1}}(v)w) &= t_n(u\,\delta_{t_{n+1}}(v)z) && \text{since } c\delta_{t_{n+1}}(v) = c(v) \\ &= t_n(uvz) && \text{by induction on } d \\ &= t_n s(uvw), \end{aligned}$$

and $c\delta_{t_{n+1}}(v) = c(v)$ yields also

$$\sigma(u\,\delta_{t_{n+1}}(v)w) = x = \sigma(uvw).$$

By (7) it follows that $t_n(u\,\delta_{t_{n+1}}(v)w) = t_n(uvw)$. $\square$

The third lemma is an analogue of [**10**, Lemma 4.3].

Lemma 4.3. *For $t \in \{\chi, i\}$, $3 \le n \le m$ and $u, v, w \in F^1$, we have*

$$\delta_{t_n}(u\,\delta_{t_m}(v)w) = \delta_{t_n}(uvw).$$

Proof. This follows immediately from Lemma 4.2. $\square$

The fourth lemma together with the first lemma completes the analogue of Lemma 3.10.

Lemma 4.4. *For $\delta \in \Delta$, we have $\delta^2 = \delta$.*

Proof. This follows from Lemma 4.3 by taking $u = w = 1$. $\square$

The fifth lemma is an analogue of [**10**, Lemma 4.4].

Lemma 4.5. *For $\delta \in \Delta$ and $u, v \in F^1$, we have $\delta(\delta(u)\delta(v)) = \delta(uv)$.*

Proof. Apply Lemma 4.3 twice. $\square$

We are now able to prove the main result of this section.

Theorem 4.6. *Let X be a nonempty set, and let $\delta \in \Delta$. On the set $\delta(F)$ define a multiplication by $u \star v = \delta(uv)$ and consider the unary operation on F restricted to $\delta(F)$. Then δ is a *-homomorphism of F onto $\delta(F)$.*

Proof. In view of Lemma 4.1, the unary operation on F maps $\delta(F)$ into itself. Hence we may keep the unary operation of F restricted to $\delta(F)$. It now follows from Lemmas 4.5 and 4.1 that δ is a *-homomorphism of F onto $\delta(F)$ with modified multiplication. $\square$

The above theorem is an analogue of a part of Theorem 3.14. That the remaining part of that theorem has a faithful analogue in the context of the *-homomorphism δ will be a consequence of Theorem 5.4 (iii).

5. Structure. The purpose of this section is to provide a representation of $\gamma_{t_n}(F)$ in terms of the structure of general *-bands. The latter can be found in [**3**] but we shall improve upon it by using the notation in [**4**] for general bands. This will make it possible to construct a *-homomorphism π mapping F onto such a *-band. In addition, we explore the relationship of π with the *-homomorphisms $\gamma_{t_n}, \delta_{t_n}$ and b studied earlier.

We start with the structure of general *-bands.

Lemma 5.1. *Let Y be a semilattice. For every $\alpha \in Y$, let X_α be a nonempty set, fix an element of X_α and denote it by α, and let $B_\alpha = X_\alpha \times X_\alpha$. Let*

$$\langle , \rangle : B_\alpha \times X_\beta \longrightarrow X_\beta$$

be a function defined whenever $\alpha \geq \beta$. Assume that $X_\alpha \cap X_\beta = \varnothing$ if $\alpha \neq \beta$. On $B = \cup_{\alpha \in Y} B_\alpha$ define a unary operation by $(x, y)^* = (y, x)$ and a multiplication by: for $a \in B_\alpha$, $b \in B_\beta$, let

$$a \circ b = (\langle a, \langle b, \alpha\beta \rangle \rangle, \langle b^* \langle a^*, \alpha\beta \rangle \rangle). \tag{8}$$

Assume

(a) *if $x, y, z \in X_\alpha$, then $\langle (x, y), z \rangle = x$,*

(b) *if $\gamma < \alpha\beta$, $a \in B_\alpha$, $b \in B_\beta$, $x \in X_\gamma$, then $\langle a, \langle b, x \rangle \rangle = \langle a \circ b, x \rangle$.*

Then B is a *-band. Conversely, every *-band is *-isomoprhic to one so constructed.

Proof. This is a reformulation of [**3**, Theorem 5.5] in the notation of [**4**, Theorem 6.1]. □

Denote the *-band constructed in Lemma 5.1 by $B(Y; B_\alpha, \langle , \rangle)$. Recall that a projection in a *-band S is an element fixed by the involution.

Note that all projections in S are of the form aa^* for any $a \in S$. Denote by $P(S)$ the set of all projections in S.

The proof of the converse of Lemma 5.1 is made explicit in the following useful result. By $S = (Y; S_\alpha)$ we mean that S is a semigroup which is a semilattice Y of semigroups S_α.

Lemma 5.2. *Let $S = (Y; S_\alpha)$ be a *-band. For each $\alpha \in Y$, let $X_\alpha = S_\alpha \cap P(S)$ and $B_\alpha = X_\alpha \times X_\alpha$. For any $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, define: for $(p,q) \in B_\alpha$, $r \in X_\beta$, let*

$$\langle (p,q), r\rangle = pqrqp. \tag{9}$$

The conditions in Lemma 5.1 *are satisfied so $B = B(Y; B_\alpha, \langle , \rangle)$ is a *-band. The mapping*

$$\xi : a \longrightarrow (aa^*, a^*a), \quad a \in S,$$

is a *-isomorphism of S onto B.

Proof. This is [**3**, Lemma 5.3] in different notation. □

We now turn to our concrete situation.

Example 5.3. Let X be a nonempty set and $\gamma \in \Gamma$, say $\gamma = \gamma_{t_n}$. On

$$S_\gamma = \{(\gamma(uu^*), \gamma(v^*v)) \mid u, v \in F, c(u) = c(v)\}$$

define a multiplication by

$$(u,v)(w,z) = (\gamma(uvwz\,wvu), \gamma(zwvu\,vwz))$$

and a unary operation by $(u,v)^* = (v,u)$. Finally define a mapping π_γ on F by

$$\pi_\gamma : w \longrightarrow (\gamma(ww^*), \gamma(w^*w)).$$

In the above notation, $c(u) = c(v)$ and $c(w) = c(z)$ imply that $c(uvwz\,wvu) = c(zwvu\,vwz)$ and u, v, w, z being projections, it follows at once that both $uvwz\,wvu$ and $zwvu\,vwz$ are also. Hence S_γ is closed under the above multiplication making it a groupoid. Clearly S_γ is closed under the unary operation *.

We prove next that π_γ is a *-homomorphism of F onto S_γ and explore the mutual relationship of the function π_γ and the operators γ, δ and b. Here $\gamma = \gamma_{t_n}$, $d = \delta_{t_n}$, $\pi_\gamma = \pi_{\gamma_{t_n}}$ and $S_\gamma = S_{\gamma_{t_n}}$. In the next theorem, for the sake of simplicity of notation, we omit the subscript for all functions and for S.

Theorem 5.4. *Let X be a nonempty set, $t \in \{\chi, i\}$ and $n \geq 3$.*

(i) *The mapping π is a *-homomorphism of F onto S.*

(ii)

$$\pi\gamma = \pi\delta = \pi b = \pi, \qquad b\gamma = b\delta \neq b^2 = b,$$
$$\gamma^2 = \gamma\delta = \gamma b = \gamma, \qquad \delta\gamma = \delta^2 = \delta b = \delta.$$

(iii) *Define a relation ρ on F by*

$$u \,\rho\, v \quad \Longleftrightarrow \quad t_{n-1}(u) \,=\, t_{n-1}(v),\, t_{n-1}(\bar{u}) \,=\, t_{n-1}(\bar{v}).$$

Then π, γ and δ induce ρ on F.

(iv) *For $\psi \in \{\pi, \gamma, \delta\}$, the mapping*

$$\psi\,|_{\zeta(F)} \colon \zeta(F) \longrightarrow \psi(F),$$

with respective multiplications, is a *-isomorphism for $\zeta \in \{\gamma, \delta\}$ and a noninjective *-epimorphism for $\zeta = b$.

Proof. (i) We show that the construction of S conforms with that of B in Lemma 5.1 and that the mapping π is the composition of γ and the mapping corresponding to ξ in Lemma 5.2.

The *-band $\gamma(F)$ corresponds to the semigroup S in Lemma 5.2. Since γ is a *-homomorphism, it follows easily that

$$P(\gamma(F)) = \{\gamma(ww^*) \mid w \in F\}$$

is the set of all projections in $\gamma(F)$. For every $A \in Y$, let

$$X_A = \{w \in P(\gamma(F)) \mid c(w) = A\}$$

and $S_A = X_A \times X_A$. For $A, B \in Y$ such that $A \subseteq B$ and $u, v, w \in F$ with $c(u) = c(v) = A$, $c(w) = B$, in view of (9), we set

$$\langle (u, v), w \rangle = \gamma(uvw\,vu).$$

We now impose a total order on X and for every $A \in Y$, let $\tilde{A} = x_1 x_2 \ldots x_n$ if $A = \{x_1, x_2, \ldots, x_n\}$ and $x_1 < x_2 < \cdots < x_n$. Let $S = \cup_{A \in Y} S_A$ with unary operation $(u, v)^* = (v, u)$ and multiplication as in (8). The latter simplifies as follows. Let $a = (u, v) \in S_A$ and $b = (w, z) \in S_B$. Note that

$$\{w \in \gamma(F) \mid c(w) = A \cup B\}$$

is a rectangular band and so the identity $xyz = xz$ holds in it. Letting $C = \widetilde{A \cup B}$, we have

$$\begin{aligned} a \circ b &= (\langle a, \langle b, C \rangle \rangle,\ \langle b^*, \langle a^*, C \rangle \rangle) \\ &= (\langle a, \gamma(wzCzw) \rangle,\ \langle b^*, \gamma(vuCuv) \rangle) \\ &= (\gamma(uvwzCzwvu),\ \gamma(zwvuCuvwz)) \\ &= (\gamma(uvwz\,wvu),\ \gamma(zwvu\,vwz)) \qquad \text{by the above remark} \end{aligned}$$

as in Construction 5.3.

Finally, for any $w \in F$,

$$\begin{aligned} \xi\gamma(w) &= (\gamma(w)(\gamma(w))^*,\ (\gamma(w))^*\gamma(w)) \\ &= (\gamma(ww^*),\ \gamma(w^*w)) \qquad \text{by Theorem 3.14} \\ &= \pi(w). \end{aligned}$$

By Lemma 5.2, ξ is a *-isomorphism of $\gamma(F)$ onto S and hence π is a *-homomorphism of F onto S.

(ii) The equality $\gamma^2 = \gamma$ is part of Lemma 3.10. By Theorem 3.14, γ induces the least $\mathcal{V}$-congruence $\eta_{\mathcal{V}}$ on F for some *-band variety $\mathcal{V}$ strictly contained in $\mathcal{B}$. By Theorem 3.15, b induces the least $\mathcal{B}$-congruence $\eta_{\mathcal{B}}$ on F. In particular, $b^2 = b$. Since $\mathcal{V} \subset \mathcal{B}$, we have $\eta_{\mathcal{B}} \subset \eta_{\mathcal{V}}$ and so $\gamma b = \gamma$. Since $b\gamma = b$ would imply $\eta_{\mathcal{V}} \subseteq \eta_{\mathcal{B}}$, we conclude that $b\gamma \neq b$.

Let $t = t_{n-1}$ and $w \in F$. Then

$$\begin{aligned}
\gamma\delta(w) &= ts\delta(w)\,\sigma\delta(w)\,\varepsilon\delta(w)\,\bar{t}e\delta(w) = tst(w)\,\sigma t(w)\,\varepsilon\bar{t}(w)\,\bar{t}e\bar{t}(w)\\
&= t^2 s(w)\,\sigma(w)\,\varepsilon(w)\,\bar{t}^2 e(w) && \text{by Lemma 3.4}\\
&= ts(w)\,\sigma(w)\,\varepsilon(w)\,\bar{t}e(w) && \text{by Lemma 3.7}\\
&= \gamma(w)
\end{aligned}$$

and thus $\gamma\delta = \gamma$.

Since $\pi = \xi\gamma$ by the proof of part (i), we obtain the equalities $\pi\gamma = \pi\delta = \pi b = \pi$.

We have $\delta^2 = \delta$ by Lemma 4.4. Also

$$\begin{aligned}
\delta\gamma(w) &= t\gamma(w)\bar{t}\gamma(w) = t\gamma(w)\bar{t}\bar{\gamma}(w) && \text{by Lemma 3.10}\\
&= t\gamma(w)\overline{t\gamma}(w) && \text{by Lemma 3.1}\\
&= t(w)\bar{t}(w) && \text{by Lemma 4.2}\\
&= \delta(w)
\end{aligned}$$

so that $\delta\gamma = \delta$. It follows that also $\delta b = \delta\gamma b = \delta\gamma = \delta$.

Finally, we have

$$\begin{aligned}
b\gamma(w) &= bs\gamma(w)\,\sigma\gamma(w)\,\varepsilon\gamma(w)\,be\gamma(w) = bts(w)\,\sigma(w)\,\varepsilon(w)\,b\bar{t}e(w)\\
&= bst(w)\,\sigma t(w)\,\varepsilon\bar{t}(w)\,be\bar{t}(w) && \text{by Lemma 3.4}\\
&= bs\delta(w)\,\sigma\delta(w)\,\varepsilon\delta(w)\,be\delta(w) = b\delta(w)
\end{aligned}$$

whence $b\gamma = b\delta$.

(iii) It follows from part (ii) that $\hat{\gamma} = \hat{\delta} \subseteq \hat{\pi}$. Let $\pi(u) = \pi(v)$. Then $\gamma(uu^*) = \gamma(vv^*)$ and $\gamma(u^*u) = \gamma(v^*v)$. Since $\hat{\gamma}$ is a *-band congruence, it follows that

$$\gamma(u) = \gamma(uu^*u^*u) = \gamma(vv^*v^*v) = \gamma(v)$$

and so $\hat{\gamma} = \hat{\pi}$.

The inclusion $\rho \subseteq \hat{\delta}$ is trivial. Conversely, let $\delta(u) = \delta(v)$. By Lemma 4.2, we have $t_{n-1}\delta = t_{n-1}$ and $\overline{t_{n-1}}\delta = \overline{t_{n-1}}$. Thus

$$t_{n-1}(u) = t_{n-1}(v), \qquad \overline{t_{n-1}}(u) = \overline{t_{n-1}}(v).$$

The latter equality is clearly equivalent to $t_{n-1}(\bar{u}) = t_{n-1}(\bar{v})$ and so we obtain $u \,\rho\, v$.

(iv) For $\nu \in \{\pi, \gamma, \delta, b\}$, we denote the product in $\nu(F)$ by $u \star_\nu v$. In the case $\nu \neq \pi$, this means that $u \star_\nu v = \nu(uv)$; for $\nu = \pi$, this signifies that the product in S is denoted by $\star_\pi$.

Now let $\psi \in \{\pi, \gamma, \delta\}$, $\zeta \in \{\gamma, \delta, b\}$ and $u, v \in \zeta(F)$. The homomorphism property has the form

$$\psi(u \star_\zeta v) = \psi(u) \star_\psi \psi(v),$$

equivalently, except for $\psi = \pi$,

$$\psi\zeta(uv) = \psi(\psi(u)\psi(v)).$$

In view of Theorem 3.14 and Lemma 4.5, this is equivalent to

$$\psi\zeta(uv) = \psi(uv) \tag{10}$$

which, by part (i), is also valid for $\psi = \pi$. We now invoke part (ii) to conclude that (10) indeed takes place. Therefore, $\psi|_{\zeta(F)}$ is a homomorphism of $\zeta(F)$ into $\psi(F)$. It follows easily from the definitions and part (i) that $\psi|_{\zeta(F)}$ preserves the * operation.

Given $w \in \psi(F)$, then $\zeta(w) \in \zeta(F)$ and, by part (ii), we obtain $\psi\zeta(w) = \psi(w) = w$. Thus $\psi|_{\zeta(F)}$ is surjective.

Let $\zeta \neq b$ and assume that $\psi(u) = \psi(v)$ for some $u, v \in \zeta(F)$. By part (iii), we get $\zeta(u) = \zeta(v)$ and thus $u = v$ since $u, v \in \zeta(F)$. Hence $\psi|_{\zeta(F)}$ is injective.

Finally, let $\zeta = b$. By part (iii), we may assume that $\psi = \gamma$. By Theorem 3.14, γ induces the least $\mathcal{V}$-congruence $\eta_{\mathcal{V}}$ on F for some *-band variety $\mathcal{V}$ strictly contained in $\mathcal{B}$. Let $u, v \in F$ be such that $\mathcal{V}$ satisfies the identity $u = v$ but $\mathcal{B}$ does not. It follows that $\gamma(u) = \gamma(v)$ and $b(u) \neq b(v)$. Since $\gamma b = \gamma$ by part (ii), it follows that $\gamma b(u) = \gamma b(v)$ and so $\gamma|_{b(F)} : b(F) \to \gamma(F)$ is not injective. $\square$

6. Free objects. From Diagram 1 we see that we have not discussed the four varieties of normal *-bands, at the bottom of the diagram, and the variety $\mathcal{B}$ of all *-bands, at the top of the diagram.

There is nothing to say about the variety of trivial *-bands. For the rest, we fix a nonempty set X, set up the needed notation for each variety separately and then state the results in a single theorem. Recall that $I = X \cup X^*$.

Let $\mathcal{RB} = [x = xyx]$ and $X_{\mathcal{RB}}$ be the set $I \times I$ with multiplication $(x, y)(w, z) = (x, z)$ and unary operation $(x, y)^* = (y, x)$. Define a mapping $\pi_{\mathcal{RB}}$ on F by

$$\pi_{\mathcal{RB}} : w \longrightarrow (h_2(w), h_2(w^*)).$$

Let $\mathcal{S} = [G_2 = I_2]$ and $X_{\mathcal{S}}$ be the set of all finite nonempty subsets of X with set theoretical union as multiplication and the identity mapping as a unary operation. Define a mapping $\pi_{\mathcal{S}}$ on F by

$$\pi_{\mathcal{S}} : w \longrightarrow c(w).$$

Let $\mathcal{NB} = [G_3 = H_3]$ and

$$X_{\mathcal{NB}} = \{(x, A, y) \in I \times X_{\mathcal{S}} \times I \mid c(x), c(y) \subseteq A\}$$

with multiplication

$$(x, A, y)(w, B, z) = (x, A \cup B, z)$$

and unary operation $(x, A, y)^* = (y, A, x)$. Define a mapping $\pi_{\mathcal{NB}}$ on F by

$$\pi_{\mathcal{NB}} : w \longrightarrow (h_2(w), c(w), h_2(w^*)).$$

For $t = h$, $n \geq 4$ and $t = i$, $n \geq 3$, let $\pi_{[G_n = T_n]} = \pi_{t_n}$ and $X_{[G_n = T_n]} = S_{t_n}$.

For the variety $\mathcal{B}$ of *-bands, we formally follow the development in Construction 5.3 where we write b for γ throughout. Here we let $X_{\mathcal{B}}$ denote the *-band S and $\pi_{\mathcal{B}}$ the relevant mapping.

We are now ready for the desired result.

Theorem 6.1. *Let X be a nonempty set and $\mathcal{V}$ be a nontrivial *-band variety. Then $\pi_{\mathcal{V}}$ is a *-homomorphism of F onto $X_{\mathcal{V}}$ which induces the least $\mathcal{V}$-congruence on F. Therefore, $X_{\mathcal{V}}$ is a $\mathcal{V}$-free *-band on X.*

Proof. We consider several cases.

$\mathcal{V} = \mathcal{RB}$. Straightforward verification shows that $\pi_{\mathcal{RB}}$ is a *-homomorphism of F onto $X_{\mathcal{RB}}$ and that $X_{\mathcal{RB}}$ is a rectangular *-band. Let ρ be a rectangular *-band congruence on F and $u, v \in F$ such that $h_2(u) = h_2(v)$ and $h_2(u^*) = h_2(v^*)$. It follows that

$$u \,\rho\, h_2(u)uh_2(u^*) \,\rho\, h_2(u)h_2(u^*) = h_2(v)\, h_2(v^*) \,\rho\, h_2(v)\, v\, h_2(v^*) \,\rho\, v$$

proving the required minimality.

$\mathcal{V} = \mathcal{S}$. Straightforward verification shows that $\pi_{\mathcal{S}}$ is a *-homomorphism of F onto $X_{\mathcal{S}}$ and that $X_{\mathcal{S}}$ is a *-semilattice. Let ρ be a *-semilattice congruence on F and $u, v \in F$ such that $c(u) = c(v) = \{x_1, x_2, \ldots, x_n\}$. Since F/ρ satisfies the identities $xy = yx$ and $x^* = x$, we obtain $u \,\rho\, x_1x_2 \ldots x_n \,\rho\, v$, which proves the desired minimality.

$\mathcal{V} = \mathcal{NB}$. Straightforward verification shows that $\pi_{\mathcal{NB}}$ is a *-homomorphism of F onto $X_{\mathcal{NB}}$ and that $X_{\mathcal{NB}}$ is a normal *-band. Let ρ be a normal *-band congruence on F and $u, v \in F$ be such that $h_2(u) = h_2(v)$, $c(u) = c(v) = \{x_1, x_2, \ldots, x_n\}$ and $h_2(u^*) = h_2(v^*)$. It is well known that then F/ρ satisfies the identity $axyb = ayxb$. By [**1**, Lemma 4.5], it satisfies the identity $axb = ax^*b$. Using these identities, we obtain

$$u \,\rho\, h_2(u)\, x_1x_2 \ldots x_n\, h_2(u^*) \,\rho\, h_2(v)\, x_1x_2 \ldots x_n\, h_2(v^*) \,\rho\, v$$

which proves the desired minimality.

For the remaining varieties, except $\mathcal{B}$, we refer to Theorems 3.14 and 5.4 (i). These deliberations carry over to the case of the variety $\mathcal{B}$ with minor modifications. □

Several observations regarding Theorem 6.1 are in order. Except in the case of $\mathcal{RB}$, the relatively free *-bands are semilattices $X_{\mathcal{S}}$ of rectangular bands. The latter are necessarily square and thus may be termed "square bands" as in [**11**].

In the case of $\mathcal{NB}$, we know that the semilattice must be strong, but from above we may deduce more than that. In fact, $X_{\mathcal{NB}}$ is a semilattice $X_{\mathcal{S}}$ of square bands $A_{\mathcal{RB}}$. Moreover, if $A \subseteq B$, then the corresponding homomorphism of $A_{\mathcal{RB}}$ into $B_{\mathcal{RB}}$ is the natural embedding of free objects. It then follows easily that $X_{\mathcal{NB}}$ is a subdirect product of $X_{\mathcal{S}}$ and $X_{\mathcal{RB}}$.

In view of Lemma 5.2 and Theorem 5.4 (i), we have

$$S_\gamma = \{(\gamma(uu^*), \gamma(u^*u)) \mid u \in F\}.$$

For the variety $\mathcal{B}$, the development in [**3**, Section 5], (where different notation is used) runs along the same lines as for general bands. One defines a mapping λ on F by $\lambda(w) = bs(w)\sigma(w)$. This is used to define a mapping μ on F by

$$\mu : w \longrightarrow (\lambda(w), \lambda(w^*)).$$

It is asserted in [**3**, Theorem 5.7] that the mapping μ is a *-homomorphism of F onto its image, where the multiplication is defined by

$$(u, v)(w, z) = (\lambda(uvw), \lambda(zwv)).$$

Hence, for any $u, v \in F$, we must have $\mu(uv) = \mu(u)\mu(v)$, and thus

$$\lambda(uv) = \lambda(\lambda(u)\,\lambda(u^*)\,\lambda(v))$$

in the first component.

Let $x, y \in X$ be distinct. Then

$$\lambda(xy) = bs(xy)\,\sigma(xy) = b(x)\,y = x^2y,$$
$$\lambda(\lambda(x)\,\lambda(x^*)\lambda(y)) = \lambda(xx^*y) = bs(xx^*y)\,\sigma(xx^*y) = b(xx^*)\,y = xx^*y.$$

Since these two elements are different, [**3**, Theorem 5.7] is false.

The above constructions give concrete copies of relatively free *-bands up to the variety $[G_3 = H_3]$ of normal *-bands. In the next result, we carry this one step further by constructing free objects in the variety $[G_3 = I_3]$ of regular *-bands.

Proposition 6.2. *Let X be a nonempty set. Then*

$$S = \{(u, v) \in i_2(F) \times i_2(F) \mid c(u) = c(v)\}$$

with multiplication

$$(u,v)(w,z) = (i_2(uw), i_2(zv))$$

and unary operation $(u,v)^* = (v,u)$, *is a free regular* **-band on* X.

Proof. Let $\gamma = \gamma_{i_3}$. Then, for any $w \in F$, we have

$$\begin{aligned}
\gamma(ww^*) &= i_2 s(ww^*)\, \sigma(ww^*)\, \varepsilon(ww^*)\, \overline{i_2} e(ww^*) \\
&= i_2 s(w)\, \sigma(w)\, \varepsilon(w^*)\, \overline{i_2} e(w^*) \\
&= i_2(w)\, \overline{i_2}(w^*) = i_2(w)\, i_2^*(w^*) \qquad \text{by Lemma 3.6} \\
&= i_2(w)(i_2(w))^*.
\end{aligned}$$

Using this, [**10**, Theorem 6.1] and Theorem 5.4 give the free object in $[G_3 = I_3]$ in the form

$$\begin{aligned}
S_\gamma &= \{(\gamma(uu^*), \gamma(vv^*)) \mid u, v \in F,\ c(u) = c(v)\} \\
&= \{(i_2(u)(i_2(u))^*, i_2(v)(i_2(v))^*) \mid u, v \in F,\ c(u) = c(v)\} \\
&= \{(pp^*, qq^*) \mid p, q \in i_2(F),\ c(p) = c(q)\}
\end{aligned}$$

with multiplication

$$\begin{aligned}
&(pp^*, qq^*)(rr^*, ss^*) \\
&\qquad = (\gamma(pp^*qq^*rr^*ss^*rr^*qq^*pp^*),\ \gamma(ss^*rr^*qq^*pp^*qq^*rr^*ss^*)) \\
&\qquad = (i_2(pr)(i_2(pr))^*,\ i_2(sq)(i_2(sq))^*)
\end{aligned}$$

and unary operation $(pp^*, qq^*)^* = (qq^*, pp^*)$.

Note that the set S in the statement of the proposition is closed for the given operations. Define a mapping

$$\eta : (p,q) \longrightarrow (pp^*, qq^*), \quad (p,q) \in S.$$

Clearly η maps S into S_γ. If $p, q \in i_2(F)$ are such that $pp^* = qq^*$, then $p = q$. It follows that the inverse mapping of η is single valued whence we conclude that η is a bijection. For any $(p,q), (r,s) \in S$, by the above we obtain

$$\begin{aligned}
(p,q)\eta\,(r,s)\eta &= (pp^*, qq^*)(rr^*, ss^*) \\
&= (i_2(pr)(i_2(pr))^*, i_2(sq)(i_2(sq))^*) \\
&= (i_2(pr), i_2(sq))\eta \\
&= ((p,q)(r,s))\eta, \\
(p,q)^*\eta = (q,p)\eta &= (qq^*, pp^*) = (pp^*, qq^*)^* = ((p,q)\eta)^*
\end{aligned}$$

and η is also a *-homomorphism. The *-isomorphism of S and S_γ implies the assertion of the proposition. $\square$

7. Joins and meets of $\widehat{t_n}$ and $\widehat{\overline{t_n}}$. The operators t_n play an important role in the definition of the fully invariant congruences on F discussed in the previous section. In this section we explore the multiple relationships among these congruences and the equivalences induced by the operators t_n.

Our first lemma is of interest for general *-semigroups. It represents an analogue of a well-known result for semigroups.

Lemma 7.1. *Let θ be an equivalence relation on a *-semigroup S. Define a relation θ^0 on S by*

$$a\,\theta^0\,b \quad \Longleftrightarrow \quad xay\,\theta\,xby,\ xa^*y\,\theta\,xb^*y \quad \text{for all } x, y \in S^1.$$

Then θ^0 is the greatest *-congruence contained in θ.

Proof. Clearly θ^0 is an equivalence relation and $\theta^0 \subseteq \theta$. Let $a\,\theta^0\,b$ and $c \in S$. For any $x, y \in S^1$, we have

$$x(ac)y\,\theta\,x(bc)y, \quad x(ac)^*y = xc^*a^*y\,\theta\,xc^*b^*y = x(bc)^*y$$

and thus $ac\,\theta^0\,bc$; dually $ca\,\theta^0\,cb$. Interchanging the roles of a and a^* and also of b and b^*, we see that $a^*\,\theta^0\,b^*$. Hence θ^0 is a *-congruence.

Let ρ be a *-congruence on S contained in θ, and let $a\,\rho\,b$. Then for any $x,\ y \in S^1$, we have $xay\,\rho\,xby$ and also $xa^*y\,\rho\,xb^*y$. This yields $xay\,\theta\,xby$ and $xa^*y\,\theta\,xb^*y$ which implies that $a\,\theta^0\,b$. Therefore $\rho \subseteq \theta^0$ proving the maximality of the latter. $\square$

We turn next to our concrete situation.

Lemma 7.2. *For any operator t on F, $\hat{t}$ is a congruence if and only if $\hat{\bar{t}}$ is a congruence.*

Proof. Assume that $\hat{t}$ is a congruence and $\bar{t}(u) = \bar{t}(v)$. Let $x, y \in F^1$. Then $t(\bar{u}) = t(\bar{v})$ and so, since $\hat{t}$ is a congruence, we have $t(\bar{y}\bar{u}\bar{x}) =$

$t(\bar{y}\bar{v}\bar{x})$. It follows that $t(\overline{xuy}) = t(\overline{xvy})$ and thus $\bar{t}(xuy) = \bar{t}(xvy)$. Hence $\hat{\bar{t}}$ is a congruence. The converse implication follows by symmetry. $\square$

Lemma 7.3. *For $t \in \{\chi, i\}$ and $n \geq 1$, $\hat{t}_n$ and $\hat{\overline{t_n}}$ are congruences.*

Proof. Using Lemma 7.2 without explicit reference, we prove that $\hat{t}_n$ is a congruence by induction on n. The case $n = 1$ is trivial for $t = i$ and almost trivial for $t = \chi$. Assume that $\widehat{t_{n-1}}$ is a congruence with $n > 1$. We show that

$$t_n(u) = t_n(v) \implies t_n(wu) = t_n(wv),\ t_n(uw) = t_n(vw)$$

for all $u, v, w \in F^1$ by secondary induction on $\sharp(uw)$. The case $\sharp(uw) = 0$ is trivial. Assume that $\sharp(uw) > 0$ and that the claim holds for all values smaller than $\sharp(uw)$. We have to show that

(a) $t_n s(wu) = t_n s(wv), \quad \sigma(wu) = \sigma(wv)$,

(b) $\overline{t_{n-1}}(wu) = \overline{t_{n-1}}(wv)$,

(c) $t_n s(uw) = t_n s(vw), \quad \sigma(uw) = \sigma(vw)$,

(d) $\overline{t_{n-1}}(uw) = \overline{t_{n-1}}(vw)$.

(a) We note first that $t_n(u) = t_n(v)$ implies that $c(u) = c(v)$ by Lemma 3.4. If $c(u) \subseteq c(w)$, then $t_n s(wu) = t_n s(w) = t_n s(wv)$ and $\sigma(wu) = \sigma(w) = \sigma(wv)$, thus we may assume that $c(u) \not\subseteq c(w)$. We can write $u = u_1 x u_2$ with $wu_1 = s(wu)$ and $x = \sigma(wu)$. Similarly, we can write $v = v_1 y v_2$ with $wv_1 = s(wv)$ and $y = \sigma(wv)$. By Lemma 3.5, we have $t_n(u) \in t_n(u_1)xF^1$ and $t_n(v) \in t_n(v_1)yF^1$. Lemma 3.5 implies that the ordering of the first occurrences of the letters is the same in $t_n(u)$ and u (also in $t_n(v)$ and v). Thus

$$i_2(u) = i_2 t_n(u) = i_2 t_n(v) = i_2(v)$$

and so $x = y$, that is, $\sigma(wu) = \sigma(wv)$. Therefore, $t_n(u) = t_n(v)$ yields $t_n(u_1) = t_n(v_1)$. Since $\sharp(u_1 w) < \sharp(uw)$, the induction hypothesis on $\sharp$ yields $t_n(wu_1) = t_n(wv_1)$, that is, $t_n s(wu) = t_n s(wv)$.

(b) By Lemma 3.7, we have $t_{n-1}\overline{t_n} = t_{n-1}$ which by Lemma 3.1 gives

$$\overline{t_{n-1}\overline{t_n}} = \overline{t_{n-1}} t_n = \overline{t_{n-1}}. \tag{11}$$

Thus $t_n(u) = t_n(v)$ implies that $t_{n-1}(u) = t_{n-1}(v)$ and the induction hypothesis on n yields $\overline{t_{n-1}}(wu) = \overline{t_{n-1}}(wv)$.

(c) If $c(w) \subseteq c(u)$, then Lemma 3.4 gives

$$t_n s(uw) = t_n s(u) = s t_n(u) = s t_n(v) = t_n s(v) = t_n s(vw),$$
$$\sigma(uw) = \sigma(u) = \sigma t_n(u) = \sigma t_n(v) = \sigma(v) = \sigma(vw),$$

thus we may assume that $c(w) \not\subseteq c(u)$. We can write $w = w_1 x w_2$ with $uw_1 = s(uw)$ and $x = \sigma(uw)$. Since $c(v) = c(u)$, we also have $vw_1 = s(vw)$ and $x = \sigma(vw)$. Thus $\sigma(uw) = \sigma(vw)$. Since $\sharp(uw_1) < \sharp(uw)$, the induction hypothesis on $\sharp$ yields $t_n(uw_1) = t_n(vw_1)$, that is, $t_n s(uw) = t_n s(vw)$.

(d) Similar to (b). □

Lemma 7.4. *For $t \in \{\chi, i\}$ and $n \geq 3$, $\rho = \hat{t}_n \vee \widehat{\overline{t_n}}$ is a fully invariant *-congruence on F.*

Proof. Let $u\,p\,v$. Then

$$u = w_0\, \rho_1\, w_1\, \rho_2\, w_2 \ldots \rho_n\, w_n = v$$

for some $w_0, w_1, \ldots, w_n \in F$ and $\rho_i \in \{\hat{t}_n, \widehat{\overline{t_n}}\}$, $i = 1, 2, \ldots, n$. To see that $u^* \rho\, v^*$, it is enough to note that

$$t_n(w) = t_n(z) \iff \overline{t_n}(w^*) = \overline{t_n}(z^*),$$

which follows directly from Lemma 3.6. Thus ρ is a *-congruence.

Let $\varphi \in \mathrm{End}\,(F)$. To prove that $\varphi(u) \rho\, \varphi(v)$, it suffices to show that, for all $w, z \in F$,

$$t_n(w) = t_n(z) \iff t_n\varphi(w) = t_n\varphi(z),$$
$$\overline{t_n}(w) = \overline{t_n}(z) \iff \overline{t_n}\varphi(w) = \overline{t_n}\varphi(z).$$

The first equality follows from Lemma 3.9. To prove the second, assume that $\overline{t_n}(w) = \overline{t_n}(z)$. Then $t_n(\bar{w}) = t_n(\bar{z})$. By Lemma 3.8, we have $\bar{\varphi} \in \mathrm{End}\,(F)$ and hence Lemma 3.9 gives $t_n\bar{\varphi}(\bar{w}) = t_n\bar{\varphi}(\bar{z})$. By Lemma 3.1, we obtain

$$\overline{t_n}\varphi(w) = \overline{t_n\overline{\varphi}}(w) = \overline{t_n\bar{\varphi}(\bar{w})} = \overline{t_n\bar{\varphi}(\bar{z})} = \overline{t_n\overline{\varphi}}(z) = \overline{t_n}\varphi(w).$$

Therefore ρ is fully invariant. $\square$

We are now ready for the theorem of this section. Recall the notation ρ^c from Section 2 and θ^0 from Lemma 7.1.

Theorem 7.5. *For $t \in \{\chi, i\}$ and $n \geq 3$, we have*

$$\widehat{\gamma_{t_n}} = \widehat{t_n} \vee \widehat{\overline{t_n}} = \widehat{t_{n-1}} \cap \widehat{\overline{t_{n-1}}} = (\widehat{t_n})^c = (\widehat{\overline{t_n}})^c = (\widehat{t_{n-1}})^0 = (\widehat{\overline{t_{n-1}}})^0.$$

Proof. Denote the seven relations above by $\rho_1, \rho_2, \ldots, \rho_7$, respectively.

$\rho_1 = \rho_3$. Since $\overline{t_{n-1}}(u) = \overline{t_{n-1}}(v)$ is equivalent to $t_{n-1}(\bar{u}) = t_{n-1}(\bar{v})$, the equality $\rho_1 = \rho_3$ follows from Theorem 5.4 (iii).

$\rho_1 = \rho_6$. Since ρ_1 is a *-congruence and $\rho_1 = \rho_3 \subseteq \widehat{t_{n-1}}$, it follows that $\rho_1 \subseteq \rho_6$. Conversely, let $u(\widehat{t_{n-1}})^0 v$. By Lemma 7.1, we have $t_{n-1}(u) = t_{n-1}(v)$ and $t_{n-1}(u^*) = t_{n-1}(v^*)$. Hence,

$$\begin{aligned} t_{n-1}(u^*) = t_{n-1}(v^*) \quad &\Longleftrightarrow \quad t^*_{n-1}(u) = t^*_{n-1}(v) \\ &\Longleftrightarrow \quad \overline{t_{n-1}}(u) = \overline{t_{n-1}}(v) \qquad \text{by Lemma 3.6} \\ &\Longleftrightarrow \quad t_{n-1}(\bar{u}) = t_{n-1}(\bar{v}) \end{aligned}$$

and so, by Theorem 5.4 (iii), $\gamma_{t_n}(u) = \gamma_{t_n}(v)$. Thus $\rho_6 \subseteq \rho_1$.

$\rho_1 \subseteq \rho_2$. Suppose that $\rho_1 \not\subseteq \rho_2$. By Theorem 3.14 and Lemma 7.4, both ρ_1 and ρ_2 are fully invariant *-congruences. Hence the structure of Diagram 1 allows us to conclude that $\rho_2 \subset \rho_1$.

Consider first the case $t = \chi$. In view of Diagram 1, we must have $\widehat{\chi_n} \cup \widehat{\overline{\chi_n}} \subseteq \widehat{\gamma_{i_{n+1}}}$. Since $\chi_n(G_{n+1}) = \chi_n(H_{n+1})$ by Lemma 3.12, it follows that $\gamma_{i_{n+1}}(G_{n+1}) = \gamma_{i_{n+1}}(H_{n+1})$ and so $\overline{\gamma_{i_{n+i}}}(G_{n+1}) = \overline{\gamma_{i_{n+1}}}(H_{n+1})$ by Lemma 3.10. Thus $\gamma_{i_{n+1}}(\overline{G_{n+1}}) = \gamma_{i_{n+1}}(\overline{H_{n+1}})$, which contradicts Lemma 3.13.

Now consider the case $t = i$. In view of Diagram 1, we must have $\hat{i}_n \cup \hat{\overline{i_n}} \subseteq \widehat{\gamma_{\chi_n}}$. Since $i_n(G_n) = i_n(I_n)$ by Lemma 3.12, it follows that $\gamma_{\chi_n}(G_n) = \gamma_{\chi_n}(I_n)$ and so $\overline{\gamma_{\chi_n}}(G_n) = \overline{\gamma_{\chi_n}}(I_n)$ by Lemma 3.10. Thus, $\gamma_{\chi_n}(\overline{G_n}) = \gamma_{\chi_n}(\overline{I_n})$, which contradicts Lemma 3.13.

We reached a contradiction in both cases. Therefore, $\rho_1 \subseteq \rho_2$.

$\rho_2 \subseteq \rho_1$. Since $\rho_1 = \rho_3$ is a *-congruence, we only need to prove that $\widehat{t_n} \cup \widehat{\overline{t_n}} \subseteq \rho_3$. By duality, it is enough to show that $\widehat{t_n} \subseteq \widehat{t_{n-1}} \cap \widehat{\overline{t_{n-1}}}$. By Lemma 3.7, we have $t_{n-1}t_n = t_{n-1}$ and also $\overline{t_{n-1}}t_n = \overline{t_{n-1}}$ by (11). Thus $t_n(u) = t_n(v)$ implies that $t_{n-1}(u) = t_{n-1}(v)$ and $\overline{t_{n-1}}(u) = \overline{t_{n-1}}(v)$, as required.

$\rho_2 = \rho_4$. Since ρ_2 is a *-congruence by Lemma 7.4, the inclusion $\rho_4 \subseteq \rho_2$ holds trivially. To prove the opposite containment, we only need to show that $\widehat{\overline{t_n}} \subseteq (\widehat{t_n})^c$. Let $\overline{t_n}(u) = \overline{t_n}(v)$. By Lemma 3.6, we have $t_n^*(u) = t_n^*(v)$, and thus

$$t_n(u^*) = (t_n^*(u))^* = (t_n^*(v))^* = t_n v^*.$$

Hence $u^* \widehat{t_n} v^*$ and so $u(\widehat{t_n})^c$ as required.

$\rho_2 = \rho_5$. This follows from $\rho_2 = \rho_4$ by duality.

$\rho_3 = \rho_7$. This follows from $\rho_3 = \rho_6$ by duality. $\square$

8. The relationship of $\widehat{t_n}$ and $\widehat{t'_n}$. In [**5**], a family of operators on the free semigroup X^+, denoted by t_n, was devised to solve the word problem for relatively free bands. In order to avoid confusion with the notation used in the present paper, we denote them by t'_n. Conforming with this convention, we now define

$$t'_n = t_n \,|_{X^+}, \quad t \in \{\chi, i\}, n \geq 2.$$

A natural question to ask at this point is: what is the relationship between the congruences $\widehat{t_n}$ on F and $\widehat{t'_n}$ on X^+? It follows that $\widehat{t'_n}$ is the restriction of $\widehat{t_n}$ to X^+. In this section we show how $\widehat{t_n}$, respectively, $\widehat{\gamma_{t_n}}$, can be obtained from $\widehat{t'_n}$, respectively $\widehat{t'_{n-1}} \cap \widehat{\overline{t'_{n-1}}}$. Toward this end, we need some notation and preliminary lemmas.

Lemma 8.1. *For $t \in \{\chi, i\}$, $n \geq 1$ and $w \in F$, we have $t_n(w) = t_n(ww^*w)$.*

Proof. We use induction on n. The case $n = 1$ is trivial. Let $n > 1$, $w \in F$, and assume that the lemma holds for $n-1$. By Lemma 3.6 and the induction hypothesis, we obtain

$$\begin{aligned}\overline{t_{n-1}}(ww^*w) = t^*_{n-1}(ww^*w) &= (t_{n-1}((ww^*w)^*))^* = (t_{n-1}(w^*ww^*))^* \\ &= (t_{n-1}(w^*))^* = t^*_{n-1}(w) = \overline{t_{n-1}}(w)\end{aligned}$$

and thus

$$\begin{aligned}t_n(ww^*w) &= t_n s(ww^*w)\,\sigma(ww^*w)\overline{t_{n-1}}(ww^*w) \\ &= t_n s(w)\,\sigma(w)\overline{t_{n-1}}(w) = t_n(w).\end{aligned} \qquad \square$$

For $w \in F$, we let

$$c'(w) = \{i \in I \mid w = u\,i\,v \text{ for some } u, v \in F^1\}$$

and define an endomorphism η of F by the requirement

$$\eta : i \longrightarrow ii^*i, \quad i \in I.$$

Until the end of this section, we assume that X is infinite.

Given $u, v \in F$, we denote by $J(u, v)$ the set of all injective mappings $f : c'(uv) \to X$ such that $f(x) = x$ for every $x \in c'(uv) \cap X$. Also by f we denote the homomorphism from $(c'(uv))^+$ into X^+ induced by the mapping $f : c'(uv) \to X$.

Lemma 8.2. *Let $t \in \{\chi, i\}$, $n \geq 1$, $u, v \in \eta(F^1)$, $f \in J(u, v)$ and $t_n(u) = t_n(v)$. Then $t'_n f(u) = t'_n f(v)$.*

Proof. We use induction on n. For $n = 1$, the case $t = i$ is trivial and the case $t = \chi$ is straightforward and may be safely omitted.

Let $n > 1$ and assume that the lemma holds for $n - 1$. We use secondary induction on $\sharp(u)$. The case $\sharp(u) = 0$ is trivial. Let $u, v \in \eta(F^1)$, $f \in J(u, v)$, and assume that $t_n(u) = t_n(v)$ and that the lemma holds for all values smaller than $\sharp(u)$. Without loss of generality, we may assume that $u, v \in \eta(F)$.

Let $w \in \{u, v\}$. Suppose that $s(w) = x_1 x_2 \dots x_m$, where $x_1, x_2, \dots, x_m \in I$, and $\sigma(w) = y$. Since $w \in \eta(F)$, we have

$$c's(w) = \{x_j, x_j^* \mid j = 1, \dots, m\}$$

and $w = x_1x_2\dots x_m\, yy^*z$ for some $z \in F^1$. Now $f \in J(u,v)$ implies that

$$sf(w) = f(x_1x_2\dots x_my)$$

and $\sigma f(w) = f(y^*)$. Hence

$$sf(u) = fs(u)f\sigma(u), \qquad sf(v) = fs(v)f\sigma(v), \tag{12}$$

$$\sigma f(u) = f((\sigma(u))^*), \qquad \sigma f(v) = f((\sigma(v))^*). \tag{13}$$

The equality $t_n(u) = t_n(v)$ implies that

$$t_ns(u) = st_n(u) = st_n(v) = t_ns(v)$$

by Lemma 3.4. We also have $s(u), s(v) \in \eta(F)$. Since

$$f\mid_{c'(s(u)s(v))} \in J(s(u), s(v))$$

and $\sharp(s(u)) < \sharp(u)$, the induction hypothesis on $\sharp$ yields that

$$t'_nfs(u) = t'_nfs(v). \tag{14}$$

In addition, Lemma 3.4 gives

$$\sigma(u) = \sigma t_n(u) = \sigma t_n(v) = \sigma(v). \tag{15}$$

Since t'_n is a restriction of t_n, it follows from Lemma 7.3 that $\widehat{t'_n}$ is a congruence on X^+. Thus (14) and (15) yield

$$t'_n(fs(u)f\sigma(u)) = t'_n(fs(v)f\sigma(v)).$$

Thus, by (12), we get

$$t'_nsf(u) = t'_nsf(v). \tag{16}$$

Further, $\sigma(u) = \sigma(v)$ implies by (13) that

$$\sigma f(u) = f((\sigma(u))^*) = f((\sigma(v))^*) = \sigma f(v). \tag{17}$$

Finally, $t_n(u) = t_n(v)$ is equivalent to $\overline{t_n}(\bar{u}) = \overline{t_n}(\bar{v})$. Since $t_{n-1}\overline{t_n} = t_{n-1}$ by Lemma 3.7, we obtain $t_{n-1}(\bar{u}) = t_{n-1}(\bar{v})$. Clearly $\bar{u}, \bar{v} \in \eta(F)$ and $f \in J(\bar{u}, \bar{v})$. By the induction hypothesis on n, we get

$$t'_{n-1}f(\bar{u}) = t'_{n-1}f(\bar{v}). \tag{18}$$

Since f sends letters to letters, direct verification shows that $\bar{f} = f$. Therefore

$$\overline{t'_{n-1}}f(w) = \overline{t'_{n-1}(\overline{f(w)})} = \overline{t'_{n-1}f(\bar{w})}$$

for every $w \in F$ and (18) gives $\overline{t'_{n-1}}f(u) = \overline{t'_{n-1}}f(v)$. Together with (16) and (17), this implies $t'_n f(u) = t'_n f(v)$. $\square$

Given a congruence τ on X^+, let τ^+ denote the least congruence ρ on F relative to the properties:

(i) $\rho \supseteq \tau \cup \{(u, uu^*u) \mid u \in F\}$,

(ii) ρ is invariant under *-endomorphism of F.

We are finally ready for the theorem of this section.

Theorem 8.3. *Let X be an infinite set and $t \in \{\chi, i\}$.*

(i) *For $n \geq 1$, we have $\widehat{t_n} = \widehat{t'_n}^{\,+}$.*

(ii) *For $n \geq 3$, we have $\widehat{\gamma_{t_n}} = (\widehat{t'_{n-1}} \cap \widehat{\overline{t'_{n+1}}})^+$.*

Proof. (i) Let $\rho = \widehat{t'_n}^{\,+}$. The inclusion $\widehat{t'_n} \subseteq \widehat{t_n}$ holds trivially and by Lemma 8.1 we have

$$\{(u, uu^*u) \mid u \in F\} \subseteq \widehat{t_n}.$$

Since $\widehat{t_n}$ is invariant for *-endomorphisms of F by Lemma 3.9, it follows that $\rho \subseteq \widehat{t_n}$.

Conversely, let $(u, v) \in \widehat{t_n}$. Since $i\,\rho\, ii^*i$ for every $i \in I$ and ρ is a congruence, we have

$$w\,\rho\,\eta(w), \quad w \in F. \tag{19}$$

Further, $\rho \subseteq \widehat{t_n}$ implies that $t_n(w) = t_n\eta(w)$ for every $w \in F$. Thus

$$t_n\eta(u) = t_n(u) = t_n(v) = t_n\eta(v).$$

Our assumption of X being infinite implies that there exists some $f \in J(\eta(u), \eta(v))$. By Lemma 8.2, we obtain $t'_n f\eta(u) = t'_n f\eta(v)$ and so $f\eta(u)\rho\, f\eta(v)$. Let $g \in \mathrm{End}\,(F)$ be an extension of the mapping f^{-1}. Since ρ is closed for *-endomorphisms of F, it follows that $gf\eta(u)\rho\, g\, f\eta(v)$, that is, $\eta(u)\,\rho\,\eta(v)$. By (19) we conclude that $u\,\rho\, v$ and $\widehat{t_n} \subseteq \rho$. Therefore, $\widehat{t_n} = \rho$, as required.

(ii) Let $\rho = (\widehat{t'_{n-1}} \cap \widehat{\overline{t'_{n-1}}})^+$. By Theorem 7.5, we have

$$\widehat{t'_{n-1}} \cap \widehat{\overline{t'_{n-1}}} \subseteq \widehat{t_{n-1}} \cap \widehat{\overline{t_{n-1}}} = \widehat{\gamma_{t_n}}.$$

Theorem 3.14 yields $\{(u, uu^*u) \mid u \in F\} \subseteq \widehat{\gamma_{t_n}}$, and also that $\widehat{\gamma_{t_n}}$ is invariant for *-endomorphisms of F. Thus $\rho \subseteq \widehat{\gamma_{t_n}}$.

Conversely, let $(u, v) \in \widehat{\gamma_{t_n}}$. We note that (19) also holds in this case. Since $\rho \subseteq \widehat{\gamma_{t_n}}$, we have that $\gamma_{t_n}(w) = \gamma_{t_n}\eta(w)$ for every $w \in F$. Thus

$$\gamma_{t_n}\eta(u) = \gamma_{t_n}(u) = \gamma_{t_n}(v) = \gamma_{t_n}\eta(v).$$

By Theorem 5.4 (iii), we get

$$t_{n-1}\eta(u) = t_{n-1}\eta(v), \qquad t_{n-1}\overline{\eta(u)} = t_{n-1}\overline{\eta(v)}.$$

The hypothesis of X being infinite implies the existence of some $f \in J(\eta(u), \eta(v))$. By Lemma 8.2, we obtain

$$t'_{n-1}f\eta(u) = t'_{n-1}f\eta(v). \tag{20}$$

We have $\overline{\eta(F)} = \eta(F)$ and $J(\eta(u), \eta(v)) = J(\overline{\eta(u)}, \overline{\eta(v)})$. Hence we also get $t'_{n-1}f(\overline{\eta(u)}) = t'_{n-1}f(\overline{\eta(v)})$. Since

$$\overline{t'_{n-1}}f\eta(w) = \overline{t'_{n-1}(\overline{f\eta(w)})} = \overline{t'_{n-1}\bar{f}(\overline{\eta(w)})} = \overline{t'_{n-1}f(\overline{\eta(w)})}$$

for every $w \in F$, we obtain $\overline{t'_{n-1}}f\eta(u) = \overline{t'_{n-1}}f\eta(v)$. Together with (20), this implies that $f\eta(u)\rho f\eta(v)$. Let $g \in \mathrm{End}\,(F)$ be an extension of the mapping f^{-1}. Since ρ is closed for *-endomorphisms of F, we get $gf\eta(u)\,\rho\, gf\eta(v)$, that is, $\eta(u)\,\rho\,\eta(v)$. By (19), we conclude that $u\,\rho\, v$ and $\widehat{\gamma_{t_n}} \subseteq \rho$. Therefore, $\widehat{\gamma_{t_n}} = \rho$, as required. $\square$

The content of Theorem 8.3 (i) can be paraphrased thus: the +-operation applied to the restriction of $\widehat{t_n}$ to X^+ gives $\widehat{t_n}$ back, that is,

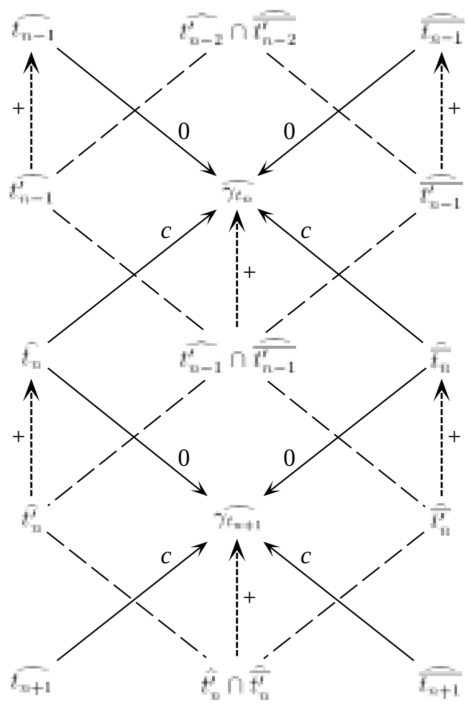


Diagram 2.

$(\widehat{t_n}\,|_{X^+})^+ = \widehat{t_n}$. As a consequence of Theorem 8.3 (ii), we obtain the following interesting formulae.

Corollary 8.4. *Let $t \in \{\chi, i\}$ and for $n \geq 2$, let $\lambda_n = \widehat{t'_n}$, $\rho_n = \widehat{\overline{t'_n}}$. Then*

$$(\lambda_n \cap \rho_n)^+ = (\lambda_n)^+ \cap (\rho_n)^+ \quad \text{for } n \geq 2,$$
$$(\lambda_n \vee \rho_n)^+ = (\lambda_n)^+ \vee (\rho_n)^+ \quad \text{for } n \geq 3.$$

Proof. The first part follows directly from Theorems 7.5 and 8.3 and the dual of Theorem 8.3 (i). The second part follows similarly by observing that $\lambda_n \vee \rho_n = \lambda_{n-1} \cap \rho_{n-1}$. □

We illustrate the situation in Theorems 7.5 and 8.3 by Diagram 2.

That the full lines in Diagram 2 represent meets and joins follows from Theorem 7.5. That the broken lines represent meets and joins follows from [**8**, Theorem 5.4] and [**9**, Theorem 5.4 (iii)]. The dotted lines connecting ts represent the transition described in Theorem 8.3 (i), those connecting $\widehat{t'_{n-1}} \cap \widehat{\widehat{t'_{n-1}}}$ and $\widehat{\gamma_{t_n}}$ are given in Theorem 8.3 (ii).

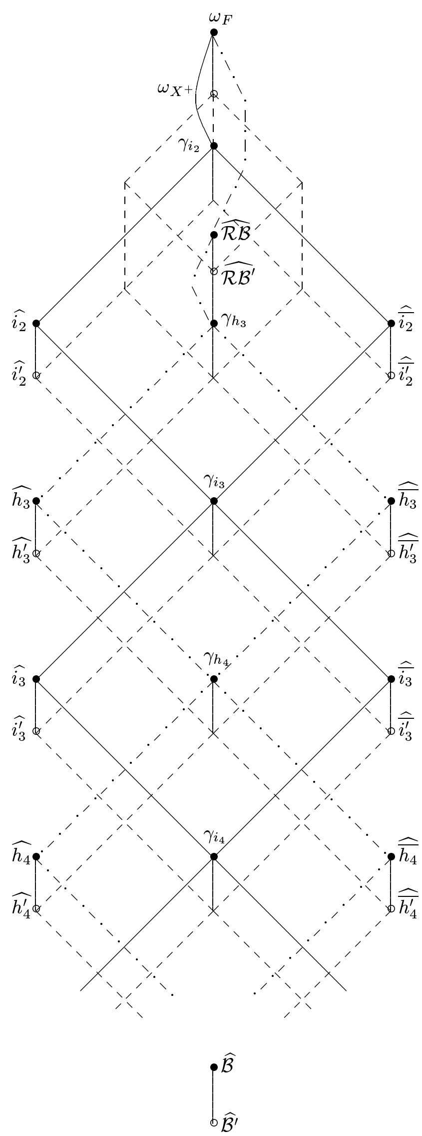
We can double Diagram 2 by drawing the cases $t = h$ and $t = i$ separately and completing its upper part. Then glue the resulting diagram onto the diagram of fully invariant congruences on a free band with countably infinite number of generators as indicated in Diagram 2 for a part of it. We thus may observe the genesis of the lattice of *-band varieties from the lattice of band varieties. This is illustrated by Diagram 3.

For an infinite set X, in Diagram 3 we have the following situation. The broken lines connect the points $\widehat{t'_n}$ and $\widehat{\widehat{t'_n}}$ for $t \in \{h, i\}$ which represent (some) fully invariant band congruences on X^+; this is the diagram of band varieties turned upside down. Full lines connect the points representing the congruences $\widehat{i_n}$ and $\widehat{\widehat{i_n}}$, dash-dot lines those for $\widehat{h_n}$ and $\widehat{\widehat{h_n}}$. Dotted lines represent the transition from some of the points of the first kind above to those of the second kind. The heavy dots in the central column represent fully invariant *-congruences on F; this is the diagram of *-band varieties turned upside down.

9. Identities. For every *-band variety, Adair [**1**] provided an identity which serves as a basis for the identities valid in that variety. She then devised an algorithm which converts an arbitrary identity on *-bands to one in her system of identities. It was proved in [**7**] that her system of identities is equivalent to the system of star-free identities depicted in Diagram 1. Combining Adair's algorithm with this result, we obtain an algorithm which, given an arbitrary identity on *-bands, produces an equivalent star-free identity.

However, the transition from a starred identity to an equivalent star-free identity is quite subtle. For *-semilattices given by the identity $x = x^*$, we get

$$xy = (xy)^* = y^*x^* = yx.$$



ω_F
ω_{X^+}
γ_{i_2}
$\widehat{\mathcal{RB}}$
$\widehat{\mathcal{RB}'}$
γ_{h_3}
$\widehat{i_2}$
$\widehat{\overline{i_2}}$
$\widehat{i_2'}$
$\widehat{\overline{i_2'}}$
γ_{i_3}
$\widehat{h_3}$
$\widehat{\overline{h_3}}$
$\widehat{h_3'}$
$\widehat{\overline{h_3'}}$
γ_{h_4}
$\widehat{i_3}$
$\widehat{\overline{i_3}}$
$\widehat{i_3'}$
$\widehat{\overline{i_3'}}$
γ_{i_4}
$\widehat{h_4}$
$\widehat{\overline{h_4}}$
$\widehat{h_4'}$
$\widehat{\overline{h_4'}}$
$\widehat{\mathcal{B}}$
$\widehat{\mathcal{B}'}$

Diagram 3.

For normal *-bands given by the identity $axa = ax^*a$, we get

$$\begin{aligned} axya = a(xy)^*a = ay^*x^*a = ay^*ay^*x^*ax^*a \\ = ay(ay^*x^*a)xa = ayax\,yaxa \end{aligned}$$

which is equivalent to the identity $axya = ayxa$ on bands. In the case of regular *-bands given by the identity $axa^*a = aa^*xa$, a somewhat longer derivation is required to deduce the validity of $axya = axaya$, see [**6**, Lemma 7.3]. Conversely, one still must show that each of the star-free identities obtained implies the given starred identity. Hence the direct conversion does not seem very promising.

By now there exist several systems of identities serving as bases for identities satisfied by various (*)-band varieties. The one of Gerhard and Petrich [**5**] was compared with that of Fennemore [**2**] in [**5**, Proposition 9.3] by means of a transformation, mapping the words figuring in the former onto those figuring in the latter. We shall now devise a function which maps the words in the Adair system to the corresponding ones in the Gerhard-Petrich system. This will establish intimate proximity of these three systems even though on the surface they appear quite different.

The Adair system of words runs as follows. Let

$$\begin{gathered} R_1 = x_1, \qquad S_1 = x_1^*x_1, \qquad R_2 = x_1x_2, \qquad S_2 = x_1x_2^*x_1x_2, \\ R_n = \overline{R_{n-2}}x_{(n+1)/2}, \qquad S_n = \overline{S_{n-2}}x_{(n+1)/2}\overline{R_{n-2}}x_{(n+1)/2} \\ \text{for} \quad n \geq 3 \text{ odd}, \\ R_n = \overline{R_{n-2}}x_{(n+2)/2}, \qquad S_n = \overline{S_{n-2}}x_{(n+2)/2}\overline{R_{n-2}}x_{(n+2)/2} \\ \text{for} \quad n \geq 4 \text{ even}. \end{gathered}$$

The identities are $R_n = S_n$ for $n \geq 1$.

For our final theorem, we shall need the following notation. Let $n \geq 1$. First let n be odd. On the semigroup $\{x_1, x_2, \dots, x_1^*\}^+$ define an endomorphism φ by the requirement

$$\varphi : \begin{cases} x_1 \to \begin{cases} x_1x_2 & \text{if } n \equiv 1 \pmod 4 \\ x_2x_1 & \text{if } n \equiv 3 \pmod 4, \end{cases} \\ x_1^* \to x_2 \\ x_i \to x_{i+1} \qquad \text{otherwise.} \end{cases}$$

Next let n be even. On the semigroup $\{x_1x_2, \dots, x_2^*\}^+$ define an endomorphism φ by the requirement

$$\varphi : \begin{cases} x_2 \to \begin{cases} x_3x_1 & \text{if } n \equiv 0 \pmod 4 \\ x_1x_3 & \text{if } n \equiv 2 \pmod 4, \end{cases} \\ x_2^* \to x_3 \\ x_i \to x_{i+1} & \text{otherwise.} \end{cases}$$

Theorem 9. *For $n \geq 1$, we have*

$$\begin{aligned} &\varphi(R_n) = \overline{G_{(n+3)/2}}, \qquad \varphi(S_n) = \overline{I_{(n+3)/2}} \quad \textit{if } n \textit{ is odd}, \\ &\varphi(R_n) = \overline{G_{(n+4)/2}}, \qquad \varphi(S_n) = \overline{H_{(n+4)/2}} \quad \textit{if } n \textit{ is even}, \end{aligned}$$

and the identities $R_n = S_n$ and $\varphi(R_n) = \varphi(S_n)$ are equivalent.

Proof. Let n be odd. We use induction on n. For a given n we may denote the mapping φ defined above by φ_n. Straightforward verification shows that $\overline{\varphi_n} = \varphi_{n-2}$ for every $n \geq 3$ odd. First

$$\varphi_1(R_1) = \varphi_1(x_1) = x_1x_2 = \overline{G_2}.$$

Assuming the statement true for $n-2$ where $n \geq 3$, we get

$$\begin{aligned} \varphi_n(R_n) &= \varphi_n(\overline{R_{n-2}}x_{(n+1)/2}) = \overline{\overline{\varphi_n}(R_{n-2})}\varphi_n\Big(x_{(n+1)/2}\Big) \\ &= \overline{\varphi_{n-2}(R_{n-2})}x_{(n+3)/2} \\ &= G_{(n+1)/2}x_{(n+3)/2} = \overline{x_{(n+3)/2}\,\overline{G_{(n+1)/2}}} = \overline{G_{(n+3)/2}}. \end{aligned}$$

Next

$$\varphi_1(S_1) = \varphi_1(x_1^*x_1) = x_2x_1x_2 = \overline{I_2}.$$

Assuming the statement true for $n-2$, we obtain

$$\begin{aligned} \varphi_n(S_n) &= \varphi_n(\overline{S_{n-2}}x_{(n+1)/2}\overline{R_{n-2}}x_{(n+1)/2}) \\ &= \overline{\overline{\varphi_n}(S_{n-2})}\varphi_n(x_{(n+1)/2})\varphi_n(R_n) \\ &= \overline{\varphi_{n-2}(S_{n-2})}x_{(n+3)/2}\overline{G_{(n+3)/2}} \\ &= I_{(n+1)/2}x_{(n+3)/2}\overline{G_{(n+3)/2}} \\ &= \overline{G_{(n+3)/2}x_{(n+3)/2}\overline{I_{(n+1)/2}}} = \overline{I_{(n+3)/2}}. \end{aligned}$$

The argument for n even is similar.

For n odd, letting $n = 2m+1$, we get $(n+3)/2 = m+2$, where $m = 0, 1, \ldots$, thereby obtaining the identities $\overline{I_2}, \overline{I_3}, \ldots$. For n even, letting $n = 2m$, we get $(n+4)/2 = m+2$ where $m = 1, 2, \ldots$, thereby obtaining the identities $\overline{H_3}, \overline{H_4}, \ldots$. On *-bands, any identity $u = v$ is equivalent to its dual $\bar{u} = \bar{v}$. Hence the identities I_n and $\overline{I_n}$, as well as the identities H_n and $\overline{H_n}$, are equivalent. The main result of [**7**, Sections 3–6] asserts that the identities $R_1 = S_1$, $R_2 = S_2$, $R_3 = S_3, \ldots$ are equivalent to the identities $G_2 = I_2$, $G_3 = H_3$, $G_3 = I_3, \ldots$ in that order. This establishes the final assertion of the theorem. □

Theorem 9.1 covers all *-band varieties except for the varieties of: trivial *-bands, rectangular *-bands and *-bands. These varieties generally require different treatment.

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