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INTERPOLATION PROBLEMS IN CSL-ALGEBRA ALG$\mathcal{L}$

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ABSTRACT. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. In this paper we obtained a necessary and sufficient condition for the existence of a solution A which is in CSL-algebra $\operatorname{Alg}\mathcal{L}$.

1. Introduction. Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$, and let x and y be vectors in $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the 'n-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [**6**]. In case $\mathcal{U}$ is a nest algebra, the interpolation problem was solved by Lance [**7**]; his result was extended by Hopenwasser [**2**] to the case that $\mathcal{U}$ is a CSL-algebra. More recently, Munch [**8**] obtained conditions for interpolation in case T is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [**3**] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contain a sufficient condition for interpolation of n-vectors, although necessity was not proved in that paper.

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of (self-adjoint) pairwise-commuting projections acting on a separable Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, $\operatorname{Alg}\mathcal{L}$ is called a CSL-algebra. The algebra $\operatorname{Alg}\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all projections in $\mathcal{L}$. Let x and y

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be vectors in a Hilbert space. Then $\langle x, y\rangle$ means the inner product of vectors x and y. In this paper, *we use the convention* $\frac{0}{0} = 0$, when necessary.

2. Interpolation problems in CSL-algebra $\operatorname{Alg}\mathcal{L}$**.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and 1. Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$. Let f be a vector in a Hilbert space $\mathcal{H}$ and $\{f_n\}$ a sequence of vectors in $\mathcal{H}$. Then $f_n \to f$ or $\lim_{n\to\infty} f_n = f$ means that the sequence $\{f_n\}$ converges to f on the norm topology on $\mathcal{H}$. Let N be the set of all natural numbers, and let $\mathbf{C}$ be the set of all complex numbers.

Theorem 1. *Let* $\mathcal{H}$ *be a Hilbert space and* $\mathcal{L}$ *a commutative subspace lattice on* $\mathcal{H}$*. Let* x *and* y *be vectors in* $\mathcal{H}$*. Then the following statements are equivalent.*

(1) *There is an operator* A *in* $\operatorname{Alg}\mathcal{L}$ *such that* $Ax = y$ *and every* E *in* $\mathcal{L}$ *reduces* A*.*

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} : l \in N,\ \alpha_i \in \mathbf{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty.$

Proof. Suppose that there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A. Then $\alpha EAx = A\alpha Ex = \alpha Ey$ for every E in $\mathcal{L}$ and for every α in $\mathbf{C}$. So $A(\sum_{i=1}^{l}\alpha_i E_i x) = \sum_{i=1}^{l}\alpha_i E_i y$ for $l \in N$, $\alpha_i \in \mathbf{C}$ and $E_i \in \mathcal{L}$. Thus $\|\sum_{i=1}^{l}\alpha_i E_i y\| = \|A(\sum_{i=1}^{l}\alpha_i E_i x)\| \le \|A\|\|\sum_{i=1}^{l}\alpha_i E_i x\|$.

If $\|\sum_{i=1}^{l}\alpha_i E_i x\| \neq 0$, then $\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} \le \|A\|$.

Hence, $\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} : l \in N,\ \alpha_i \in \mathbf{C} \text{ and } E_i \in \mathcal{L}\right\} \le \|A\|$.

If $\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} : l \in N,\ \alpha_i \in \mathbf{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$, then, without loss of generality, we may assume that

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} : l\in N,\ \alpha_i\in \mathbf{C} \text{ and } E_i\in\mathcal{L}\right\}=1.$$

So $\|\sum_{i=1}^{l}\alpha_i E_i y\| \le \|\sum_{i=1}^{l}\alpha_i E_i x\|,\ l\in N,\ \alpha_i\in\mathbf{C}$ and $E_i\in\mathcal{L}\ldots(*)$. Let $\mathcal{M} = \{\sum_{i=1}^{l}\alpha_i E_i x : l\in N, \alpha_i \in \mathbf{C}$ and $E_i \in \mathcal{L}\}$. Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M}\to\mathcal{H}$ by $A(\sum_{i=1}^{l}\alpha_i E_i x) = \sum_{i=1}^{l}\alpha_i E_i y$. Then A is well defined. For, if $\sum_{i=1}^{l}\alpha_i E_i x = \sum_{j=1}^{m}\beta_j E_j x$, then $\sum_{i=1}^{l}\alpha_i E_i x + \sum_{j=1}^{m}(-\beta_j)E_j x = 0$. So $\|\sum_{i=1}^{l}\alpha_i E_i x + \sum_{j=1}^{m}(-\beta_j)E_j x\| = 0$ and hence $\|\sum_{i=1}^{l}\alpha_i E_i y + \sum_{j=1}^{m}(-\beta_j)E_j y\| = 0$ by $(*)$. Thus $\sum_{i=1}^{l}\alpha_i E_i y = \sum_{j=1}^{m}\beta_j E_j y$, i.e., $A(\sum_{i=1}^{l}\alpha_i E_i x) = A(\sum_{j=1}^{m}\beta_j E_j x)$. Extend A to $\overline{\mathcal{M}}$ by continuity, and define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. Clearly, $Ax = y$ and $\|A\|\le 1$.

Now we must show that E reduces A or $AE = EA$ for every E in $\mathcal{L}$. Let f be in $\mathcal{H}$, and let $f = \sum_{i=1}^{l}\alpha_i E_i x \oplus g$, where $g\in\overline{\mathcal{M}}^{\perp}$. Then, for every E in $\mathcal{L}$,

$$\begin{aligned} AEf &= AE\left(\sum_{i=1}^{l}\alpha_i E_i x + g\right)\\ &= A\left(\sum_{i=1}^{l}\alpha_i E E_i x\right) + AEg\\ &= \sum_{i=1}^{l}\alpha_i E E_i y \quad \text{because } Eg\in\overline{\mathcal{M}}^{\perp}\\ &= E\left(\sum_{i=1}^{l}\alpha_i E_i y\right)\\ &= EA\left(\sum_{i=1}^{l}\alpha_i E_i x + g\right) \quad \text{because } Ag = 0. \end{aligned}$$

So $AE = EA$ for every $E\in\mathcal{L}$.

If we modify the proof of Theorem 1 a little, we can prove the following theorem.

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ a commutative subspace lattice on $\mathcal{H}$. Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ be two sequences of vectors in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *There is an operator A in $\operatorname{Alg}\mathcal{L}$ such that $Ax_i = y_i$, $i = 1, \ldots, n$, and every E in $\mathcal{L}$ reduces A.*

(2)

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \leq n,\ E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbf{C}\right\} < \infty.$$

Proof. Suppose that there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that $Ax_i = y_i$, $i = 1, 2, \ldots, n$, and every E in $\mathcal{L}$ reduces A. Then $\alpha EAx_i = A\alpha Ex_i = \alpha Ey_i$ for every E in $\mathcal{L}$ and for every α in $\mathbf{C}$, $i = 1, 2, \ldots, n$. So $A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i) = \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i$, $m_i \in N$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $\alpha_{k,i} \in \mathbf{C}$. Thus $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\| \leq \|A\|\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|$. If $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\| \neq 0$, then

$$\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} \leq \|A\|.$$

Hence

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \leq n,\ E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbf{C}\right\} < \infty.$$

If

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \leq n,\ E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbf{C}\right\} < \infty,$$

then, without loss of generality, we may assume that

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \leq n,\ E_{k,i} \in \mathcal{L}\text{and } \alpha_{k,i} \in \mathbf{C}\right\} = 1.$$

So

$$\left\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\right\| \le \left\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\right\|,\ m_i\in N,\ l\le n,\ E_{k,i}\in\mathcal{L}$$

and $\alpha_{k,i}\in\mathbf{C}\cdots(*)$. Let $\mathcal{M}=\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i\in N, l\le n, \alpha_{k,i}\in\mathbf{C}$ and $E_{k,i}\in\mathcal{L}\}$. Then $\mathcal{M}$ is a linear manifold. Define $A:\mathcal{M}\to\mathcal{H}$ by $A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i)=\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i$. Then A is well defined.

For, if $\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i = \sum_{k=1}^{m_j}\sum_{j=1}^{t}\beta_{k,j}E_{k,j}x_j$, then $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i+\sum_{k=1}^{m_j}\sum_{j=1}^{t}(-\beta_{k,j})E_{k,j}x_j\|=0$ and hence $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i+\sum_{k=1}^{m_j}\sum_{j=1}^{t}(-\beta_{k,j})E_{k,j}y_j\|=0$ by $(*)$. Thus, $\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i=\sum_{k=1}^{m_j}\sum_{j=1}^{t}\beta_{k,j}E_{k,j}y_j$.

Extend A to $\overline{\mathcal{M}}$ by continuity and define $A|_{\overline{\mathcal{M}}^\perp}=0$. Clearly, $Ax_i=y_i$, $i=1,2,\dots,n$, and $\|A\|\le 1$.

Now we must show that E reduces A or $AE=EA$ for every E in $\mathcal{L}$. Let f be in $\mathcal{H}$, and let $f=(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i)\oplus g$, where $g\in\overline{\mathcal{M}}^\perp$. Then, for every E in $\mathcal{L}$,

$$\begin{aligned}AEf&=AE\bigg(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i+g\bigg)\\&=A\bigg(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}EE_{k,i}x_i\bigg)+AEg\\&=\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}EE_{k,i}y_i\quad\text{because } Eg\in\overline{\mathcal{M}}^\perp\end{aligned}$$

and

$$\begin{aligned}EAf&=EA\bigg(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i+g\bigg)\\&=E\bigg(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\bigg)+Ag\\&=\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}EE_{k,i}y_i\quad\text{because } Ag=0.\end{aligned}$$

So $AE = EA$ for every $E \in \mathcal{L}$.

If we modify the proof of Theorem 2 a little, we can get the following theorem. So we omit its proof.

Theorem 3. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *There is an operator A in* $\operatorname{Alg}\mathcal{L}$ *such that $Ax_n = y_n$, $n = 1, 2, \dots,$ and every E in $\mathcal{L}$ reduces A.*

(2)

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i,\, l\in N,\, E_{k,i}\in\mathcal{L} \text{ and } \alpha_{k,i}\in\mathbf{C}\right\} < \infty.$$

Theorem 4. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $x_1, \dots, x_n$ and y be vectors in $\mathcal{H}$. If there are operators $A_1, \dots, A_n$ in* $\operatorname{Alg}\mathcal{L}$ *such that $\sum_{k=1}^{n} A_k x_k = y$ and every E in $\mathcal{L}$ reduces A_k, $k = 1, 2, \dots, n$, then*

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\sum_{k=1}^{n}\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} : l\in N,\, E_i\in\mathcal{L} \text{ and } \alpha_i\in\mathbf{C}\right\} < \infty.$$

Proof. Since $\sum_{k=1}^{n} A_k x_k = y$ and $A_k E = E A_k$ for every $k = 1, 2, \dots, n$, and every E in $\mathcal{L}$, $\sum_{k=1}^{n} A_k(\sum_{i=1}^{l}\alpha_i E_i x_k) = \sum_{i=1}^{l}\alpha_i E_i y$, $l \in N$, $\alpha_i \in \mathbf{C}$ and $E_i \in \mathcal{L}$. So

$$\left\|\sum_{i=1}^{l}\alpha_i E_i y\right\| \le [\sup_k \|A_k\|]\left(\sum_{k=1}^{n}\left\|\sum_{i=1}^{l}\alpha_i E_i x_i\right\|\right).$$

If $\sum_{k=1}^{n}\|\sum_{i=1}^{l}\alpha_i E_i x_k\| \neq 0$, then

$$\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\sum_{k=1}^{n}\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} < \sup_k \|A_k\|.$$

Hence,

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\sum_{k=1}^{n}\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} : l \in N,\ E_i \in \mathcal{L} \text{ and } \alpha_i \in \mathbf{C}\right\} < \infty.$$

Theorem 5. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, \dots, x_n$ and y be vectors in $\mathcal{H}$. If*

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} : l \in N,\ E_i \in \mathcal{L} \text{ and } \alpha_i \in \mathbf{C}\right\} < \infty$$

for all $k = 1, 2, \dots, n$, then there are operators $A_1, \dots, A_n$ in $\operatorname{Alg}\mathcal{L}$ *such that $\sum_{k=1}^{n} A_k x_k = y$ and $EA_k = A_k E$ for every E in $\mathcal{L}$ and $k = 1, 2, \dots, n$.*

Proof. Put $y/n = y_k$, $k = 1, 2, \dots, n$. Since

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} : l \in N,\ E_i \in \mathcal{L} \text{ and } \alpha_i \in \mathbf{C}\right\} < \infty,$$
$$\sup\left\{\frac{\|\sum_{i=1}^{l}\alpha_i E_i y_k\|}{\|\sum_{i=1}^{l}\alpha_i E_i x_k\|} : l \in N,\ E_i \in \mathcal{L} \text{ and } \alpha_i \in \mathbf{C}\right\} < \infty,$$

and hence there is an operator A_k in $\operatorname{Alg}\mathcal{L}$ such that $A_k x_k = y_k$ and $EA_k = A_k E$ for every E in $\mathcal{L}$ and all $k = 1, 2, \dots, n$ by Theorem 1. So $A_1 x_1 + A_2 x_2 + \cdots + A_n x_n = y_1 + y_2 + \cdots + y_n = y$.

We want to apply Theorem 2 to concrete examples.

Example 1. Let $\mathcal{H}$ be a Hilbert space with an orthonormal base $\{e_1, e_2, \dots, e_n\}$ and $\mathcal{L} = \{[0], [e_1], [e_1 e_2], [e_1 e_2 e_3], \dots, [e_1, e_2, \dots, e_n]\}$.

Let $\mathbf{A} = \operatorname{Alg}\mathcal{L}$. Then B is in $\operatorname{Alg}\mathcal{L}$ if and only if B has the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, \dots, e_n\}$, where all nonstarred entries are zero.

Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ be two sequences of vectors in $\mathcal{H}$. Assume that

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \le t,\ \alpha_{k,i} \in \mathbf{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty.$$

Then there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that (i) $Ax_i = y_i$, $i = 1, 2, \dots, t$, (ii) every E reduces A, (iii) A is diagonal, and (iv) $\beta_{ji} = \alpha_{jj}\alpha_{ji}$, $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, n$, where $x_i = (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni})^t$ and $y_i = (\beta_{1i}, \beta_{2i}, \dots, \beta_{ni})^t$. For, by arguments similar to those of the proof of Theorem 2, we can get the above results.

Example 2. Let $\mathcal{H}$ be a Hilbert space with an orthonormal base $\{e_1, e_2, \dots, e_{2n+1}\}$. Let $\mathcal{L} = \{[0], [e_1], [e_1e_2e_3], [e_1e_2e_3e_4e_5], \dots, [e_1, e_2, e_3, \dots, e_{2n+1}]\}$.

Let B be in $\mathcal{B}(\mathcal{H})$. Then B is in $\operatorname{Alg}\mathcal{L}$ if and only if B has the form

$$\begin{pmatrix}
* & * & * & * & * & \cdots & * & * \\
 & * & * & * & * & \cdots & * & * \\
 & * & * & * & * & \cdots & * & * \\
 & & & * & * & \cdots & * & * \\
 & & & * & * & \cdots & * & * \\
 & & & & & \ddots & \vdots & \vdots \\
 & & & & & & * & * \\
 & & & & & & * & *
\end{pmatrix}$$

with respect to the basis $\{e_1, e_2, \dots, e_{2n+1}\}$, where all nonstarred entries are zero.

Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ be two sequences of vectors in $\mathcal{H}$. Assume that

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\ l \le t,\ \alpha_{k,i} \in \mathbf{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty.$$

Then there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that

(i) $Ax_i = y_i$, $i = 1, 2, \dots, t$, (ii) every E in $\operatorname{Alg}\mathcal{L}$ reduces A, (iii) A has the form

$$\begin{pmatrix} * & & & & & & & \\ & * & * & & & & & \\ & * & * & & & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ & & & & & \ddots & & \\ & & & & & & * & * \\ & & & & & & * & * \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, \dots, e_{2n+1}\}$, where all nonstarred entries are zero, and (iv) $\beta_{1j} = a_{11}\alpha_{1j}, \beta_{2p,j} = a_{2p,2p}\alpha_{2p,j} + a_{2p,2p+1}\alpha_{2p+1,j}$ and $\beta_{2p+1,j} = a_{2p+1,2p}\alpha_{2p,j} + a_{2p+1,wp+1}\alpha_{2p+1,j}$, $j = 1, 2, \dots, t$ and $p = 1, 2, \dots, n$, where $A = (a_{pp})$, $x_j = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{2n+1,j})^t$ and $y_j = (\beta_{1j}, \beta_{2j}, \dots, \beta_{2n+1,j})^t$.

For, by Theorem 2, there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that (i) $Ax_i = y_i$, $i = 1, 2, \dots, t$, and every E in $\mathcal{L}$ reduces A. If we put $E = [e_1]$, $E = [e_1, e_2, e_3]$, $E = [e_1, e_2, e_3, e_4, e_5], \dots$, $E = [e_1, e_2, \dots, e_{2n+1}]$ in turn in the equation $AE = EA$, and if we compare components of AE with those of EA, then (iii) A has the desired form. We know that $AEx_i = Ey_i$ in the proof of Theorem 1, $i = 1, 2, \dots, t$. (iv) If we put $E = [e_1]$, $E = [e_1, e_2, e_3]$, $E = [e_1, e_2, e_3, e_4, e_5], \dots$, $E = [e_1, e_2, \dots, e_{2n+1}]$ in turn in the equation $AEx_i = Ey_i$, $i = 1, 2, \dots, t$, and if we compare each component of AEx_i with that of Ey_i, then we can get $\beta_{1j} = a_{11}\alpha_{1j}$, $\beta_{2p,j} = a_{2p,2p}\alpha_{2p,j} + \alpha_{2p,2p+1}\alpha_{2p+1,j}$ and $\beta_{2p+1,j} = a_{2p+1,2p}\alpha_{2p,j} + a_{2p+1,2p+1}\alpha_{2p+1,j}$, $j = 1, 2, \dots,$ and $p = 1, 2, \dots, n$, where $A = (a_{pp})$, $x_i = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{2n+1,j})^t$ and $y_i = (\beta_{1j}, \beta_{2,j}, \dots, \beta_{2n+1,j})^t$.

Example 3. Let $\mathcal{H}$ be a Hilbert space with an orthonormal base $\{e_1, e_2, \dots, e_9\}$. Let $\mathcal{L}$ be the lattice generated by $\{[0], [e_1, e_2], [e_1, e_2, e_3], [e_4, e_5], [e_1, \dots, e_8], [e_9]\}$.

Let B be a $\mathcal{B}(\mathcal{H})$. Then B is in $\operatorname{Alg}\mathcal{L}$ if and only if B has the form

$$\begin{pmatrix} * & * & * & & & * & * & * & \\ * & * & * & & & * & * & * & \\ & & * & & & * & * & * & \\ & & & * & * & * & * & * & \\ & & & * & * & * & * & * & \\ & & & & & * & * & * & \\ & & & & & * & * & * & \\ & & & & & * & * & * & \\ & & & & & & & & * \end{pmatrix}$$

with respect to the basis $\{e_1, \dots, e_9\}$, where all nonstarred entries are zero.

Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ be two sequences of vectors in $\mathcal{H}$, $t \leq 9$. Assume that

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in N,\, l \leq t,\, \alpha_{k,i} \in \mathbf{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty.$$

Then there is an operator A in $\operatorname{Alg}\mathcal{L}$ such that

(i) $Ax_i = y_i$, $i = 1, 2, \dots, t$, (ii) every E in $\mathcal{L}$ reduces A, (iii) A has the form

$$\begin{pmatrix} * & * & & & & & & & \\ * & * & & & & & & & \\ & & * & & & & & & \\ & & & * & * & & & & \\ & & & * & * & & & & \\ & & & & & * & * & * & \\ & & & & & * & * & * & \\ & & & & & * & * & * & \\ & & & & & & & & * \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, \dots, e_9\}$, where all nonstarred entries are zero, and

(iv)

$$\begin{aligned}
&\beta_{1j} = a_{11}\alpha_{1j} + a_{12}\alpha_{2j}, \beta_{2j} = a_{21}\alpha_{1j} + a_{22}\alpha_{2j},\\
&\beta_{3j} = a_{33}\alpha_{3j},\\
&\beta_{4j} = a_{44}\alpha_{4j} + a_{45}\alpha_{5j},\\
&\beta_{5j} = a_{54}\alpha_{4j} + a_{55}\alpha_{5j},\\
&\beta_{6j} = a_{66}\alpha_{6j} + a_{67}\alpha_{7j} + a_{68}\alpha_{8j},\\
&\beta_{7j} = a_{76}\alpha_{6j} + a_{77}\alpha_{7j} + a_{78}\alpha_{8j}\\
&\beta_{8j} = a_{86}\alpha_{6j} + a_{87}\alpha_{7j} + a_{88}\alpha_{8j} \text{ and}\\
&\beta_{9j} = a_{99}\alpha_{9j}, \quad j = 1, 2, \dots, 9.
\end{aligned}$$

For, by arguments similar to those of the proof of Examples 1 and 2, we can get the above results.

We can get much information from Examples 1, 2 and 3 about A, x_i and y_i.

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