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GRAPHS OF CONVEX FUNCTIONS ARE σ1-STRAIGHT

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ABSTRACT. A set $E \subseteq \mathbf{R}^n$ is s-straight for $s > 0$ if E has finite Method II outer s-measure equal to its Method I outer s-measure. If E is Method II s-measurable, this means E has finite Hausdorff s-measure equal to its Hausdorff s-content. The graph Γ of a convex function $f : [a, b] \to \mathbf{R}$ is shown to be a countable union of 1-straight sets, and to contain a 1-straight set maximal in the sense that its Hausdorff 1-measure equals the diameter of Γ.

1. Introduction. In [**7**], Foran introduced the notion of an s-straight set (Definition 2), that is, a set whose (finite) Hausdorff s-measure and Hausdorff s-content are equal. In [**1**], [**2**] we continued the first analysis of such sets, among other results proving that a quarter circle is a countable union of 1-straight sets, verifying a conjecture of Foran. Here, by a different argument we extend that result, proving that the graph of any convex function $f : [a, b] \to \mathbf{R}$ is a countable union of 1-straight sets (Theorem 7). In [**4**], using yet another different argument, we extend this result further to graphs of continuously differentiable, absolutely continuous, and increasing continuous functions, as well as to regular 1-sets in $\mathbf{R}^2$. Finally, in [**3**] we prove a general theorem which implies that every set of finite s-measure is a countable union of s-straight sets.

Before proceeding to the main results, we provide some necessary background information. Let d be the standard distance function on $\mathbf{R}^n$ where $n \geq 1$. The diameter of an arbitrary nonempty set $U \subseteq \mathbf{R}^n$ is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$, with $|\varnothing| = 0$. Given $0 < \delta \leq \infty$, let C^n_δ represent the collection of subsets of $\mathbf{R}^n$ with diameter less than δ.

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Definition 1. For $s > 0$ and $E \subseteq \mathbf{R}^n$, let

$$s\text{-}m_\delta^*(E) = \inf\left\{ \sum |E_i|^s : E \subseteq \bigcup E_i \text{ where } E_i \in C_\delta^n \text{ for } i = 1, 2, ... \right\}.$$

Define $s\text{-}m_I^*(E) = s\text{-}m_\infty^*(E)$ and $s\text{-}m_{II}^*(E) = \sup_{\delta>0} s\text{-}m_\delta^*(E)$. The outer measure $s\text{-}m_I^*(E)$ is constructed by what is called Method I, and is called Hausdorff s-content. The outer measure $s\text{-}m_{II}^*(E)$ is constructed by what is called Method II, and when restricted to the σ-field of $s\text{-}m_{II}^*$-measurable sets is called Hausdorff s-measure, or $\mathcal{H}^s$-measure. A set $E \subseteq \mathbf{R}^n$ is called an s-set if it is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$.

Note that $\mathcal{H}^s$-measure is a metric outer measure, so that closed, and hence compact, sets are $\mathcal{H}^s$-measurable.

Definition 2 [**1**], [**2**]. Define a set $E \subseteq \mathbf{R}^n$ to be s-straight if

$$s\text{-}m_I^*(E) = s\text{-}m_{II}^*(E) < \infty.$$

So, when $s = 1$, we say 1-straight. A set which is the countable union of s-straight sets will be called σs-straight. When $s = 1$, we say $\sigma 1$-straight.

In [**7**], Foran proves the following theorem which provides a useful equivalent definition of an s-straight set that does not require the calculation of $s\text{-}m_I^*$. Henceforth we will often use this result without reference.

Theorem 1 [**7**, p. 733]. *Let $E \subseteq \mathbf{R}^n$ satisfy $s\text{-}m_{II}^*(E) < \infty$. Then E is s-straight if and only if $s\text{-}m_{II}^*(A) \leq |A|^s$ for each $s\text{-}m_{II}^*$-measurable $A \subseteq E$. This last condition can be written*

$$\mathcal{H}^s(A) \leq |A|^s.$$

In particular, sets of zero $\mathcal{H}^s$-measure are s-straight.

In the same paper [**7**], the following corollary appears. A proof is provided in [**1**], [**2**].

Corollary 1 [**7**, p. 734]. *$\mathcal{H}^s$-measurable subsets A of an s-straight set $E \subseteq \mathbf{R}^n$ are s-straight. In particular, intersections of s-straight sets are s-straight.*

By Theorem 1, to show that a set E is s-straight it then suffices to show for all (bounded) $\mathcal{H}^s$-measurable subsets $A \subseteq E$ that $\mathcal{H}^s(A) \leq |A|^s$.

Theorem 2 [**1**], [**2**]. *Let $E \subseteq \mathbf{R}^n$ have finite s-measure. Every $\mathcal{H}^s$-measurable subset of positive $\mathcal{H}^s$-measure of E contains an s-straight set of positive $\mathcal{H}^s$-measure if and only if E is σs-straight.*

Definition 3. A (closed) line segment in $\mathbf{R}^n$ is the image under an isometry of a closed (non-degenerate) interval in $\mathbf{R}$. The length $\mathcal{L}(E)$ of a line segment E with endpoints x and y is defined by $\mathcal{L}(E) = |E| = d(x, y)$. Following [**9**, p. 197], an arc in $\mathbf{R}^n$ is defined to be the image of a homeomorphism $f : [0, 1] \to \mathbf{R}^n$. In particular, an arc does not cross itself. The length of an arc Λ is defined to be $\mathcal{L}(\Lambda) = \sup \sum_{i=1}^{m} d(f(t_{i-1}), f(t_i))$, where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0, 1]$.

A well-known fact will be helpful.

Theorem 3 [**5**, p. 29]. *If $\Lambda \subseteq \mathbf{R}^n$ for $n \geq 1$ is an arc, then $\mathcal{H}^1(\Lambda) = \mathcal{L}(\Lambda)$.*

The next two results are also proved in [**1, 2**].

Theorem 4 [**1**], [**2**]. *If $E \subseteq \mathbf{R}^n$ for $n \geq 1$ is a (non-degenerate) line segment, then $0 < |E| = \mathcal{L}(E) = \mathcal{H}^1(E) < \infty$, and E is a 1-straight 1-set.*

Theorem 5 [**1**], [**2**]. *Let $E_1, E_2 \subseteq \mathbf{R}^n$ be nonoverlapping line segments. The set $E = E_1 \cup E_2$ is a 1-straight 1-set if and only if $|E_1 \cup E_2| \geq |E_1| + |E_2|$.*

2. Main results.

Definition 4 [**6**, p. 363]. A function $f : [a,b] \to \mathbf{R}$ is convex if for $x_1, x_2, x_3 \in [a,b]$ where $x_1 < x_2 < x_3$ it follows that

$$f(x_2) \leq f(x_1) \cdot \frac{x_3 - x_2}{x_3 - x_1} + f(x_3) \cdot \frac{x_2 - x_1}{x_3 - x_1}.$$

If $f : [a,b] \to \mathbf{R}$ is convex, let $\Gamma = \{(x, f(x)) : x \in [a,b]\}$ denote its graph. Denote the length of Γ by $\mathcal{L}(\Gamma)$, as in Definition 3. Let $\Gamma(u,v)$ represent the closed arc of Γ between the points $u, v \in \Gamma$. Let the line segment between any two points on Γ be called a secant. That f is convex means every such point $(x_2, f(x_2))$ in the definition is below or on the secant connecting the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$. By [**6**, p. 364], if $f : [a,b] \to \mathbf{R}$ is convex, then f is continuous on (a,b), and differentiable except at most at a countable set of points.

Lemma 1. *Let $f : [a,b] \to \mathbf{R}$ be a convex function whose graph Γ contains no line segments. Then Γ can be written as the union of at most two isolated endpoints and at most two continuous arcs such that for any points p_1 and p_2 in a given arc, $|\Gamma(p_1,p_2)| = d(p_1,p_2) > (1/2)\mathcal{L}(\Gamma(p_1,p_2))$.*

Proof. Since f is continuous on a closed interval, it attains both a maximum and a minimum value. So Γ is circumscribed by the rectangle formed by the supporting lines $x = a$, $x = b$, $y = \min\{f(x) : x \in [a,b]\}$, and $y = \max\{f(x) : x \in [a,b]\}$. Since f is a convex function, and Γ contains no line segments, Γ intersects the lines $x = a$, $x = b$, or $y = \min\{f(x) : x \in [a,b]\}$ in at most one point, and the line $y = \max\{f(x) : x \in [a,b]\}$ in at most two points. The graph Γ then consists of at most two isolated endpoints, and at most two continuous arcs, say Γ_1, Γ_2, each of which intersects adjacent sides of the rectangle. For say Γ_1, the secant of length $|\Gamma_1|$ connecting its endpoints is the hypotenuse of a right triangle formed with adjacent sides of the rectangle. Since Γ_1 is contained in this triangle, for any points $p_1, p_2 \in \Gamma_1$ the property $|\Gamma_1(p_1,p_2)| = d(p_1,p_2)$ holds because a corresponding right triangle can be circumscribed about $\Gamma_1(p_1,p_2)$. Finally, let r, s be the lengths of the sides of the right triangle for which $t = |\Gamma_1(p_1,p_2)| = d(p_1,p_2)$ is the length of the hypotenuse. Since $\Gamma_1(p_1,p_2)$ is the graph of an increasing, or decreasing, function contained in this triangle, it is well-known and follows from the definition of the length of a convex arc that $\mathcal{L}(\Gamma_1(p_1, p_2)) \leq r + s < t + t = 2 \cdot d(p_1, p_2)$. Thus in a given arc, $d(p_1, p_2) > (1/2)\mathcal{L}(\Gamma(p_1, p_2))$ as desired. □

Theorem 6. *Let $f : [a, b] \to \mathbf{R}$ be a convex function whose graph Γ contains no line segments. Then Γ contains a perfect 1-straight 1-set P which is maximal in the sense that $\mathcal{H}^1(P) = |\Gamma|$.*

Proof. By Lemma 1 we can take Γ to be a (non-degenerate) continuous arc such that for any points $p_1, p_2 \in \Gamma$ we have $|\Gamma(p_1, p_2)| = d(p_1, p_2)$. We now construct a particular subset $P \subseteq \Gamma$. Let $a^* = (a, f(a))$ and $b^* = (b, f(b))$. Note that $|\Gamma| = d(a^*, b^*)$. At stage 0 of the construction, let the points $a_{0,1}, b_{0,1} \in \Gamma$ satisfy $d(a^*, a_{0,1}) = (1/2) \cdot |\Gamma| = d(b_{0,1}, b^*)$, and remove the open arc $\Gamma(a_{0,1}, b_{0,1})$ from Γ. (Intuitively, divide the secant from a^* to b^* in half and rotate each half secant toward Γ about the endpoints a^* and b^*, respectively, until they intersect Γ in two new points.) At stage 1 of the construction, let the additional points $a_{1,1}, a_{1,2}, b_{1,1}, b_{1,2} \in \Gamma$ satisfy $d(a^*, a_{1,1}) = d(a_{1,2}, a_{0,1}) = (1/2^2) \cdot |\Gamma| = d(b_{0,1}, b_{1,2}) = d(b_{1,1}, b^*)$, and remove from the two remaining arcs of Γ two open arcs $\Gamma(a_{1,1}, a_{1,2}) \subseteq \Gamma(a^*, a_{0,1})$ and $\Gamma(b_{1,2}, b_{1,1}) \subseteq \Gamma(b_{0,1}, b^*)$. (Intuitively, divide each of the two equal length secants from stage 0 in half and rotate these half secants toward Γ about the points of intersection with Γ until they meet Γ in four new points.) In general at stage m of the construction remove from the 2^m remaining arcs of Γ a collection of 2^m open arcs of Γ in the same manner. Call the perfect set which remains, P. We claim that P is a 1-straight 1-set. For each $m = 0, 1, 2, \ldots$, let B_m represent the union of the collection of 2^{m+1} disjoint equal length secants, as described above, corresponding to the remaining arcs in the construction of P at stage m. For each $m = 0, 1, 2, \ldots$, we have $\mathcal{H}^1(B_m) = |\Gamma|$. Note that since P is perfect and bounded, it is compact. Thus, any open cover of P has a finite subcover, and for sufficiently large m it follows that B_m is contained in that open subcover. So $\mathcal{H}^1(B_m) \leq \mathcal{H}^1(P)$. But since P can be covered with sets having the same diameter as the secant lines in B_m, it follows that $\mathcal{H}^1(P) \leq \mathcal{H}^1(B_m)$. Therefore we also have $\mathcal{H}^1(P) = \mathcal{H}^1(B_m) = |\Gamma|$. So P is a 1-set. By Theorem 1, the set P will be 1-straight if for each

$\mathcal{H}^1$-measurable $A \subseteq P$ it follows that $\mathcal{H}^1(A) \leq |A|$. Here we can write that $A = P \cap \Gamma(p_1, p_2)$ for some $p_1, p_2 \in P$ or its closure. So it suffices to show that $\mathcal{H}^1(P \cap \Gamma(p_1, p_2)) \leq |P \cap \Gamma(p_1, p_2)|$. Let $B_m(p_1, p_2)$ be the union of the disjoint equal length secants in B_m lying strictly between p_1 and p_2, not including the two disjoint equal length secants in B_m which subtend arcs containing p_1 and p_2. By Lemma 4 (following this proof) B_m is 1-straight. By Corollary 1, it follows that $B_m(p_1, p_2) \subseteq B_m$ is also 1-straight. So for each $m = 0, 1, 2, \dots,$ we have $\mathcal{H}^1(B_m(p_1, p_2)) \leq |B_m(p_1, p_2)| \leq |P \cap \Gamma(p_1, p_2)| = d(p_1, p_2)$. Let the two disjoint equal length secants in B_m which subtend arcs containing p_1 and p_2 be denoted respectively by $B_m(p_1)$ and $B_m(p_2)$. It then follows that $\mathcal{H}^1(B_m(p_1, p_2)) \leq \mathcal{H}^1(P \cap \Gamma(p_1, p_2)) \leq \mathcal{H}^1(B_m(p_1, p_2) \cup B_m(p_1) \cup B_m(p_2)) = \mathcal{H}^1(B_m(p_1, p_2)) + \mathcal{H}^1(B_m(p_1)) + \mathcal{H}^1(B_m(p_2))$. So, because $\lim_{m\to\infty} \mathcal{H}^1(B_m(p_1)) = \lim_{m\to\infty} \mathcal{H}^1(B_m(p_2)) = 0$, we have $\mathcal{H}^1(A) = \mathcal{H}^1(P \cap \Gamma(p_1, p_2)) = \lim_{m\to\infty} \mathcal{H}^1(B_m(p_1, p_2)) \leq |P \cap \Gamma(p_1, p_2)| = |A|$. Since $A = P \cap \Gamma(p_1, p_2) \subseteq P$ is arbitrary, P is 1-straight. □

Lemmas 2 and 3 are technical and used to prove Lemma 4.

Lemma 2. *Let $x_1, x_2 \in \mathbf{R}^n$ and $x_1 \neq x_2$. Let $E_1, E_2 \subseteq \mathbf{R}^n$ be line segments such that x_1 is an endpoint of E_1 and x_2 is an endpoint of E_2, with $|E_1| = |E_2| \leq (1/2)d(x_1, x_2)$. Then $E = E_1 \cup E_2$ is a* 1*-straight* 1*-set.*

Proof. Since $|E_1 \cup E_2|$ is determined by a pair of endpoints from line segments E_1 or E_2, we have $|E_1 \cup E_2| \geq d(x_1, x_2) \geq |E_1| + |E_2|$. So, by Theorem 5, it follows that $E = E_1 \cup E_2$ is a 1-straight 1-set. □

Figure 1 is an aid to visualizing the statement and proof of Lemma 3.

Lemma 3. *Let $f : [a, b] \to \mathbf{R}$ be a convex function whose graph Γ contains no line segments. Let $p_i = (x_i, f(x_i)) \in \Gamma$ for $x_i \in [a, b]$ and $i = 1, \dots, 6$, such that $a < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < b$. Let $q_{12}, q_{34}, q_{36}, q_{56}$ be four points such that q_{jk} lies on the secant between p_j and p_k. Then, if $d(p_3, q_{36}) = d(p_3, q_{34})$ it follows that $d(q_{12}, q_{34}) > d(q_{12}, q_{36})$, and if $d(q_{36}, p_6) = d(q_{56}, p_6)$ it follows that*

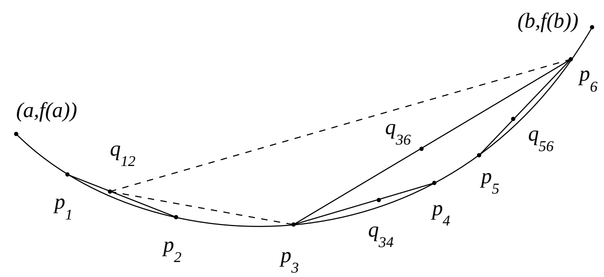

FIGURE 1.

$d(q_{12}, q_{56}) > d(q_{12}, q_{36})$.

Proof. Let $m(u, v)$ represent the slope, defined as usual, of the secant between points u and v in $\mathbf{R}^2$. Since f is convex, by [**8**, p. 194] we have that $m(p_3, p_4) < m(p_3, p_6) < m(p_5, p_6)$. Let the notation angle$\,(rst)$ represent the angle with vertex s, formed by rays through r and t. We use the fact that the slope of a line in $\mathbf{R}^2$ equals the tangent of the angle measured counterclockwise which that line makes with the x-axis. Then

$$\begin{aligned}\text{angle}\,(q_{12}p_3q_{34}) &= \tan^{-1}(m(q_{12}, p_3)) - \tan^{-1}(m(p_3, p_4))\\ &> \tan^{-1}(m(q_{12}, p_3)) - \tan^{-1}(m(p_3, p_6))\\ &= \text{angle}\,(q_{12}p_3q_{36}).\end{aligned}$$

Since in a triangle a larger angle is opposite a larger side, if $d(p_3, q_{36}) = d(p_3, q_{34})$ it follows that $d(q_{12}, q_{34}) > d(q_{12}, q_{36})$. Similarly

$$\begin{aligned}\text{angle}\,(q_{12}p_6q_{56}) &= \tan^{-1}(m(p_5, p_6)) - \tan^{-1}(m(q_{12}, p_6))\\ &> \tan^{-1}(m(p_3, p_6)) - \tan^{-1}(m(q_{12}, p_6))\\ &= \text{angle}\,(q_{12}p_6q_{36}),\end{aligned}$$

from which if $d(q_{36}, p_6) = d(q_{56}, p_6)$ it follows that $d(q_{12}, q_{56}) > d(q_{12}, q_{36})$. □

Lemma 4 establishes that the sets B_m defined in the proof of Theorem 6 are each 1-straight.

Lemma 4. *For each $m = 0, 1, 2, \dots$, the union B_m of the collection of disjoint equal length secants corresponding to the remaining arcs at stage m in the construction of the perfect set P in Theorem 6 is a 1-straight 1-set.*

Proof. The proof is by induction on m. At stage $m = 0$, the set B_0 is the union of two disjoint equal length secants, which by construction and Lemma 2 is a 1-straight 1-set. Now suppose $m = j \geq 0$ and by the induction hypothesis assume B_j, which is the union of 2^{j+1} disjoint equal length secants, is a 1-straight 1-set. Call a pair of secants contained in B_{j+1} adjacent if two of their four endpoints are the endpoints of the same secant contained in B_j. (Intuitively, two secants form an adjacent pair if they are the rotated halves of the same secant in the previous step of the construction.) By Lemma 2, the union of such an adjacent pair of secants is a 1-straight 1-set. The set B_{j+1} will be 1-straight by Theorem 1 if for each $\mathcal{H}^1$-measurable $A \subseteq B_{j+1}$ it follows that $\mathcal{H}^1(A) \leq |A|$. Let $\overline{A}$ represent the disjoint union of the smallest line segments in B_{j+1} containing A. Then $|\overline{A}| = |A|$ and $\mathcal{H}^1(A) \leq \mathcal{H}^1(\overline{A})$. Assume that $\overline{A}$ is not contained in the union of a pair of adjacent secants in B_{j+1}. Let $A' \subseteq B_j$ be the exact set of (closed) line segments whose image in the construction of B_{j+1} as described in Theorem 6 is $\overline{A} \subseteq B_{j+1}$. So $\mathcal{H}^1(\overline{A}) = \mathcal{H}^1(A')$. Since B_j is 1-straight, by Corollary 1 then $A' \subseteq B_j$ is also 1-straight. Thus, $\mathcal{H}^1(A') \leq |A'|$. Let $x_1', x_2' \in A'$ be such that $|A'| = d(x_1', x_2')$, and suppose that in the construction of B_{j+1}, we have that x_1' corresponds to $x_1 \in \overline{A}$ and x_2' corresponds to $x_2 \in \overline{A}$. (See Figure 2, where the pair of thick dashed line segments represent a set A', and the pair of thick solid line segments represent a set $\overline{A}$.)

Since $\overline{A}$ is not contained in the union of an adjacent pair of secants in B_{j+1}, it cannot happen that both $x_1 = x_1'$ and $x_2 = x_2'$. If say $x_2 \neq x_2'$, let $B' \subseteq B_j$ be the line segment containing x_2' whose image in the construction of B_{j+1} is $B \subseteq B_{j+1}$ containing x_2. Let $x_0 = x_0'$ be the common endpoint of B' and B. Then by Lemma 3, since $d(x_0, x_2') = d(x_0, x_2)$, we conclude that $d(x_1', x_2) > d(x_1', x_2')$. If also $x_1 = x_1'$ then this last inequality becomes $d(x_1, x_2) > d(x_1', x_2')$. So by the definition of diameter as a supremum, $|A'| \leq |\overline{A}|$ and hence $\mathcal{H}^1(A) \leq \mathcal{H}^1(\overline{A}) = \mathcal{H}^1(A') \leq |A'| \leq |\overline{A}| = |A|$. If both $x_2 \neq x_2'$ and $x_1 \neq x_1'$, then by an argument similar to that for

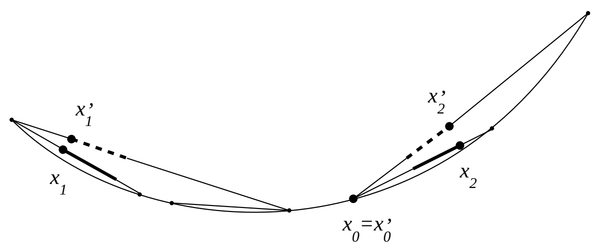

FIGURE 2.

the case $x_2 \neq x_2'$ and $x_1 = x_1'$, using Lemma 3 we conclude that $d(x_2, x_1) > d(x_2, x_1')$. Together with $d(x_1', x_2) > d(x_1', x_2')$ this last inequality yields $d(x_1, x_2) > d(x_1', x_2')$. So again $|A'| \leq |\overline{A}|$, and likewise we conclude $\mathcal{H}^1(A) \leq |A|$. Since $A \subseteq B_{j+1}$ is arbitrary, B_{j+1} is 1-straight. Therefore by induction each set B_m is a 1-straight 1-set. □

Theorem 7. *Let* $f : [a, b] \to \mathbf{R}$ *be a convex function with graph* Γ*. Then* Γ *is the countable union of perfect* 1*-straight* 1*-sets along with a set of* $\mathcal{H}^1$*-measure zero; that is,* Γ *is* $\sigma 1$*-straight.*

Proof. Since Γ can contain at most a countable number of line segments, which by Theorem 4 are 1-straight, we can take Γ to be a continuous arc. Let $E \subseteq \Gamma$ be an $\mathcal{H}^1$-measurable set with $\mathcal{H}^1(E) > 0$. Let $q_1, q_2 \in E$ or its closure, such that $E \subseteq \Gamma(q_1, q_2)$ and $|E| = |\Gamma(q_1, q_2)| = d(q_1, q_2)$. Construct as in Theorem 6 above, a perfect 1-straight 1-set $P_1 \subseteq \Gamma(q_1, q_2)$. If $\mathcal{H}^1(P_1 \cap E) = 0$, then using Theorem 3 and Lemma 1 it follows that $0 < \mathcal{H}^1(E) \leq \mathcal{H}^1(\Gamma(q_1, q_2)) - \mathcal{H}^1(P_1) = \mathcal{L}(\Gamma(q_1, q_2)) - d(q_1, q_2) < (1/2)\mathcal{L}(\Gamma(q_1, q_2))$. Next, within each of the countable number of open arcs removed in the construction of P_1, construct a perfect 1-straight 1-set as above. The countable union of these sets is a perfect $\sigma 1$-straight 1-set $P_2 \subseteq \Gamma(q_1, q_2)$. If $\mathcal{H}^1(P_2 \cap E) = 0$, then using Lemma 1 again, it follows that $0 < \mathcal{H}^1(E) < (1/2^2)\mathcal{L}(\Gamma(q_1, q_2))$. Continue this process. Since a least $k \geq 1$ exists such that $(1/2^k)\mathcal{L}(\Gamma(q_1, q_2)) \leq \mathcal{H}^1(E)$, and the countable union of $\sigma 1$-straight sets is again $\sigma 1$-straight, there eventually exists a $\sigma 1$-straight 1-set $P_k \subseteq \Gamma(q_1, q_2)$ and a 1-straight set $F \subseteq P_k \cap E \subseteq E$ such that

$\mathcal{H}^1(F) > 0$. Since E is arbitrary, by Theorem 2, it follows that Γ is a $\sigma 1$-straight 1-set. □

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