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EQUIVARIANT BIVARIANT CYCLIC THEORY AND EQUIVARIANT CHERN-CONNES CHARACTER

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ABSTRACT. We construct an equivariant bivariant cyclic theory, as a combination of equivariant cyclic and noncommutative de Rham theories for unital G-Banach algebras, where G is a compact Lie group. By incorporating the JLO formula and the superconnection formalism of Quillen, an equivariant bivariant Chern Connes character of Kasparov's G-bimodule is defined, with values in the bivariant cyclic theory.

1. Introduction. It is known, due to Connes [**9**] and an equivalent but convenient version which is due to Jaffe, Lesniewski and Osterwalder and known as a JLO formula [**14**], that the Chern character of a θ summable Fredholm module $(\mathcal{H}, D)$ over a unital C^* algebra A, takes value in the entire cyclic cohomology of A. On the other hand, bivariant Chern-Connes character of Kasparov's kk bimodule, takes values in the bivariant cyclic theory [**27, 11, 21, 29, 30**].

Explicit formula of an equivariant Chern-Connes character, associated to the invariant Dirac operator, in the presence of a countable discrete group action on a smooth compact spin Riemannian manifold, was given by Azmi, [**1, 2**]. Moreover, in [**2**] it was shown that this equivariant cocycle is an element of the delocalized cohomology, and it pairs with an equivariant K-theory idempotent. In the case G is a compact Lie group, Chern and Hu [**6**] gave an explicit formula of an equivariant Chern-Connes character, associated to G-equivariant θ-summable Fredholm module.

In this paper we define an equivariant bivariant Chern-Connes character in the presence of a compact lie group, which acts by continuous automorphism on certain algebras. As a first step, we construct an equivariant bivariant cyclic theory. To motivate our construction, we recall some equivariant and bivariant cyclic theories.

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There are several different approaches to the bivariant cyclic theory. Jones and Kassel [**15**] introduced a bivariant cyclic theory for unital algebras using the (b,B) bicomplex. Their approach uses differential homological algebra. Another approach is due to Connes [**7**], the bivariant groups in the category of cyclic modules are defined using classical homological algebra. Using the fact that noncommutative de Rham homology is an algebraic generalization of the de Rham cohomology of smooth manifolds [**8, 16**], Lott [**20**] defined a bivariant cyclic theory, as a combination of entire cyclic (co)homology and noncommutative de Rham homology of graded differential algebra (Ω, d). Similar construction was also carried out by Quillen [**27**], where he presented a new approach to the algebraic formalism of cyclic cohomology. He defined an Hom complex from differential graded algebra to another algebra and established a bivariant character by incorporating the JLO formula.

Twisted cyclic (co)homology arises in the studies of cyclic (co)homology of crossed product of algebras. It involves an automorphism of the algebra and differs from the ordinary cyclic (co)homology by the cyclic and boundary operators which contains the action of automorphism.

In [**23**] Nistor defined certain twisted modules (quasi-cyclic object) and computed its cyclic homology. The smooth G-action on the twisted module and the isomorphism between the twisted module and the crossed product algebra, enabled him to define the cyclic cohomology of the crossed product by a compact Lie group G. Benameur's quest to construct cyclic Lefschetz formula for foliations [**5**] led him to define a notion of Γ-equivariant cyclic cohomology which pairs with equivariant K-theory. His equivariant cyclic cocycle is in fact the cyclic cocycle on the discrete crossed product with certain twist assumption. In [**4**] he constructed an equivariant cyclic cocycle out of closed Γ-invariant current. This enabled him to prove a fixed point formula in the cyclic homology of the smooth convolution algebra of the foliation.

Klimek, Kondracki and Lesniewski [**18**] defined an equivariant version of the entire cyclic cohomology. For technical simplicity they dealt with the case G is a finite group, cf. [**13**] for G a compact group. They defined a complex $\mathcal{L}^n(A, \mathcal{F}(G))$, which consists of n linear mappings from $A \times \cdots \times A$ with values in the space $\mathcal{F}(G)$, of continuous functions on G. As G acts by automorphism on A, the action extends to $\mathcal{L}^n(A, \mathcal{F}(G))$. The G-equivariant complex $C^n_G(A)$, with boundary operators b, B defined in a certain way, gives rise to equivariant entire cyclic cohomology.

Our quest to define an equivariant bivariant cyclic theory of unital G-Banach algebras and not of crossed product algebra, led us to adopt the equivariant cyclic theory defined by the last authors, together with the construction of the bivariant cyclic theory provided by Lott.

Let G be a compact Lie group, and let U and $\mathcal{B}$ be unital G-Banach algebras. The action of G is extended to the differential graded algebra (DGA) $(\Omega(\mathcal{B}), d)$. Let $C_n^G(U) = \{C_n^g(U) | g \in G\}$ where $C_n^g(U) = C_n(U)/\{Id - \alpha_g^*\}$ and $C_n(U) = U \otimes \bar{U}^{\otimes n}$, the symbol $\otimes$ means the completed projective tensor product. The b, B operators defined on $C_n^G(U)$ are very similar to the one in [**18**]. An Hom complex $CE^G(U, \Omega(\mathcal{B}))$ with boundary operator ∂ is defined, which consists of continuous linear maps with certain properties from $(CE_*^G(U), b + B)$ to the space $C(G, (\Omega(\mathcal{B}), d))$, of continuous maps from G to $\Omega_*(\mathcal{B})$. The homology of the complex $(CE^G(U, \Omega(\mathcal{B})), \partial)$ is the equivariant bivariant cyclic homology denoted by $HE_*^G(U, \Omega(\mathcal{B}))$.

In the special case, when $\mathcal{B} = \mathbf{C}$ the complex numbers, and G is a finite group. The equivariant bivariant cyclic homology gets reduced to the equivariant cyclic cohomology in [**18**].

There are several different approaches to bivariant Chern-Connes character. Therefore, to extend the bivariant character to accommodate the equivariant case, and to motivate our framework, let us first recall some of these approaches.

Nistor introduced bivariant Chern-Connes character for finitely summable Kasparov's KK-modules, with values in Jones-Kassel bivariant group [**22**]. Furthermore in [**24**], a new bivariant character with values in Connes bivariant cyclic group was defined, and it was shown to be compatible with the periodicity operator in cyclic (co)homology and with Kasparov's product.

Giving up the compatibility with the periodicity operator and Kasparov's product, Wu [**30**] constructed a bivariant Chern-Connes character $Ch(\mathcal{M}, \mathcal{D})$ for (a special class) of θ summable modules, by incorporating the JLO formula and the superconnection formalism of Quillen [**26**]. His construction is influenced by the work of Connes, cf. [**8, 11**], see also [**27**]; in fact, when $(\mathcal{H}, D)$ is the dual Dirac on a locally symmetric space, then $Ch(\mathcal{M}, \mathcal{D})$ is essentially the bivariant character of

Connes, cf. [**10, 11**].

Wu's bivariant character takes value in the bivariant cyclic theory described by Lott [**20, 21**]. As our construction of equivariant bivariant cyclic theory is influenced by Lott's bivariant theory. Therefore, we adopt Wu's method and extend it to the equivariant case. Moreover, it is convenient to deal with heat kernel, supercurvature and the JLO formula.

We believe that there are other methods of extending bivariant character to the equivariant case; we hope to exploit this fact in the future.

We remark that in the case of families of Dirac operators, Azmi [**3**] gave an explicit expression of the cyclic cocycle formula for such families, by incorporating Wu's bivariant Chern-Connes character and Bismut's superconnection. On the other hand, Perrot [**25**] gave a simple formula expressing certain BRS cocycles (which are obtained by transgression of the Chern character of an index bundle) as generalized forms, involving a G-equivariant family of Dirac operators.

The paper is divided into two parts. The first part deals with defining an equivariant bivariant cyclic theory, while the second part deals with defining an equivariant bivariant Chern-Connes character.

Let A and B be $G-C^*$ algebras, and let $(A, \mathcal{H}, D)$ be a G-equivariant θ-summable Fredholm module over A. Let $\mathcal{A}$ and $\mathcal{B}$ be dense $*$ subalgebras of A and B respectively which are Banach algebras under a certain norm. Let $U = \mathcal{A} \otimes \mathcal{B}$ be the projective tensor product of Banach algebras with the projective tensor product norm. There is an obvious G action on U. Denote by $\mathcal{M} = \mathcal{H} \otimes \mathcal{B}$, the unbounded G-bimodule as in [**30**]. Let $\mathbf{A} = \nabla + \mathcal{D}$ be a G-superconnection on $\mathcal{M}$, where ∇ is a flat G-invariant $\mathcal{B}$ connection and $\mathcal{D} = D \otimes I + L$, with L being a G invariant operator with certain properties.

The nth component of the equivariant bivariant Chern-Connes character $Ch^n_G(\mathcal{M}, \mathcal{D})$ is defined by

$$
\begin{aligned}
&Ch^n_G(\mathcal{M}, \mathcal{D})(u_0, \dots, u_n)(g) \\
&\qquad = \int_{\Delta_n} Tr_s(\tilde{\rho}_g\, u_0 e^{-s_0 \mathbf{A^2}} [\mathbf{A}, u_1] \cdots [\mathbf{A}, u_n] e^{-s_n \mathbf{A^2}}) ds_0 \cdots ds_n,
\end{aligned}
$$

where $u_i \in U$ and $g \in G$. The total equivariant bivariant character is denoted by $Ch^*_G(\mathcal{M}, \mathcal{D})$, and it is shown to be closed. Hence, it defines a class in the equivariant bivariant cyclic homology $HE^G_{ev}(U, \Omega(\mathcal{B}))$.

1. Equivariant bivariant cyclic theory. In the present chapter we define the equivariant bivariant cyclic homology, as a combination of equivariant (normalized) entire cyclic homology and noncommutative de Rham theory for unital G-Banach algebras U and $\mathcal{B}$, where G is a compact lie group.

1.1 Equivariant cyclic homology. Let U be a unital Banach algebra on which G acts by continuous automorphism. Thus, there is a continuous map $\alpha : G \to \mathrm{Aut}\,(U)$, such that for each $g \in G$ we have

$$|\alpha_g(u)| \leq c\,|u|, \quad \text{where } c \text{ is some constant.}$$

Let $C_n(U) = U \otimes \bar{U}^{\otimes n}$, where $\bar{U} = U/\mathbf{C}$, thus $C_n(U)$ is a (normalized) complex. The symbol $\otimes$ means the completed projective tensor product. Any $u \in C_n(U)$ is of the form $u = u_0 \otimes u_1 \otimes \cdots \otimes u_n$ and it will be denoted by $u = (u_0, u_1, \ldots, u_n)$. The norm $|\cdot|_\pi$ on $C_n(U)$ is the n-fold projective tensor product norm of the norm on U.

There is a G-action on $C_n(U)$, given by

$$\alpha^*_h(u_0, u_1, \ldots, u_n) = (\alpha_h^{-1} u_0, \alpha_h^{-1} u_1, \ldots, \alpha_h^{-1} u_n). \tag{1}$$

Note that $\alpha^*_h : C_n(U) \to C_n(U)$ satisfies $|\alpha^*_h(u)|_\pi \leq C|u|_\pi$.

For each $g \in G$, we let $C^g_n(U)$ be the quotient space of $C_n(U)$ by the closure of the images of the map $Id - \alpha^*_g$. Thus

$$C^g_n(U) = C_n(U)/\{Id - \alpha^*_g\}. \tag{2}$$

Let $C^G_n(U) = \{C^g_n(U) |\, g \in G\}$ be a (normalized) equivariant complex. For each $g \in G$, we define some operators on $C^G_n(U)$.

The cyclic operator $t^g_n : C^G_n(U) \to C^G_n(U)$ is defined by

$$t^g_n(u_0, u_1, \ldots, u_n) = (\alpha_g^{-1} u_n, u_0, \ldots, u_{n-1}), \tag{3}$$

let $T^g_n = (-1)^n t^g_n$. One can show that $(T^g_n)^{n+1} = I$ which follows from (2), since

$$\begin{aligned}(T^g_n)^{n+1}(u_0, u_1, \ldots, u_n) &= (-1)^{n(n+1)}(\alpha_g^{-1} u_0, \alpha_g^{-1} u_1, \ldots, \alpha_g^{-1} u_n)\\ &= (u_0, u_1, \ldots, u_n).\end{aligned}$$

As G acts on $C_n^G(U)$, hence for any $h \in G$, we have $T_n^g\,\alpha_h^* = \alpha_h^*\,T_n^{hgh^{-1}}$. Moreover $|T_n^g(u_0, \ldots, u_n)|_\pi \leq C\,|(u_0, \ldots, u_n)|_\pi$ for some constant C.

The norm operator $N_n^g : C_n^G(U) \to C_n^G(U)$ is defined by $N_n^g = \sum_{j=0}^{n} (T_n^g)^j$ and it satisfies the relation

$$(1 - T_n^g)N_n^g = N_n^g(1 - T_n^g) = 0. \tag{4}$$

Let $(b_n^g)' : C_n^G(U) \to C_{n-1}^G(U)$ be defined by

$$(b_n^g)'(u_0, u_1, \ldots, u_n) = \sum_{j=0}^{n-1}(-1)^j(u_0, u_1, \ldots, u_j u_{j+1}, \ldots, u_n).$$

One can easily deduce the following relation

$$(b_n^g)' \circ (b_{n+1}^g)' = 0. \tag{5}$$

The degree -1 boundary operator $b_n^g : C_n^G(U) \to C_{n-1}^G(U)$ is given by

$$b_n^g = (b_n^g)' + V_n^g \tag{6}$$

where $V_n^g(u_0, u_1, \ldots, u_n) = (-1)^n((\alpha_g^{-1}u_n)\,u_0, u_1, \ldots, u_{n-1})$, and for any $h \in G$, $V_n^g\alpha_h^* = \alpha_h^* V_n^{hgh^{-1}}$. One can show that $b_n^g \circ b_{n+1}^g = 0$, which follows from (5).

The homotopy operator $S_n^g : C_n^G(U) \to C_{n+1}^G(U)$ is given by

$$S_n^g(u_0, u_1, \ldots, u_n) = (1, u_0, u_1, \ldots, u_n) \tag{7}$$

and a degree $+1$ boundary operator $B_n^g : C_n^G(U) \to C_{n+1}^G(U)$ is defined by

$$B_n^g = (I - T_{n+1}^g)S_n^g\,N_n^g. \tag{8}$$

Explicitly B_n^g is given by:

$$\begin{aligned} B_n^g&(u_0, u_1, \ldots, u_n) \\ &= \sum_{j=0}^{n}(-1)^{nj}(1, \alpha_g^{-1}(u_j), \ldots, \alpha_g^{-1}(u_n), u_0, \ldots, u_{j-1}). \end{aligned} \tag{9}$$

Proposition 1.1. *With all the operators as mentioned above, the following relations hold*:

1. $(b^g_{n+1})' S^g_n + S^g_{n-1} (b^g_n)' = I$.
2. $(b^g_n)'(I - T^g_n) = (I - T^g_{n-1})\, b_n$, *and* $N^g_{n-1}\,(b^g_n)' = b^g_n\, N^g_n$.
3. $B^g_{n+1} \circ B^g_n = 0$.
4. $b^g_{n+1} B^g_n + B^g_{n-1}\, b^g_n = 0$.

Proof. Parts 1 and 2 follow by direct computations. Part 3 follows from the definition and (4). As for part 4,

$$\begin{aligned} b^g_{n+1} B^g_n + B^g_{n-1}\, b^g_n &= b^g_{n+1}(I - T^g_{n+1}) S^g_n\, N^g_n + (I - T^g_n) S^g_{n-1}\, N^g_{n-1}\, b^g_n \\ &= (I - T^g_n)\,(b^g_{n+1})'\, S^g_n\, N^g_n + (I - T^g_n) S^g_{n-1}\,(b^g_n)'\, N^g_n \\ &= (I - T^g_n)\,((b^g_{n+1})' S^g_n + S^g_{n-1}(b^g_n)')\, N^g_n = 0. \end{aligned}$$

In the last step we have used part 1 and (4). □

Given an equivariant mixed complex $\{(C^G_*(U), b^g, B^g) \mid g \in G\}$, one can form the associated complex:

$$\tilde{C}^G_n(U) = C^G_n(U) \oplus C^G_{n-2}(U) \oplus C^G_{n-4} \oplus \cdots$$

with boundary operator $(b^g + B^g)_n : \tilde{C}^G_n(U) \to \tilde{C}^G_{n-1}(U)$ of degree -1. Clearly $(b^g + B^g)^2 = 0$, which follows from $(b^g)^2 = (B^g)^2 = 0$ and $b^g\, B^g + B^g\, b^g = 0$.

The associated complex $\{(\tilde{C}^G_n(U), b^g + B^g) \mid g \in G\}$ gives rise to the equivariant cyclic complex.

1.2 The equivariant entire cyclic complex. Let $C^G(U) = \oplus_n\, C^G_n(U)$. For each $r > 0$, denote by $C^G_{r,*}(U)$ the completion of $C^G(U)$ with respect to the norm

$$||\bar{u}||_r = \sum_{n \geq 0} r^n \, \frac{||\bar{u}_n||_\pi}{\Gamma(n/2)}, \quad \text{where } \bar{u} = (\bar{u}_n) \in \oplus_{n\geq 0} C^G_n(U).$$

Let $CE^G_*(U) = \cup_{r>0} C^G_{r,*}(U)$. Thus $CE^G_*(U)$ is a Z_2 graded Fréchet space, and $CE^G_*(U) = CE^G_{ev}(U) \oplus CE^G_{odd}(U)$ with boundary operator

$b+B$ of degree -1, where $CE^G_{ev}(U)$, respectively $CE^G_{odd}(U)$, denote the even and odd spaces. The homology of the complex $(CE^G_*(U), b+B)$ is the equivariant entire cyclic homology of U, denote it by $HE^G_*(U)$.

1.3 Noncommutative de Rham homology. Let $\mathcal{B}$ be a unital Banach algebra. Consider the graded differential algebra $(\Omega(\mathcal{B}), d)$. This means the following:

1. $\Omega_0 = \mathcal{B}$, and Ω_n for $n \geq 0$ is a Banach space with norm $||.||_n$;

2. the multiplication map $\Omega_p.\Omega_q \to \Omega_n$, where $n = p+q$, is continuous, i.e., there is a constant $C(p,q)$ such that

$$||\omega_1\, \omega_2||_n \leq C(p,q)\, ||\omega_1||_p\, ||\omega_2||_q,\ \omega_i \in \Omega_{k_i}.$$

3. The differential $d : \Omega_n \to \Omega_{n+1}$ is continuous with $d^2 = 0$ and satisfies the derivation property in the graded sense:

$$d(\omega_1\, \omega_2) = (d\omega_1).\omega_2 + (-1)^{|\omega_1|}\omega_1\, (d\, \omega_2),$$

where $\omega_i \in \Omega_i$ and $|\omega_1|$ is the degree of ω_1.

Let $\Omega = \prod_{n\geq 0} \Omega_n$ be endowed with the product topology. Also, let $[\Omega, \Omega]_k = \oplus_{j=0}^k [\Omega_j, \Omega_{k-j}]$ be the closure of the subspace of the commutators of degree k. Define $\hat{\Omega} := \prod_{n\geq 0}(\Omega_n/[\Omega,\Omega]_n)$. The homology of the complex $(\hat{\Omega}, d)$ is called the noncommutative de Rham homology of (Ω, d), cf. [**16**].

One interesting example of DGA over $\mathcal{B}$ is the universal DGA of $\mathcal{B}$, it is obtained by completing the algebraic universal DGA of $\mathcal{B}$.

Thus, $\Omega_0^{alg} = \mathcal{B}$ and $\Omega_1^{alg} = \ker(\mathcal{B} \otimes_{alg} \mathcal{B} \to^m \mathcal{B})$ where $m(b_1 \otimes_{alg} b_2) = b_1\, b_2$, here $\otimes_{alg}$ is the algebraic tensor product. Let $\Omega_n^{alg} = \otimes_{alg} \Omega_1^{alg}$, n times. As a vector space, $\Omega_n^{alg}(\mathcal{B})$ is isomorphic to the algebraic tensor product $\mathcal{B} \otimes_{alg} (\otimes^n_{alg}(\mathcal{B}/\mathbf{C}))$. The isomorphism is given by

$$b_0 db_1 \cdots db_n \mapsto b_0 \otimes_{alg} b_1 \cdots \otimes_{alg} b_n.$$

Under this isomorphism, the multiplication map and the derivation d are continuous with respect to the projective tensor product norms on $\mathcal{B} \otimes_{alg} (\otimes^n_{alg}(\mathcal{B}/\mathbf{C}))$. We endow $\Omega_n^{alg}(\mathcal{B})$ with the projective tensor product topology induced from this identification. The completion is denoted by $\Omega_n(\mathcal{B})$.

Let $\Omega(\mathcal{B}) = \prod \Omega_n(\mathcal{B})$ be the topological product, then this is the universal DGA of the Banach algebra $\mathcal{B}$. The noncommutative de Rham homology of $(\Omega(\mathcal{B}), d)$ is denoted by $H_*^{dr}(\mathcal{B})$.

1.4 Equivariant bivariant cyclic homology. Let $(\hat{\Omega}(\mathcal{B}), d)$ be the DGA over $\mathcal{B}$, and let $(CE_*^G(U), b^g + B^g | g \in G)$ be the equivariant entire cyclic homology as defined above.

G acts on $\mathcal{B}$ by continuous automorphism, i.e., there is a continuous map $\gamma : G \to \operatorname{Aut}(\mathcal{B})$. We will assume that the map γ_g commutes with the operator d. Any element ω_k of $\Omega_k(\mathcal{B})$ can be written as a finite sum $\sum b_0 db_1 \cdots db_k$. Therefore G acts on ω_k with the obvious action and $\gamma_g : \Omega_k(\mathcal{B}) \to \Omega_k(\mathcal{B})$.

Let us recall the Hom complex, if (X, d_1) and (Y, d_2) are chain complexes, then $\operatorname{Hom}(X, Y)$ is the chain complex which in degree n is given by

Hom $(X, Y)_n = \prod_{p \geq 0} \operatorname{Hom}(X_p, Y_{p+n})$ and the boundary operator ∂ is given by $\partial f = d_2 f + (-1)^{|f|} f d_1$. Where $|f|$ is the parity of the homogeneous map f and the differential is then extended to general f by linearity.

Consider the Z_2 graded Hom complex $CE^G(U, \Omega(\mathcal{B}))$, of continuous linear maps from $CE_*^G(U)$ to $C(G, \hat{\Omega}(\mathcal{B}))$, where $C(G, \hat{\Omega}(\mathcal{B}))$ is the space of all continuous maps from G to $\hat{\Omega}(\mathcal{B})$. Thus

$$CE^G(U, \Omega(\mathcal{B})) = \operatorname{Hom}(CE_*^G(U), C(G, \hat{\Omega}(\mathcal{B})) \tag{10}$$

For $\phi \in CE^G(U, \Omega(\mathcal{B}))$ and for any $n \geq 0$, let $\phi^n \in \operatorname{Hom}(CE_n^G(U), C(G, \Omega_*(\mathcal{B}))$. Thus, for $Z = (u_0, \cdots, u_n) \in C_n^G(U)$ and for each $k \geq 0$, the degree k component of ϕ^n is the continuous map $\phi_k^n(Z) : G \to \Omega_k(\mathcal{B})$, in the sense

$$\sup_{g \in G} ||\phi_k^n(u_0, \ldots, u_n)(g)||_k \leq C_k ||(u_0, \ldots, u_n)||_\pi \tag{11}$$

and the norm of ϕ_k^n is equal to

$$||\phi_k^n|| = \sup_{\substack{g \in G \\ u_i \in U}} ||\phi_k^n(u_0, \ldots, u_n)(g)||_k. \tag{12}$$

Moreover, for any g, $h \in G$, we assume this holds

$$\phi_k^n(\alpha_h^*(u_0, \dots, u_n))(g) = \phi_k^n(u_0, \dots, u_n)(h\, g\, h^{-1}). \tag{13}$$

For each $k \geq 0$, ϕ_k has components

$$\phi_k = \{\phi_k^0, \phi_k^1, \dots, \phi_k^n, \dots\}.$$

If the constant C_k in (11) depends only on k and not on n, then we have a continuous map $\phi_k : C_*^G(U) \to C(G, \Omega_k(\mathcal{B}))$, and the collection of all such maps for $k \geq 0$ will imply the continuity of the map $\phi \in CE^G(U, \Omega(\mathcal{B}))$.

The Z_2 degree of (a homogeneous) $\phi \in CE^G(U, \Omega(\mathcal{B}))$ is even, respectively odd, if ϕ preserves, respectively reverses, the Z_2 grading. The boundary operator ∂ is given

$$(\partial\phi)(Z)(g) = d(\phi(Z)(g)) + (-1)^{|\phi|}\phi((b + B)(Z))(g) \tag{14}$$

where $\phi((b + B)(Z))(g)$ is in fact $\phi((b^g + B^g)(Z))(g)$. Here $|\phi|$ is the degree of the map ϕ and the differential is extended to general ϕ by linearity. One can easily show that $\partial^2 = 0$ which follows from the above properties of b, B and d.

Denote by $HE^G(U, \Omega)$, the equivariant bivariant cyclic homology of the complex $(CE_*^G(U, \Omega(\mathcal{B})), \partial)$, i.e.,

$$HE^G(U, \Omega) = H(CE^G(U, \Omega), \partial).$$

Example 1.2. In the case $\mathcal{B} = \mathbf{C}$ the complex numbers, the Hom complex becomes $\mathrm{Hom}\,(CE_*^G(U), C(G, \mathbf{C}))$. In addition, if G is a finite group then we recover the equivariant entire cyclic cohomology in [**18**].

Example 1.3. Let $G = \{e\}$ be the trivial group, so G acts trivially on both U and $\mathcal{B}$, and $\mathrm{Hom}\,(CE_*^G(U), C(G, \Omega(\mathcal{B})))$ becomes $\mathrm{Hom}\,(CE_*(U), \Omega(\mathcal{B}))$. Therefore $HE^G(U, \Omega(\mathcal{B}))$ reduces to the bivariant cyclic theory $HE(U, \Omega(\mathcal{B}))$ discussed in [**30**].

2. The equivariant bivariant Chern-Connes character. First we recall the definition of the G-equivariant θ-summable Fredholm module over a unital C^* algebra A, where G is a compact lie group, cf. [**19**]. It is a triple $(A, \mathcal{H}, D)$ where the following condition holds:

1. A is a unital $G - C^*$ algebra; that is, G acts on A by continuous automorphism $\alpha : G \to \mathrm{Aut}\,(A)$ such that α_g is unitary for all $g \in G$.

2. $\mathcal{H}$ is a Z_2 graded Hilbert space with grading operator γ, such that $\gamma^2 = I$ and there is an even unitary representation of G on $\mathcal{H}$; $\rho : G \to \mathcal{L}(\mathcal{H})$, $\rho : g \to \rho_g$ (here $\mathcal{L}(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$), moreover ρ_g commutes with γ. There is also the induced G-action on $\mathcal{L}(\mathcal{H})$ given by $\rho_{g,*}\, P = \rho_g\, P\, \rho_g^{-1}$.

3. $\mathcal{H}$ is a Z_2 graded G-equivariant A module; there is a unital $*$ representation of A on $\mathcal{H}$; $\mu : A \to \mathcal{L}(\mathcal{H})$ such that $\mu(\alpha_g\, a) = \rho_{g,*}(\mu(a))$.

4. D is an unbounded odd self-adjoint operator which is densely defined on $\mathcal{H}$, and it is G-invariant, that is, $D\,\rho_g = \rho_g\, D$ for all $g \in G$.

5. There is a dense subalgebra $\mathcal{A} \subset A$, such that for any $a \in \mathcal{A}$, the graded commutator $[D, \mu(a)]$ is densely defined, extending to a bounded operator on $\mathcal{H}$ and there is a constant $N(D)$ such that

$$||\mu(a)|| + ||[D, \mu(a)]|| \leq N(D)\, ||a||_{\mathcal{A}}.$$

6. $\mathrm{tr}\,(e^{-D^2}) < \infty$.

Let $(A, \mathcal{H}, D)$ be a G-equivariant θ-summable Fredholm module as above. The dense subalgebra $\mathcal{A}$ is a Banach algebra with the norm

$$|\mu(a)| := ||\mu(a)|| + ||\,[D, \mu(a)]\,||, \; \forall\, a \in \mathcal{A}.$$

Let $\mathcal{B}$ be a dense G invariant $*$ subalgebra of a G-C^* algebra B, and it is endowed with a Banach algebra norm which is greater than the C^* norm on B.

Let $U = \mathcal{A} \otimes \mathcal{B}$ be the projective tensor product of G-Banach algebras $\mathcal{A}$ and $\mathcal{B}$, with the projective tensor product norm. The continuous action $\tilde{\alpha}$ of G on U is as follows: for

$$u = a \otimes b, \;\text{ then } \tilde{\alpha}_g(u) = \alpha_g(a) \otimes \gamma_g(b). \tag{15}$$

Consider the $(U - \mathcal{B})$ Kasparov bimodule $\mathcal{M} = \mathcal{H} \otimes \mathcal{B}$, where U acts on the left of $\mathcal{M}$ by letting $\mathcal{A}$ act on $\mathcal{H}$ and $\mathcal{B}$ acts on $\mathcal{B}$ by multiplication from the left, while $\mathcal{B}$ acts on $\mathcal{M}$ by multiplication from the right. Also there is a continuous $\mathcal{B}$ valued inner product on $\mathcal{M}$ which is G invariant.

There is an obvious continuous action $\tilde{\rho}$ of G on $\mathcal{M}$, by letting G act on $\mathcal{H}$ via ρ and on $\mathcal{B}$ via γ. Also there is an induced action $\tilde{\rho_*}$ of G on $\mathcal{L}(\mathcal{H}) \otimes \mathcal{B}$. Hence, $\mathcal{M}$ is a G-equivariant $(U-\mathcal{B})$ bimodule, such that for $g \in G$ and $u = a \otimes b \in U$, the following holds:

$$\tilde{\mu}(\,\tilde{\alpha}_g(u)) = \tilde{\rho}_{g,*}\tilde{\mu}(u). \tag{16}$$

Here $\tilde{\mu} : U \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ and $\mathrm{End}_{\mathcal{B}}(\mathcal{M})$ is the algebra of bounded linear operators on $\mathcal{M}$ commuting with the right action of $\mathcal{B}$ and having adjoint. We have the inclusions

$$\mathcal{L}^1(\mathcal{H}) \otimes \mathcal{B} \subset \mathcal{L}(\mathcal{H}) \otimes \mathcal{B} \subset \mathrm{End}_{\mathcal{B}}(\mathcal{M}).$$

Here $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is the ideal of trace class operators on $\mathcal{H}$.

For $p \geq 1$, let $\Omega^p(\mathcal{M}) := \mathcal{L}^p(\mathcal{H})\hat{\otimes}\Omega(\mathcal{B})$ be the space of 'Schatten class' forms on $\mathcal{M}$, where $\hat{\otimes}$ means the graded projective tensor product. $\mathcal{L}^p(\mathcal{H})$ is the p-Schatten class on $\mathcal{H}$ with norm $||.||_p$. Let $||.||_k$ be the norm on Ω_k. Denote by $||.||_{p,k}$ the projective tensor product norm of $||.||_p$ and $||.||_k$ on $\mathcal{L}^p(\mathcal{H})\hat{\otimes}\Omega_k(\mathcal{B})$. The topology on $\Omega^p(\mathcal{M})$ is by definition the product topology

$$\Omega^p(\mathcal{M}) = \prod_{k\geq 0}(\mathcal{L}^p(\mathcal{H})\hat{\otimes}\Omega_k(\mathcal{B})).$$

The $\mathbf{C}$ valued graded supertrace on the ideal $\mathcal{L}^1(\mathcal{H})$ extends to a continuous graded supertrace linear map

$$Tr_s := tr_s \otimes I : \Omega^1(\mathcal{M}) \to \hat{\Omega}_*(\mathcal{B}),$$

where $\hat{\Omega} = \Omega/[\Omega, \Omega]$ and $tr_s : \mathcal{L}^1(\mathcal{H}) \to \mathbf{C}$. By graded supertrace we mean that for any $A, B \in \Omega^1(\mathcal{M})$ this holds

$$\mathrm{Tr}_s(A\,B) = (-1)^{|A||B|}\mathrm{Tr}_s(B\,A). \tag{17}$$

Next, we define a flat G-invariant $\mathcal{B}$ connection as follows:

$$\begin{gathered}\nabla : \mathcal{M} \to \mathcal{M}\otimes_{\mathcal{B}}\Omega_1(\mathcal{B}), \quad \text{by} \\ \nabla(\xi \otimes b) = \xi \otimes db, \quad \forall\, \xi \in \mathcal{H},\, b \in \mathcal{B}\end{gathered} \tag{18}$$

where flatness means $\nabla^2 = 0$. Note that ∇ is not U-linear but for any $P \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{B}$, the commutator $[\nabla, P] \in \mathcal{L}(\mathcal{H}) \hat{\otimes} \Omega_1(\mathcal{B})$. The connection ∇ is G-invariant which means

$$\tilde{\rho}_g \nabla(\xi \otimes b) = \tilde{\rho}_g(\xi \otimes db) = \rho_g \xi \otimes d\,\gamma_g(b) = \nabla(\tilde{\rho}_g(\xi \otimes b)). \tag{19}$$

Moreover, the connection ∇ extends uniquely to an operator on $\mathcal{M} \otimes \Omega$ satisfying

$$\nabla\Big(\tilde{\rho}_g((\xi \otimes \omega_1)\omega_2)\Big) = \tilde{\rho}_g\Big(\nabla(\xi \otimes \omega_1))\omega_2 + (-1)^{|\omega_1|}\xi \otimes \omega_1(d\omega_2)\Big),$$

for any homogeneous $\omega_1 \in \Omega$.

The operator D on $\mathcal{H}$ is G invariant, and so is $D \otimes I$ on $\mathcal{M}$. Mostly, we will denote $D \otimes I$ by D and it will be clear from the context. Moreover, for any $u \in U$ we have

$$\tilde{\rho}_{g,*}\,[D, \mu(u)] = [D, \tilde{\rho}_{g,*}\mu(u)].$$

Let $\mathcal{V}$ be the space

$$\mathcal{V} := \{A \in (\mathcal{L}(\mathcal{H}) \hat{\otimes} \Omega) |\ \ A \text{ is an odd } G\text{-invariant, } [D, A] \in \mathcal{L}(\mathcal{H}) \hat{\otimes} \Omega(\mathcal{B})\}.$$

Consider the operator of the form [**30**]

$$\mathcal{D} = D \otimes I + A, \quad \text{where } \ A \in \mathcal{V} \tag{20}$$

and

$$\mathcal{D}^2 = D^2 \otimes I + V, \quad \text{where } \ V := [D, A] + A^2, \tag{21}$$

It is clear that both $\mathcal{D}$ and $\mathcal{D}^2$ are G-invariant operators.

The heat operator $e^{-t\mathcal{D}^2}$ for $t > 0$, is formally given by Duhamel's expansion

$$e^{-t\mathcal{D}^2} := \sum_{n \geq 0} t^n \int_{\Delta_n} e^{-ts_0\mathcal{D}^2} V e^{-ts_1\mathcal{D}^2} \cdots V e^{-ts_n\mathcal{D}^2}\, ds, \tag{22}$$

where Δ_n is the n simplex

$$\Delta_n = \Big\{ s = (s_0, s_1, \dots, s_n) \in \mathbf{R}^{n+1} \Big| \sum_{i=0}^{n} s_i = 1,\ s_i \geq 0,\ i = 0, 1, \dots, n \Big\}.$$

Convention. For any $L \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega(\mathcal{B})$, then as in Section 1.4, we let $(L)_k \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega_k(\mathcal{B})$ denote the degree k component of the operator L. Therefore

$$L = \{(L)_0, (L)_1, \dots, (L)_k, \dots\} \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega(\mathcal{B}).$$

Lemma 2.1. *For $\mathcal{D}$ as in* (20)*, the series* (22) *defining the heat operator $e^{-t\mathcal{D}^2}$ is convergent in $\Omega^1\mathcal{M}$ and*

$$\sup_{g \in G} ||\tilde{\rho}_g(e^{-t\mathcal{D}^2})_k)||_{t^{-1},k} \leq C_k e^{t(k+1)c_k} (\mathrm{Tr}\,(e^{-D^2}))^t, \quad \textit{for } 0 < t \leq 1,$$

where $C_k > 0$ and $c_k = \sup_{0 \leq i \leq k} ||([D \otimes I + A] + A^2)_i||_{\infty,i}$, the norm $||.||_\infty$ is the operator norm.

Proof. An easy extension of Lemma 1.3 in [**30**]. □

Consider the following superconnection on the module $\mathcal{M}$

$$\mathbf{A} := \nabla + \mathcal{D} \tag{23}$$

where $\mathcal{D}$ as in (20) and ∇ as in (18). Clearly $\mathbf{A}$ is a G-invariant operator. The curvature of $\mathbf{A}$ is given by

$$\mathbf{A^2} = \mathcal{D}^2 + [\nabla, \mathcal{D}] = \mathcal{D}^2 + [\nabla + A]. \tag{24}$$

One can define the heat operator $e^{-t\mathbf{A^2}}$ by Duhamel's expansion

$$e^{-t\mathbf{A^2}} := \sum_{n \geq 0} t^n \int_{\Delta_n} e^{-ts_0\mathcal{D}^2}[\nabla, A]e^{-ts_1\mathcal{D}^2} \cdots [\nabla, A]e^{-ts_n\mathcal{D}^2}\, ds. \tag{25}$$

Similarly as in Lemma 2.1, one can show that $e^{-t\mathbf{A^2}}$ is convergent in $\Omega^1(\mathcal{M})$ and

$$\sup_{g\in G} ||\tilde{\rho}_g(e^{-t\mathbf{A^2}})_k)||_{t^{-1},k} \leq C_k e^{tc_k} \,(\mathrm{Tr}\,(e^{-D^2}))^t, \quad 0 < t \leq 1. \tag{26}$$

Lemma 2.2. *For any $L \in \Omega^1(\mathcal{M})$ and for each $g \in G$, we have $d\,\mathrm{Tr}_s(\tilde{\rho}_g\, L) = \mathrm{Tr}_s(\tilde{\rho}_g[\nabla, D])$, where $[\nabla, D] \in \Omega^1(\mathcal{M})$ is the graded commutator.*

Proof. By continuity and linearity of the map Tr_s, it is enough to check for operators L of the form $L = T \otimes \omega$, where $T \in \mathcal{L}^1(\mathcal{H})$ and $\omega \in \Omega_k$. For each $g \in G$ and for any $\xi \otimes \omega_1 \in \mathcal{H} \otimes \Omega$, we have

$$\begin{aligned}
&\tilde{\rho}_g\, \nabla(T \otimes \omega)(\xi \otimes \omega_1) \\
&\qquad = \tilde{\rho}_g\Big(\nabla(T(\xi) \otimes \omega\, \omega_1)\Big) = \rho_g T(\xi) \otimes \gamma_g\, d(\omega\, \omega_1) \\
&\qquad = \rho_g T(\xi) \otimes (d\gamma_g\, \omega)\gamma_g \omega_1 + (-1)^{|\omega|} \rho_g T(\xi) \otimes \gamma_g \omega\, (d\gamma_g\, \omega_1) \\
&\qquad = (-1)^{|\omega|} \tilde{\rho}_g(T \otimes \omega)\, \nabla(\xi \otimes \omega_1) + \tilde{\rho}_g(T \otimes\, d\omega)(\xi \otimes \omega_1).
\end{aligned}$$

Hence

$$\tilde{\rho}_g[\nabla, T \otimes \omega] = \tilde{\rho}_g(T \otimes d\omega) \tag{27}$$

and

$$\mathrm{Tr}_s(\tilde{\rho}_g[\nabla, T \otimes \omega]) = \mathrm{Tr}_s(\tilde{\rho}_g(T \otimes d\omega)) = d\,\mathrm{Tr}_s(\tilde{\rho}_g(T \otimes \omega)). \qquad \square$$

For any $A_i \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega_*(\mathcal{B})$, $i = 1, \ldots, n$ and $g \in G$, we define

$$\begin{aligned}
&\langle\langle A_0, A_1, \ldots, A_n\rangle\rangle(g) \\
&\quad := \int_{\Delta_n} \mathrm{Tr}_s(\tilde{\rho}_g\, A_0 e^{-s_0\mathbf{A^2}} A_1\, e^{-s_1\mathbf{A^2}} \cdots A_n e^{-s_n\mathbf{A^2}})\, ds_0 \cdots ds_n.
\end{aligned} \tag{28}$$

Thus for each $g \in G$, $\langle\langle A_0, A_1, \ldots, A_n\rangle\rangle(g) \in \Omega(\mathcal{B})$. The next lemma is a generalization of Lemma 2.2 in [**12**] to an equivariant case.

Lemma 2.3. *Let* $A_j \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega(\mathcal{B})$ *for* $j = 0, 1, \dots, n$. *Let* $\varepsilon_j = (|A_0| + |A_1| + \cdots + |A_{j-1}|)(|A_j| + \cdots + |A_n|)$. *Then for each* $g \in G$ *the following holds*:

$$1 \text{ - } \langle\langle A_0, A_1, \dots, A_j, \dots, A_{n-1}, A_n\rangle\rangle(g) = (-1)^{\varepsilon_j}\langle\langle \tilde{\rho}_{g,*}^{-1}A_j, \tilde{\rho}_{g,*}^{-1}A_{j+1}, \dots, \tilde{\rho}_{g,*}^{-1}A_n, A_0, A_1, A_2, \dots, A_{j-1}\rangle\rangle(g). \tag{29}$$

$$2 \text{ - } \langle\langle A_0, A_1, A_2, \dots, A_{n-1}, A_n\rangle\rangle(g) = \sum_{j=0}^{n}(-1)^{\varepsilon_j}\langle\langle 1,\ \tilde{\rho}_{g,*}^{-1}A_j, \tilde{\rho}_{g,*}^{-1}A_{j+1}, \dots, \tilde{\rho}_{g,*}^{-1}A_n, A_0, A_1, \dots, A_{j-1}\rangle\rangle(g). \tag{30}$$

$$3 \text{ - } d(\langle\langle A_0, A_1, \dots, A_n\rangle\rangle(g)) = -\sum_{j=0}^{n}(-1)^{|A_0|+|A_1|+\cdots+|A_{j-1}|} \langle\langle A_0, A_1, \dots, [\mathbf{A},\ A_j], A_{j+1}, \dots, A_n\rangle\rangle(g). \tag{31}$$

4 (i) *for* $j = 0, \dots, n-1$;

$$\langle\langle A_0, A_1, \dots, [\mathbf{A}^2, A_j], \dots, A_n\rangle\rangle(g) = \langle\langle A_0, \dots, A_{j-1}A_j, \dots, A_n\rangle\rangle(g) - \langle\langle A_0, \dots, A_jA_{j+1}, \dots, A_n\rangle\rangle(g) \tag{32}$$

(ii) *for* $j = n$;

$$\langle\langle A_0, A_1, \dots, [\mathbf{A}^2, A_n]\rangle\rangle(g) = \langle\langle A_0, \dots, A_{n-1}A_n\rangle\rangle(g) - \langle\langle(\tilde{\rho}_{g,*}^{-1}A_n)\, A_0, \dots, A_{n-1}\rangle\rangle(g).$$

Proof. The equivariant cyclic symmetry in part 1 follows from

$$\begin{aligned}
&\langle\langle A_0, A_1, \dots, A_n\rangle\rangle(g)\\
&= \int_{\Delta_n} \mathrm{Tr}_s(\tilde{\rho}_g\, A_0 e^{-s_0\mathbf{A}^2} A_1 \cdots A_n e^{-s_n\mathbf{A}^2})\, ds\\
&= \int_{\Delta_n} (-1)^{\varepsilon_n}\mathrm{Tr}_s(A_n e^{-s_n\mathbf{A}^2}\tilde{\rho}_g A_0 e^{-s_0\mathbf{A}^2} A_1 e^{-s_1\mathbf{A}^2}\cdots A_{n-1}e^{-s_{n-1}\mathbf{A}^2})ds\\
&= (-1)^{\varepsilon_n}\langle\langle \tilde{\rho}_{g,*}^{-1}A_n, A_0, \dots, A_{n-1}\rangle\rangle.
\end{aligned}$$

To prove part 2, first note that by similar computation as in [**12**] we get

$$\langle\langle A_0, A_1, \dots, A_n\rangle\rangle(g) = \sum_{j=0}^{n} \langle\langle A_0, \dots, A_j, 1, A_{j+1}, \dots, A_n\rangle\rangle(g); \tag{33}$$

then applying part 1 to each term in (33) will give the desired result.

As $\mathbf{A} = \nabla + \mathcal{D}$, and by [**12**] we have

$$\sum_{j=0}^{n} (-1)^{|A_0|+\cdots+|A_{j-1}|} \langle\langle A_0, \dots, [\mathcal{D},\ A_j], \dots, A_n\rangle\rangle(g) = 0.$$

Thus, part 3 becomes

$$\begin{aligned} &d(\langle\langle A_0, \dots, A_n\rangle\rangle(g)) \\ &\qquad = -\sum_{j=0}^{n} (-1)^{|A_0|+\cdots+|A_{j-1}|} \langle\langle A_0, \dots, [\nabla,\ A_j], \dots, A_n\rangle\rangle(g). \end{aligned}$$

and the result follows from Lemma 2.2.

Part 4 (i) is a similar computation as in [**12**]. The case $j = n$ follows by first applying the cyclic symmetry as in part 1, and then part 4(i) will give the result. □

For any $u \in U$, then $[\mathbf{A}, \tilde{\mu}(u)] \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega_*(\mathcal{B})$. Furthermore, for any $k \geq 0$,

$$||([\mathbf{A}, \tilde{\mu}(u)])_k||_{\infty,k} = ||([\nabla + \mathcal{D}, u])_k||_{\infty,k} \leq C_k\, ||u||_\pi, \quad \forall\, u \in U, \tag{34}$$

where as before, $([\mathbf{A}, \tilde{\mu}(u)])_k \in \mathcal{L}(\mathcal{H})\hat{\otimes}\Omega_k(\mathcal{B})$ denotes the degree k component of $[\mathbf{A}, u]$ and $||.||_\infty$ is the operator norm.

Definition 2.4. The nth component of the equivariant bivariant Chern-Connes character of the module $(\mathcal{M}, \mathcal{D})$ is defined by

$$Ch^n_G(\mathcal{M}, \mathcal{D})(u_0, \dots, u_n)(g) := \langle\langle u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_n]\rangle\rangle(g) \in \Omega_*(\mathcal{B}),$$

where $u_i \in U$ and $g \in G$, for simplicity of notation we write $\tilde{\mu}(u_i)$ as u_i. And the total equivariant bivariant Chern-Connes character is

$$Ch^*_G(\mathcal{M}, \mathcal{D}) := \{Ch^0_G(\mathcal{M}, \mathcal{D}), Ch^1_G(\mathcal{M}, \mathcal{D}), \dots , Ch^n_G(\mathcal{M}, \mathcal{D}), \dots\}.$$

The character $Ch^*_G(\mathcal{M}, \mathcal{D})$ is a linear map

$$Ch^*_G(\mathcal{M}, \mathcal{D}) : \oplus_{n\geq 0} C^G_n(U) \to C(G, \hat{\Omega}(\mathcal{B})).$$

The next lemma shows that the above character extends continuously to a map $Ch^*_G(\mathcal{M}, \mathcal{D}) : CE^G_*(U) \to C(G, \hat{\Omega}(\mathcal{B}))$.

For each n, the degree k-component of $Ch^n_G(\mathcal{M}, \mathcal{D})$ is denoted by $Ch^n_{G,k}(\mathcal{M}, \mathcal{D})$. Thus for each $k \geq 0$, we have a linear map

$$Ch^n_{G,k}(\mathcal{M}, \mathcal{D}) : C^G_n(U) \to C(G, \Omega_k(\mathcal{B})).$$

Lemma 2.5 [**30**]. *For each $k \geq 0$, there exists C, c such that for all $u_i \in U$ and $i = 1, \dots , n$*

$$\sup_{g\in G} ||Ch^n_{G,k}(u_0, u_1, \dots , u_n)(g)||_k \leq \frac{C_k(k+1)^n}{n!}\, e^{c_k}\, Tr(e^{-D^2}) \prod_{i=0}^{n} ||u_i||,$$

where C and c are constants depending on k only and not on n.

Proof. For each $g \in G$ and $u_i \in U$, we have

$$\text{(35)} \quad Ch^n_{G,k}(\mathcal{M}, \mathcal{D})(u_0, \dots , u_n)(g)$$
$$= \int_{\Delta_n} \mathrm{Tr}_s \left(\tilde{\rho}_g\, u_0 e^{-s_0 \mathbf{A^2}} [\mathbf{A}, u_1]\, e^{-s_1 \mathbf{A^2}} \cdots [\mathbf{A}, u_n]\, e^{-s_n \mathbf{A^2}}\right)_k ds_0 \cdots ds_n.$$

Therefore,

$$\text{(36)} \quad ||(\tilde{\rho}_g\, u_0 e^{-s_0 \mathbf{A^2}} [\mathbf{A}, u_1]\, e^{-s_1 \mathbf{A^2}} \cdots [\mathbf{A}, u_n]\, e^{-s_n \mathbf{A^2}})_k||_{1,k}$$
$$\leq C_1 \sum_{k_0+\cdots+k_n=k} ||(u_0 e^{-s_0 \mathbf{A^2}})_{k_0}||_{s^{-1},k_0} \prod_{i=1}^{n} ||[\mathbf{A}, u_i]\, e^{-s_i \mathbf{A^2}})_{k_i}||_{s_i^{-1},k_i}$$
$$\leq C(k+1)^n e^{c_k}\, \mathrm{Tr}\,(e^{-D^2}) \prod_{i=0}^{n} ||u_i||;$$

the last step follows from (26) and the constant depends only on k. Finally, integrating the above over the simplex Δ_n will give the result. □

The discussion in Section 1.4 implies that $Ch^*_G(\mathcal{M},\mathcal{D})\in CE^G(U,\Omega(\mathcal{B}))$.

Theorem 2.6. *The equivariant bivariant Chern-Connes character $Ch^*_G(\mathcal{M},\mathcal{D})$ is closed. Hence, it defines a homology class in $HE^G_{ev}(U,\Omega(\mathcal{B}))$, i.e., for any $n\geq 0$, $u_i\in U$ and $g\in G$, then*

$$\Big(Ch^{n-1}_G b^g + Ch^{n+1}_G B^g\Big)(u_0,\dots,u_n)(g) + d\Big(Ch^n_G(u_0,\dots,u_n)(g)\Big) = 0$$

Proof. We apply Lemma 2.3 part 3 with $A_0=u_0$ and $A_i=[\mathbf{A},u_i]$ for $i=1,\dots,n$. This gives the following:

$$\begin{aligned} &d\Big(Ch^n_G(\mathcal{M},\mathcal{D})(u_0,\dots,u_n)(g)\Big)\\ &\quad= d\Big(\langle\langle u_0,[\mathbf{A},u_1],\dots,[\mathbf{A},u_{n-1}],[\mathbf{A},u_n]\rangle\rangle(g)\Big)\\ &\quad= -\langle\langle[\mathbf{A},u_0],[\mathbf{A},u_1],\dots,[\mathbf{A},u_n]\rangle\rangle(g) \qquad (37)\\ &\qquad -\sum_{j=1}^{n}(-1)^{j-1}\langle\langle u_0,[\mathbf{A},u_1],\dots,[\mathbf{A}^2,u_j],\dots,[\mathbf{A},u_n]\rangle\rangle(g). \qquad (38)\end{aligned}$$

Applying Lemma 2.3 part 2 to (37), we get (the notation as in (16))

$$\begin{aligned}&\langle\langle[\mathbf{A},u_0],\dots,[\mathbf{A},u_n]\rangle\rangle(g)\\ &\quad=\sum_{j=0}^{n}(-1)^{\varepsilon_j}\langle\langle 1,[\mathbf{A},\tilde{\alpha}_g^{-1}u_j],\dots,[\mathbf{A},\tilde{\alpha}_g^{-1}u_n],[\mathbf{A},u_0],\dots,[\mathbf{A},u_{j-1}]\rangle\rangle(g),\end{aligned}$$

which is just $(Ch^{n+1}_G B^g(u_0,\dots,u_n))(g)$.

Lemma 2.3 part 4 implies that (38) is equal to $(Ch^{n-1}_G b^g(u_0,\dots,u_n))(g)$. Indeed, for each $0<j\leq n-1$, and by Lemma 2.3 part 4(i) we have

$$\begin{aligned}&(-1)^{j-1}\langle\langle u_0,[\mathbf{A},u_1],\dots,[\mathbf{A}^2,u_j],\dots,[\mathbf{A},u_n]\rangle\rangle(g)\\ &\quad=(-1)^{j-1}\langle\langle u_0,[\mathbf{A},u_1],\dots,[\mathbf{A},u_{j-1}]\,u_j,\dots,[\mathbf{A},u_n]\rangle\rangle(g)\\ &\qquad+(-1)^{j}\langle\langle u_0,[\mathbf{A},u_1],\dots,u_j\,[\mathbf{A},u_{j+1}],\dots,[\mathbf{A},u_n]\rangle\rangle(g)\end{aligned}$$

while for $j = n$, we have

$$\begin{aligned}(-1)^{n-1}&\langle\langle u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_{n-1}], [\mathbf{A}^2, u_n]\rangle\rangle(g)\\ &= (-1)^{n-1}\langle\langle u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_{n-1}]u_n\rangle\rangle(g)\\ &\quad + (-1)^n \langle\langle(\tilde{\alpha}_g^{-1}u_n)\, u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_{n-1}]\rangle\rangle(g)\end{aligned}$$

Summing up over all j and using the fact

$$[\mathbf{A}, u_j\, u_{j+1}] = [\mathbf{A}, u_j]u_{j+1} + u_j\, [\mathbf{A}, u_{j+1}],$$

Therefore, what is left after the cancellation is

$$\sum_{j=1}^{n-1}(-1)^{j-1}\langle\langle u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_j\, u_{j+1}], \dots, [\mathbf{A}, u_n]\rangle\rangle(g)$$
$$+ (-1)^n\langle\langle(\tilde{\alpha}_g^{-1}u_n)u_0, [\mathbf{A}, u_1], \dots, [\mathbf{A}, u_{n-1}]\rangle\rangle(g)$$

which is $(Ch_G^{n-1}b^g(u_0, \dots, u_n))(g)$. □

Example 2.7. With the hypotheses of Example 1.2, then $\mathcal{M}$ is the Hilbert space $\mathcal{H}$. And as there is no flat B-connection, the operator $\mathbf{A}$ reduces to $\mathcal{D}$. Moreover, if G is a finite group then the equivariant bivariant Chern-Connes character reduces to the one in [**19**].

Example 2.8. With the hypotheses of Example 1.3, the G-equivariant θ-summable Fredholm module over the C^* algebra A, reduces to θ-summable Fredholm module over A. And we recover the bivariant Chern-Connes character in [**30**].

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