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OSCILLATION CRITERIA FOR SYSTEMS OF PARABOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

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ABSTRACT. Sufficient conditions are established for the oscillations of systems of parabolic equations with functional arguments of the form

$$\begin{aligned}\frac{\partial}{\partial t}u_i(x,t) &= a_i(t)\Delta u_i(x,t) + \sum_{k=1}^{m}\sum_{j=1}^{s} a_{ikj}(t)\Delta u_k(x,\rho_j(t))\\ &\quad - \sum_{k=1}^{m}\sum_{h=1}^{l} q_{ikh}(x,t)u_k(x,\sigma_h(t)),\\ &\quad (x,t)\in\Omega\times[0,\infty)\equiv G,\quad i=1,2,\dots,m,\end{aligned}$$

under boundary conditions of Dirichlet and Neumann type, where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and Δ is the Laplacian in Euclidean n-space R^n. These results are illustrated by some examples.

1. Introduction. Recently, the oscillation theory for systems of partial functional differential equations has been studied extensively [**3**–**7**]. In this paper, we study the oscillation of systems of parabolic differential equations with functional arguments of the form

$$\begin{aligned}\frac{\partial}{\partial t}u_i(x,t) &= a_i(t)\Delta u_i(x,t) + \sum_{k=1}^{m}\sum_{j=1}^{s} a_{ikj}(t)\Delta u_k(x,\rho_j(t))\\ &\quad - \sum_{k=1}^{m}\sum_{h=1}^{l} q_{ikh}(x,t)u_k(x,\sigma_h(t)),\\ &\quad (x,t)\in\Omega\times[0,\infty)\equiv G,\quad i=1,2,\dots,m,\end{aligned}\tag{1}$$

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where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u_i(x,t) = \sum_{r=1}^{n}(\partial^2 u_i(x,t)/\partial x_r^2)$, $i = 1, 2, \dots, m$.

Suppose that the following conditions hold:

(C1) $a_i \in C([0,\infty);[0,\infty)), a_{ikj} \in C([0,\infty);R), a_{iij}(t) > 0$ and

$$A_j(t) = \min_{1\le i\le m}\left\{a_{iij}(t) - \sum_{k=1,k\neq i}^{m} \left|a_{kij}(t)\right|\right\} \geq 0,$$
$$i,k \in I_m = \{1,2,\dots,m\}, \quad j \in I_s = \{1,2,\dots,s\};$$

(C2) $q_{ikh} \in C(\overline{G};R)$, and $q_{iih}(x,t) > 0$; $q_{iih}(t) = \min_{x\in\overline{\Omega}} q_{iih}(x,t)$,

$$\bar{q}_{ikh}(t) = \max_{x\in\overline{\Omega}} \mid q_{ikh}(x,t) \mid, \quad \text{and}$$

$$Q_h(t) = \min_{1\le i\le m}\left\{q_{iih}(t) - \sum_{k=1,k\neq i}^{m} \bar{q}_{kih}(t)\right\} \geq 0,$$
$$i,k \in I_m, \quad h \in I_l = \{1,2,\dots,l\};$$

(C3) $\rho_j, \sigma_h \in C([0,\infty);[0,\infty))$, $\lim_{t\to\infty}\rho_j(t) = \lim_{t\to\infty}\sigma_h(t) = \infty$, $j \in I_s$, $h \in I_l$.

Consider the following boundary conditions:

$$(2) \qquad \frac{\partial u_i(x,t)}{\partial N} + g_i(x,t)u_i(x,t) = 0, \quad (x,t) \in \partial\Omega\times[0,\infty), \quad i \in I_m,$$

where N is the unit exterior normal vector to $\partial\Omega$ and $g_i(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, $i \in I_m$, and

$$(3) \qquad u_i(x,t) = 0, \quad (x,t) \in \partial\Omega\times[0,\infty), \quad i \in I_m.$$

Definition 1.1. The vector function $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ is said to be a solution of the problem (1),(2) (or (1), (3)) if it satisfies (1) in $G = \Omega\times[0,\infty)$ and boundary condition (2) (or (3)).

Definition 1.2. The vector solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to be oscillatory in the domain $G = \Omega\times[0,\infty)$ if at least one of its nontrivial components is oscillatory in G. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

2. Oscillation of the problem (1), (2).

Theorem 2.1. *If the differential inequality*

$$V'(t) + \sum_{h=1}^{l} Q_h(t)V(\sigma_h(t)) \le 0, \quad t \ge 0, \tag{4}$$

has no eventually positive solution, then every solution $u(x,t)$ of the problem (1),(2) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2). We assume that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0$, $i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t), Z_i(x,t) = \delta_i u_i(x,t)$. Then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$, $i \in I_m$. From (C3) there exists a number $t_1 \ge t_0$ such that $Z_i(x,t) > 0$, $Z_k(x, \rho_j(t)) > 0$ and $Z_i(x, \sigma_h(t)) > 0$ in $\Omega \times [t_1, \infty)$, $i, k \in I_m$, $j \in I_s, h \in I_l$.

Integrating (1) with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_i(x,t)\,dx &= a_i(t) \int_\Omega \Delta u_i(x,t)\,dx \\ &\quad + \sum_{k=1}^{m} \sum_{j=1}^{s} a_{ikj}(t) \int_\Omega \Delta u_k(x, \rho_j(t))\,dx \\ &\quad - \sum_{k=1}^{m} \sum_{h=1}^{l} \int_\Omega q_{ikh}(x,t) u_k(x, \sigma_h(t))\,dx, \\ t \ge t_1, \quad i \in I_m. \end{aligned} \tag{5}$$

Hence, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega Z_i(x,t)\,dx &= a_i(t)\int_\Omega \Delta Z_i(x,t)\,dx\\ &\quad+\sum_{k=1}^{m}\sum_{j=1}^{s} a_{ikj}(t)\frac{\delta_i}{\delta_k}\int_\Omega \Delta Z_k(x,\rho_j(t))\,dx\\ &\quad-\frac{\delta_i}{\delta_k}\sum_{k=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ikh}(x,t)Z_k(x,\sigma_h(t))\,dx,\\ &\quad t\ge t_1,\quad i\in I_m.\end{aligned}\tag{6}$$

Green's formula and boundary condition (2) yield

$$\begin{aligned}\int_\Omega \Delta Z_i(x,t)\,dx &= \int_{\partial\Omega}\frac{\partial Z_i(x,t)}{\partial N}\,dS\\ &= -\int_{\partial\Omega} g_i(x,t)Z_i(x,t)\,dS \le 0,\end{aligned}\tag{7}$$

and

$$\begin{aligned}\int_\Omega \Delta Z_k(x,\rho_j(t))\,dx &= \int_{\partial\Omega}\frac{\partial Z_k(x,\rho_j(t))}{\partial N}\,dS\\ &= -\int_{\partial\Omega} g_k(x,\rho_j(t))Z_k(x,\rho_j(t))\,dS,\\ &\quad t\ge t_1,\quad i,k\in I_m,\quad j\in I_s,\end{aligned}\tag{8}$$

where dS is the surface element on $\partial\Omega$.

Now combining (6)–(8), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega Z_i(x,t)\,dx+\sum_{k=1}^{m}\sum_{j=1}^{s} a_{ikj}(t)\frac{\delta_i}{\delta_k}\int_{\partial\Omega} g_k(x,\rho_j(t))Z_k(x,\rho_j(t))\,dS\\ &\qquad+\sum_{h=1}^{l} q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))\,dx\\ &\qquad-\sum_{h=1}^{l}\sum_{k=1,k\ne i}^{m}\overline{q}_{ikh}(t)\int_\Omega Z_k(x,\sigma_h(t))\,dx\le 0,\\ &\qquad t\ge t_1,\quad i\in I_m.\end{aligned}\tag{9}$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega} Z_i(x,t)\,dx + \sum_{j=1}^{s} a_{iij}(t)\int_{\partial\Omega} g_i(x,\rho_j(t))Z_i(x,\rho_j(t))\,dS \\
&\quad - \sum_{\substack{k=1\\k\neq i}}^{m}\sum_{j=1}^{s} |a_{ikj}(t)|\int_{\partial\Omega} g_k(x,\rho_j(t))Z_k(x,\rho_j(t))\,dS \\
&\quad + \sum_{h=1}^{l} q_{iih}(t)\int_{\Omega} Z_i(x,\sigma_h(t))\,dx \\
&\quad - \sum_{h=1}^{l}\sum_{k=1,k\neq i}^{m} \bar{q}_{ikh}(t)\int_{\Omega} Z_k(x,\sigma_h(t))\,dx \le 0, \\
&\qquad t \ge t_1, \quad i \in I_m.
\end{aligned} \tag{10}
$$

Set

$$
V_i(t) = \int_{\Omega} Z_i(x,t)\,dx, U_i(t) = \int_{\partial\Omega} g_i(x,t)Z_i(x,t)\,dS, \quad t \ge t_1, \quad i \in I_m.
$$

From (10) we have

$$
\begin{aligned}
&V_i'(t) + \sum_{j=1}^{s}[a_{iij}(t)U_i(\rho_j(t)) - \sum_{\substack{k=1\\k\neq i}}^{m} |a_{ikj}(t)|U_k(\rho_j(t))] \\
&\quad + \sum_{h=1}^{l}\Big[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{k=1,k\neq i}^{m} \bar{q}_{ikh}(t)V_k(\sigma_h(t))\Big] \le 0, \\
&\qquad t \ge t_1, \quad i \in I_m.
\end{aligned} \tag{11}
$$

Let

$$
V(t) = \sum_{i=1}^{m} V_i(t), U(t) = \sum_{i=1}^{m} U_i(t), \quad t \ge t_1.
$$

From (11) we have

(12)

$$V'(t) + \sum_{j=1}^{s} \left\{ \sum_{i=1}^{m} [a_{iij}(t) U_i(\rho_j(t)) - \sum_{\substack{k=1 \\ k \neq i}}^{m} |a_{ikj}(t)| U_k(\rho_j(t))] \right\}$$
$$+ \sum_{h=1}^{l} \left\{ \sum_{i=1}^{m} [q_{iih}(t) V_i(\sigma_h(t)) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \bar{q}_{ikh}(t) V_k(\sigma_h(t))] \right\} \leq 0, \quad t \geq t_1.$$

Noting that

$$\begin{aligned}
&\sum_{i=1}^{m} \Big[q_{iih}(t) V_i(\sigma_h(t)) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \bar{q}_{ikh}(t) V_k(\sigma_h(t)) \Big] \\
&\quad = \Big[q_{11h}(t) V_1(\sigma_h(t)) - \sum_{\substack{k=1 \\ k \neq 1}}^{m} \bar{q}_{1kh}(t) V_k(\sigma_h(t)) \Big] \\
&\qquad + \Big[q_{22h}(t) V_2(\sigma_h(t)) - \sum_{\substack{k=1 \\ k \neq 2}}^{m} \bar{q}_{2kh}(t) V_k(\sigma_h(t)) \Big] \\
&\qquad + \cdots\cdots \\
&\qquad + \Big[q_{mmh}(t) V_m(\sigma_h(t)) - \sum_{\substack{k=1 \\ k \neq m}}^{m} \bar{q}_{mkh}(t) V_k(\sigma_h(t)) \Big] \\
&\quad = \Big[q_{11h}(t) - \sum_{\substack{k=1 \\ k \neq 1}}^{m} \bar{q}_{k1h}(t) \Big] V_1(\sigma_h(t)) + \Big[q_{22h}(t) - \sum_{\substack{k=1 \\ k \neq 2}}^{m} \bar{q}_{k2h}(t) \Big] V_2(\sigma_h(t)) \\
&\qquad + \cdots\cdots \\
&\qquad + \Big[q_{mmh}(t) - \sum_{\substack{k=1 \\ k \neq m}}^{m} \bar{q}_{kmh}(t) \Big] V_m(\sigma_h(t)) \\
&\quad \geq \min_{1 \leq i \leq m} \Big\{ q_{iih}(t) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \bar{q}_{kih}(t) \Big\} \sum_{i=1}^{m} V_i(\sigma_h(t)) \\
&\quad = Q_h(t) V(\sigma_h(t)), \quad t \geq t_1, \quad h \in I_l.
\end{aligned}$$

Similarly we have

$$\sum_{i=1}^{m}\Big[a_{iij}(t)U_i(\rho_j(t)) - \sum_{\substack{k=1\\k\neq i}}^{m}|a_{ikj}(t)|U_k(\rho_j(t))\Big]$$
$$\geq \min_{1\leq i\leq m}\Big\{a_{iij}(t) - \sum_{k=1}^{m}|a_{kij}(t)|\Big\}\sum_{i=1}^{m}U_i(\rho_j(t))$$
$$= A_j(t)U(\rho_j(t)), \quad t\geq t_1, \quad j\in I_s.$$

Then from (12), we get

$$V'(t) + \sum_{j=1}^{s}A_j(t)U(\rho_j(t)) + \sum_{h=1}^{l}Q_h(t)V(\sigma_h(t)) \leq 0, \quad t\geq t_1. \tag{13}$$

It is easy to see that

$$U(\rho_j(t)) = \sum_{i=1}^{m}U_i(\rho_j(t)) \geq 0, \quad t\geq t_1, \quad j\in I_s.$$

Therefore,

$$V'(t) + \sum_{h=1}^{l}Q_h(t)V(\sigma_h(t)) \leq 0, \quad t\geq t_1,$$

which contradicts the assumption that (4) has no eventually positive solutions. This completes the proof. □

We now give two lemmas which are useful for the proof of the following results.

Lemma 2.1 [**7**]. *Consider the differential inequality*

$$x'(t) + p(t)x(g(t)) \leq 0. \tag{14}$$

Assume that $p \in C(R;[0,\infty))$, $g\in C(R;R)$, $g(t)\leq t$ *and* $g(t)$ *is nondecreasing,* $\lim_{t\to\infty} g(t) = \infty$ *and suppose that*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}p(s)\,ds > \frac{1}{e}. \tag{15}$$

Then the inequality (14) *has no eventually positive solutions.*

Lemma 2.2 [**4**]. *Consider the differential inequality* (14). *Assume that* $p, g \in C([0,\infty); [0,\infty))$, $g(t) \le t$, $t \ge 0$ *and* $g(t)$ *is nondecreasing,* $\lim_{t\to\infty} g(t) = \infty$, *and suppose that when* $L < 1$ *and* $0 < K \le 1/e$ *the following conditions hold*

$$L > \frac{\ln \mu_1 + 1}{\mu_1} - \frac{1 - K - \sqrt{1 - 2K - K^2}}{2}, \tag{16}$$

where

$$K = \liminf_{t\to\infty} \int_{g(t)}^{t} p(s)\, ds, \quad L = \limsup_{t\to\infty} \int_{g(t)}^{t} p(s)\, ds,$$

and μ_1 *is the smaller root of the equation*

$$\mu = e^{K\mu}.$$

Then the inequality (14) *has no eventually positive solutions.*

Theorem 2.2. *If there exists* $h_0 \in I_l$ *such that* $\sigma_{h_0}(t) \le t$, $\sigma_{h_0}(t)$ *is nondecreasing in* $[0,\infty)$ *and*

$$\liminf_{t\to\infty} \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\, ds > \frac{1}{e}, \tag{17}$$

then every solution $u(x,t)$ *of the problem* (1), (2) *is oscillatory in* G.

Proof. We prove that the inequality (4) has no eventually positive solution if the conditions of Theorem 2.2 hold. Suppose $V(t)$ is an eventually positive solution of the inequality (4). Then there exists a number $t_1 \ge t_0$ such that $V(\sigma_h(t)) > 0$, $t \ge t_1$, $h \in I_l$. Therefore, we have

$$V'(t) + Q_{h_0}(t) V(\sigma_{h_0}(t)) \le 0, \quad t \ge t_1. \tag{18}$$

By Lemma 2.1 we obtain that the inequality (18) has no eventually positive solutions, which contradicts the fact that $V(t) > 0$ is a solution of the inequality (18).

By using Lemma 2.2, the proof of the following theorem is similar to that of Theorem 2.2 and we omit it.

Theorem 2.3. *If there exists $h_0 \in I_l$ such that $\sigma_{h_0}(t) \le t$, $\sigma_{h_0}(t)$ is nondecreasing in $[0,\infty)$ and suppose that when $\overline{L} < 1$ and $0 < \overline{K} \le 1/e$ the following conditions hold*

$$\overline{L} > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \overline{K} - \sqrt{1 - 2\overline{K} - \overline{K}^2}}{2}, \tag{19}$$

where

$$\overline{K} = \liminf_{t\to\infty} \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds, \quad \overline{L} = \limsup_{t\to\infty} \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds,$$

and λ_1 is the smaller root of the equation

$$\lambda = e^{\overline{K}\lambda}.$$

Then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Theorem 2.4. *If there exists $h_0 \in I_l$ such that $\sigma_{h_0}(t) \le t$, $\sigma_{h_0}(t)$ is nondecreasing in $[0,\infty)$ and*

$$\limsup_{t\to\infty} \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds > 1, \tag{20}$$

then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Proof. As in the proof of Theorem 2.2 we obtain (18). Integrating the inequality (18) from $\sigma_{h_0}(t)$ to t we have

$$V(t) - V(\sigma_{h_0}(t)) + \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s) V(\sigma_{h_0}(s))\,ds \le 0, \quad t \ge t_1. \tag{21}$$

Noting that $V'(t) \le 0$, $\sigma_{h_0}(t) \le t$, $\sigma_{h_0}(t)$ is nondecreasing in $[t_1,\infty)$, from (21) we have

$$V(t) - V(\sigma_{h_0}(t)) + V(\sigma_{h_0}(t)) \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds \le 0, \quad t \ge t_1. \tag{22}$$

Therefore,

$$\int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds \le 1 - \frac{V(t)}{V(\sigma_{h_0}(t))} < 1.$$

And, hence,

$$\limsup_{t\to\infty} \int_{\sigma_{h_0}(t)}^{t} Q_{h_0}(s)\,ds \le 1,$$

which violates the condition (20). This completes the proof of Theorem 2.4.

3. Oscillation of the problem (1), (3). It is known that the smallest eigenvalue α_0 of the Dirichlet problem

$$\begin{cases} \Delta\omega(x) + \alpha\omega(x) = 0 & \text{in } \Omega, \\ \omega(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where α is a constant, is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Theorem 3.1. *If the differential inequality*

$$(23)\quad V'(t) + \alpha_0 \sum_{j=1}^{s} A_j(t)V(\rho_j(t)) + \sum_{h=1}^{l} Q_h(t)V(\sigma_h(t)) \le 0, \quad t \ge t_1$$

has no eventually positive solutions, then every solution of the problem (1), (3) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (3). We assume that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0, i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t), Z_i(x,t) = \delta_i u_i(x,t)$. Then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0,\infty)$, $i \in I_m$. From (C3) there exists a number $t_1 \ge t_0$ such that $Z_i(x,t) > 0$, $Z_k(x,\rho_j(t)) > 0$ and $Z_i(x,\sigma_h(t)) > 0$ in $\Omega \times [t_1,\infty)$, $i,k \in I_m$, $j \in I_s$, $h \in I_l$.

Multiplying both sides of (1) by $\varphi(x) > 0$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_i(x,t)\varphi(x)\,dx = a_i(t)&\int_\Omega \Delta u_i(x,t)\varphi(x)\,dx\\ &+\sum_{k=1}^m\sum_{j=1}^s a_{ikj}(t)\int_\Omega \Delta u_k(x,\rho_j(t))\varphi(x)\,dx\\ &-\sum_{k=1}^m\sum_{h=1}^l\int_\Omega q_{ikh}(x,t)u_k(x,\sigma_h(t))\varphi(x)\,dx,\\ &t\ge t_1,\quad i\in I_m.\end{aligned}\tag{24}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\int_\Omega Z_i(x,t)\varphi(x)\,dx = a_i(t)&\int_\Omega \Delta Z_i(x,t)\varphi(x)\,dx\\ &+\sum_{j=1}^s a_{iij}(t)\int_\Omega \Delta Z_i(x,\rho_j(t))\varphi(x)\,dx\\ &+\sum_{\substack{k=1\\k\ne i}}^m\sum_{j=1}^s a_{ikj}(t)\int_\Omega \Delta Z_k(x,\rho_j(t))\varphi(x)\,dx\\ &-\sum_{h=1}^l\int_\Omega q_{iih}(x,t)Z_i(x,\sigma_h(t))\varphi(x)\,dx\\ &-\frac{\delta_i}{\delta_k}\sum_{\substack{k=1\\k\ne i}}^m\sum_{h=1}^l\int_\Omega q_{ikh}(x,t)Z_k(x,\sigma_h(t))\varphi(x)\,dx,\\ &t\ge t_1,\quad i\in I_m.\end{aligned}\tag{25}$$

Green's formula and boundary (3) yield

$$\begin{aligned}\int_\Omega \Delta Z_i(x,t)\varphi(x)\,dx &= \int_\Omega Z_i(x,t)\Delta\varphi(x)\,dx\\ &= -\alpha_0\int_\Omega Z_i(x,t)\varphi(x)\,dx \le 0,\end{aligned}\tag{26}$$

and

$$\begin{aligned}\int_\Omega \Delta Z_k(x,\rho_j(t))\varphi(x)\,dx &= \int_\Omega Z_k(x,\rho_j(t))\Delta\varphi(x)\,dx \\ &= -\alpha_0 \int_\Omega Z_k(x,\rho_j(t))\varphi(x)\,dx, \\ &\quad t \ge t_1, \quad i,k \in I_m, \quad j \in I_s.\end{aligned} \tag{27}$$

Now from (25), (26) and (27), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega Z_i(x,t)\varphi(x)\,dx \le &-\alpha_0 \sum_{j=1}^{s} a_{iij}(t)\int_\Omega Z_i(x,\rho_j(t))\varphi(x)\,dx \\ &+ \alpha_0 \sum_{\substack{k=1\\k\neq i}}^{m}\sum_{j=1}^{s} |a_{ikj}(t)|\int_\Omega Z_k(x,\rho_j(t))\varphi(x)\,dx \\ &- \sum_{h=1}^{l} q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))\varphi(x)\,dx \\ &+ \sum_{\substack{k=1\\k\neq i}}^{m}\sum_{h=1}^{l} \overline{q}_{ikh}(t)\int_\Omega Z_k(x,\sigma_h(t))\varphi(x)\,dx, \\ &\quad t \ge t_1, \quad i \in I_m.\end{aligned} \tag{28}$$

Setting

$$V_i(t) = \int_\Omega Z_i(x,t)\varphi(x)\,dx, \quad t \ge t_1, i \in I_m,$$

we have

$$\begin{aligned}V_i'(t) + \alpha_0 \sum_{j=1}^{s}&\Big[a_{iij}(t)V_i(\rho_j(t)) - \sum_{\substack{k=1\\k\neq i}}^{m} |a_{ikj}(t)|V_k(\rho_j(t))\Big] \\ &+ \sum_{h=1}^{l}\Big[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{\substack{k=1\\k\neq i}}^{m} \overline{q}_{ikh}(t)V_k(\sigma_h(t))\Big] \le 0, \\ &\quad t \ge t_1, \quad i \in I_m.\end{aligned} \tag{29}$$

Let

$$V(t) = \sum_{i=1}^{m} V_i(t), t \geq t_1.$$

From (29) we have

(30)

$$\begin{aligned} V'(t) + \alpha_0 \sum_{j=1}^{s} \Big\{ \sum_{i=1}^{m} \Big[a_{iij}(t) V_i(\rho_j(t)) - \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)| V_k(\rho_j(t)) \Big] \Big\} \\ + \sum_{h=1}^{l} \Big\{ \sum_{i=1}^{m} \Big[q_{iih}(t) V_i(\sigma_h(t)) - \sum_{\substack{k=1 \\ k\neq i}}^{m} \bar{q}_{ikh}(t) V_k(\sigma_h(t)) \Big] \Big\} \leq 0, \\ t \geq t_1. \end{aligned}$$

As in the proof of Theorem 2.1, from (30) we obtain

$$V'(t) + \alpha_0 \sum_{j=1}^{s} A_j(t) V(\rho_j(t)) + \sum_{h=1}^{l} Q_h(t) V(\sigma_h(t)) \leq 0, \quad t \geq t_1.$$

The above inequality shows that $V(t) = \sum_{i=1}^{m} V_i(t) > 0$ is an eventually positive solution of the inequality (23), which contradicts the assumption that (23) has no eventually positive solutions. This completes the proof of Theorem 3.1. $\square$

The proofs of the following theorems are similar to that of Theorem 2.2, Theorem 2.3 and Theorem 2.4.

Theorem 3.2. *If there exists $j_0 \in I_s$ such that $\rho_{j_0}(t) \leq t$, $\rho_{j_0}(t)$ is nondecreasing in $[0, \infty)$ and*

$$\liminf_{t\to\infty} \alpha_0 \int_{\rho_{j_0}(t)}^{t} A_{j_0}(s)\, ds > \frac{1}{e}. \tag{31}$$

Then every solution $u(x,t)$ of the problem (1), (3) *is oscillatory in G.*

Theorem 3.3. *If there exists $j_0 \in I_s$ such that $\rho_{j_0}(t) \leq t$, $\rho_{j_0}(t)$ is nondecreasing in $[0, \infty)$ and suppose that when $\underline{L_1} < 1$ and*

$0 < \overline{K_1} \leq 1/e$ *the following conditions hold*

$$\overline{L_1} > \frac{\ln \gamma_1 + 1}{\gamma_1} - \frac{1 - \overline{K_1} - \sqrt{1 - 2\overline{K_1} - \overline{K_1}^2}}{2}, \tag{32}$$

where

$$\overline{K_1} = \liminf_{t\to\infty} \alpha_0 \int_{\rho_{j_0}(t)}^{t} A_{j_0}(s)\, ds, \overline{L_1} = \limsup_{t\to\infty} \alpha_0 \int_{\rho_{j_0}(t)}^{t} A_{j_0}(s)\, ds,$$

and γ_1 *is the smaller root of the equation*

$$\gamma = e^{\overline{K_1}\gamma};$$

then every solution $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

Theorem 3.4. *If there exists* $j_0 \in I_s$ *such that* $\rho_{j_0}(t) \leq t$, $\rho_{j_0}(t)$ *is nondecreasing in* $[0,\infty)$ *and*

$$\limsup_{t\to\infty} \alpha_0 \int_{\rho_{j_0}(t)}^{t} A_{j_0}(s)\, ds > 1, \tag{33}$$

then every solution $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

Theorem 3.5. *If the conditions of Theorem* 2.2 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

Theorem 3.6. *If the conditions of Theorem* 2.3 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

Theorem 3.7. *If the conditions of Theorem* 2.4 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

4. Examples.

Example 4.1. Consider the system of parabolic equations

(34)

$$\begin{cases} \dfrac{\partial}{\partial t}\, u_1(x,t) = \Delta u_1(x,t) + (1+e^{-t})\Delta u_1(x,t-\dfrac{\pi}{2}) \\ \qquad\qquad\qquad +e^{-\pi/2}\Delta u_2(x,t-\dfrac{\pi}{2}) - (1+e^{-t})u_1(x,t-\pi) \\ \qquad\qquad\qquad -e^{-\pi}u_2(x,t-\pi), \\ \dfrac{\partial}{\partial t}\, u_2(x,t) = (1+e^{\pi})\Delta u_2(x,t) + \dfrac{1}{3}e^{-t}\Delta u_1(x,t-\dfrac{\pi}{2}) \\ \qquad\qquad\qquad +\dfrac{4}{3}e^{-\pi/2}\Delta u_2(x,t-\dfrac{\pi}{2}) - e^{-t}u_1(x,t-\pi) \\ \qquad\qquad\qquad -(1+e^{-\pi})u_2(x,t-\pi), \quad (x,t)\in(0,\pi)\times[0,\infty), \end{cases}$$

with boundary condition

$$\frac{\partial}{\partial x}\, u_i(0,t) = \frac{\partial}{\partial x}\, u_i(\pi,t) = 0, \quad t \geq 0, i = 1,2. \tag{35}$$

Here $n=1$, $m=2$, $s=1$, $l=1$, $a_1(t)=1$, $a_{111}(t)=1+e^{-t}$, $a_{121}(t)=e^{-\pi/2}$, $\rho_1(t)=t-(\pi/2)$, $q_{111}(x,t)=1+e^{-t}$, $q_{121}(x,t)=e^{-\pi}$, $\sigma_1(t)=t-\pi$, $a_2(t)=1+e^{\pi}$, $a_{211}(t)=(1/3)e^{-t}a_{221}=(4/3)e^{-t}e^{-\pi/2}$, $q_{211}(x,t)=e^{-t}$, $q_{221}(x,t)=1+e^{-\pi}$, $\Omega=(0,\pi)$. It is easy to see that the conditions of Theorem 2.2 are verified. Thus all solutions of the problem (34), (35) are oscillatory in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=\cos x\sin t$, $u_2(x,t)=-e^{-t}\cos x\sin t$ is such a solution.

Example 4.2. Consider the system of parabolic equations

(36)

$$\begin{cases} \dfrac{\partial}{\partial t}\, u_1(x,t) = 3\Delta u_1(x,t) + \Delta u_1(x,t-\dfrac{3\pi}{2}) + \Delta u_2(x,t-\dfrac{3\pi}{2}) \\ \qquad\qquad\qquad -4u_1(x,t-\pi) - (-2)u_2(x,t-\pi), \\ \dfrac{\partial}{\partial t}\, u_2(x,t) = \dfrac{9}{2}\Delta u_2(x,t) + \dfrac{1}{2}\Delta u_1(x,t-\dfrac{3\pi}{2}) \\ \qquad\qquad\qquad +2\Delta u_2(x,t-\dfrac{3\pi}{2}) - 3u_1(x,t-\pi) - 4u_2(x,t-\pi), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in(0,\pi)\times[0,\infty), \end{cases}$$

with boundary condition

$$u_i(0,t) = u_i(\pi,t) = 0, \quad t \geq 0, i = 1,2. \tag{37}$$

It is easy to see that all conditions of Theorem 3.7 are fulfilled. Then every solution of the problem (36), (37) oscillates in $(0,\pi) \times [0,\infty)$. In fact, $u_1(x,t) = \sin x \cos t$, $u_2(x,t) = \sin x \sin t$ is such a solution.

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