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A NOTE ON THE TANGENTIAL ENVELOPES OF VERONESE VARIETIES

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ABSTRACT. If $X \subseteq \mathbf{P}^m$ is a projective variety, then its tangential envelope has expected dimension $\min\{2\dim X, m\}$. We calculate the dimensions of some higher tangential envelopes of the Veronese varieties and show that they violate this expectation.

The classical Veronese varieties are known to exhibit various kinds of pathological behavior from the point of view of differential geometry. It is known that many of them have deficient secant varieties, the Veronese surface in $\mathbf{P}^5$ being perhaps the most widely known example. (See [**2**, Corollary 7.5], where the dimension of the secant variety is listed in each possible case.) In this paper, we discuss a similar pathology, namely, the deficiency of their tangential envelopes.

Consider the d-fold Veronese imbedding

$$v_d : \mathbf{P}^n \longrightarrow \mathbf{P}^N, \quad N = \binom{n+d}{d} - 1$$

given by the complete linear system of degree d hypersurfaces in $\mathbf{P}^n$. Let $T^{(0)}$ denote its image and recursively define $T^{(r)}$ as the tangential envelope of $T^{(r-1)}$. Now if $X \subseteq \mathbf{P}^m$ is a projective variety, then the expected dimension of its tangential envelope is $\min\{2\dim X, m\}$. Here we calculate $\dim T^{(r)}$ for $r \leq 4$ and all but a handful of pairs (n,d). It will turn out that almost always we have a strict inequality $\dim T^{(r)} < 2\dim T^{(r-1)}$. The calculations are summarized in Theorem 4.3 at the end of the paper.

In the first section we establish notation and describe a calculus to represent a general element of $T^{(r)}$. This will be the basis of all subsequent dimension computations.

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1. Preliminaries. The base field k will be algebraically closed of characteristic zero. For an irreducible variety $X \subseteq \mathbf{P}^N$ we define the tangential envelope (or developable) TX to be the closure of the union of tangent spaces to *smooth* points of X. It is an irreducible projective subvariety of $\mathbf{P}^N$ and $\dim TX \leq \min\{2\dim X, N\}$. We say that X has a deficient tangential envelope if strict inequality holds; this happens if and only if a general point of a general tangent space to X lies on at least ∞^1 tangent spaces. The numerical difference $\min\{2\dim X, N\} - \dim TX$ is the tangential deficiency of X.

Let $S = \bigoplus_{d\geq 0} S_d = k[x_0, \dots, x_n]$ denote the polynomial ring with the usual grading. Identify the Veronese variety $T^{(0)}$ with the subset of $\mathbf{P}S_d$ consisting of all forms L^d (modulo scalars), where L is a nonzero linear form. To avoid trivialities assume $d \geq 2$. We define $T^{(r)} = T(T^{(r-1)})$ and give a concrete description of a general point of $T^{(r)}$ as an element of $\mathbf{P}S_d$. We begin by clarifying the connection between the Zariski and projective tangent space of a variety, cf. [**4**, Section 14].

For a k-vector space V, let

$$\mathbf{A}V = \mathrm{Spec}\,(\mathrm{Sym}^\bullet V^*) \quad \text{and} \quad \mathbf{P}V = \mathrm{Proj}\,(\mathrm{Sym}^\bullet V^*).$$

Let $X \subseteq \mathbf{P}V$ be a projective variety and $CX \subseteq \mathbf{A}V$ the cone over X. For a (closed) point $x \in X$, let $\lambda_x \subseteq V$ be the corresponding one-dimensional space. The Zariski tangent space $T_{X,x}$ is canonically a subspace of $T_{\mathbf{P},x} = \mathrm{Hom}\,(\lambda_x, V/\lambda_x)$. Consider the space $\{\mathrm{image}\,(\tau) : \tau \in T_{X,x}\} \subseteq V/\lambda_x$; it lifts to a $(\dim T_{X,x} + 1)$-dimensional space $W_{X,x} \subseteq V$. The projectivization $\mathbf{P}W_{X,x} \subseteq \mathbf{P}V$ is the projective tangent space $\mathbf{T}_{X,x}$.

Now for a nonzero $v \in \lambda_x$, the Zariski tangent space $T_{CX,v}$ is canonically isomorphic to $W_{X,x}$ as a subspace of $T_{\mathbf{A}V,v} = V$. If TCX denotes the closure of $\cup_{x \text{ smooth}} W_{X,x}$ in $\mathbf{A}V$,

$$CTX = TCX, \quad \text{hence} \dim TX = \dim TCX - 1.$$

Now let X be the Veronese variety $T^{(0)}$ and consider the morphism $\phi_0 : S_1 \to S_1^d$, $L_0 \to {L_0}^d$. A tangent vector to S_1 at L_0 may be written $L_0 + \varepsilon L_1$, and its image $(L_0 + \varepsilon L_1)^d = L_0^d + \varepsilon d L_0^{d-1} L_1$, since $\varepsilon^2 = 0$. Hence we have a morphism

$$\phi_1 : (S_1)^2 \longrightarrow S_d, \quad (L_0, L_1) \longrightarrow L_0^{d-1} L_1,$$

such that the closure of the image of ϕ_1 is $CT^{(1)}$.

Letting $L_0 \to L_0 + \varepsilon L_2$, $L_1 \to L_1 + \varepsilon L_3$,

$$(L_0 + \varepsilon L_2)^{d-1}(L_1 + \varepsilon L_3) = L_0^{d-1} L_1 + \varepsilon((d-1)L_0^{d-2} L_1 L_2 + L_0^{d-1} L_3);$$

hence a morphism

$$\phi_2 : (S_1)^4 \longrightarrow S_d, \quad (L_0, L_1, L_2, L_3) \longrightarrow L_0^{d-2}(L_0 L_3 + L_1 L_2).$$

The closure of the image of ϕ_2 is $CT^{(2)}$. To recapitulate, we have a morphism of affine spaces

$$\phi_r : (S_1)^{2^r} \longrightarrow S_d, \quad (L_0, \dots, L_{2^r-1}) \longrightarrow F_r(L_i) \tag{1}$$

with F_r a homogeneous degree d form. The closure of its image is $CT^{(r)}$. By generic smoothness ([**5**, Chapter III]), the corresponding morphism on tangent spaces is surjective over a general point of $CT^{(r)}$. Hence we take a tangent vector $(L_0 + \varepsilon L_{2^r}, \dots, L_{2^r-1} + \varepsilon L_{2^{r+1}-1})$ at a general point of $(S_1)^{2^r}$ and follow its image via ϕ_r. A moment's reflection will show that this is nothing more than the classical polarization process, i.e., letting

$$F_{r+1} = \sum_{j=0}^{2^r-1} \frac{\partial F_r}{\partial L_j} L_{(j+2^r)}, \tag{2}$$

we get ϕ_{r+1}. This gives an inductive procedure to represent a general point of $CT^{(r)}$. The map $d\phi_r$ on tangent spaces at the point $(L_0, \dots, L_{2^r-1})$ is given by

$$d\phi_r : (S_1)^{2^r} \longrightarrow S_d, \quad (M_0, \dots, M_{2^r-1}) \longrightarrow \sum_{j=0}^{2^r-1} \frac{\partial F_r}{\partial L_j} M_j. \tag{3}$$

and $\dim CT^{(r)} = \operatorname{rank} d\phi_r$ at a general point of $(S_1)^{2^r}$. To forestall any confusion, note that ϕ_r is a morphism of affine varieties whereas $d\phi_r$ is a linear map of vector spaces. The image of ϕ_r may not be a variety, although we know it to be a constructible subset of S_d.

We list the F_r to be used in the sequel:

(4)

$$
\begin{aligned}
F_1 &= L_0^{d-1}L_1, \quad F_2 = L_0^{d-2}(L_0L_3 + L_1L_2),\\
F_3 &= L_0^{d-3}(L_0^2L_7 + L_0(L_1L_6 + L_2L_5 + L_3L_4) + L_1L_2L_4) \quad \text{for } d \geq 3,\\
F_4 &= L_0^{d-4}(L_0^3L_{15} + 2L_0^2(L_1L_{14} + L_2L_{13} + L_3L_{12} + L_4L_{11} + L_5L_{10}\\
&\qquad + L_6L_9 + L_7L_8) + L_0(L_1L_6L_8 + L_2L_5L_8 + L_3L_4L_8 + L_1L_2L_{12}\\
&\qquad + L_1L_4L_{10} + L_2L_4L_9) + L_1L_2L_4L_8) \quad \text{for } d \geq 4,\\
F_4 &= L_0^2L_{15} + L_0(L_1L_{14} + L_2L_{13} + L_3L_{12} + L_4L_{11} + L_5L_{10} + L_6L_9\\
&\qquad + L_7L_8) + (L_1L_6L_8 + L_2L_5L_8 + L_3L_4L_8 + L_1L_2L_{12} + L_1L_4L_{10}\\
&\qquad + L_2L_4L_9) \quad \text{for } d = 3, \quad \text{and}\\
F_r &= \sum_{j=0}^{2^{r-1}-1} L_jL_{(2^r-j-1)} \quad \text{for } d = 2.
\end{aligned}
$$

The last formula follows from (2) by a simple induction.

Wherever possible, we have absorbed purely numerical factors such as $d-1, d-2$ etc. by rescaling the L_i. For this and for the generic smoothness argument, we need the characteristic to be zero.

2. Low dimensional cases. In this section we treat a medley of cases, all distinguished by the fact that one of the numbers n, d or r is small enough to make a direct geometric argument feasible.

Proposition 2.1. *For n arbitrary,* $\dim T^{(1)} = 2n$.

Proof. The morphism $\phi_1 : (L_0, L_1) \to L_0^{d-1}L_1$ has one-dimensional fibers, hence dim $CT^{(1)} = 2n+1$. $\square$

Consequently $T^{(0)}$ has no tangential deficiency.

Proposition 2.2.

1. *For $d \geq 3$ and n arbitrary,* $\dim T^{(2)} = 4n-1$.
2. *For $d = 2$, $n \geq 2$,* $\dim T^{(2)} = 4n-3$.

In particular, $T^{(1)}$ is always tangentially deficient.

Proof.

1. Here we assume $n \geq 2$; the case $n = 1$ is covered by Theorem 2.4. Write $F_2 = L_0^{d-2}(L_0L_3 + L_1L_2) = L_0^{d-2}Q$.

Firstly assume $n = 2$. By a change of variables, we may let $L_0 = x_0$. Any quadratic form in x_0, x_1, x_2 may be written as $x_0L_3 + L_1L_2$, where the L_i are linear forms. We have shown that for a fixed L_0 the expression Q can be made to represent a general plane conic. Since projectively L_0 depends on two parameters and Q on five, $\dim T^{(2)} = 7$.

Now assume $n \geq 3$. By choosing the L_i generally, we may assume that rank $(Q) = 4$. Firstly assume $n = 3$, this implies that Q is smooth. Since the plane $L_0 = 0$ intersects Q in the pair of lines $L_0 = L_1 = 0$ and $L_0 = L_2 = 0$, it must be tangent to Q. Alternately if Q' is any smooth quadric tangent to L_0, then its equation can be written in the form $L_0M_3 + M_1M_2$. Define an incidence correspondence

$$\Sigma \subseteq \mathbf{P}S_1 \times \mathbf{P}S_2, \quad \Sigma = \{(L, Q) : Q \text{ is smooth and } L \text{ is tangent to } Q\}.$$

The general fiber of the projection $\Sigma \to \mathbf{P}S_2$ is two-dimensional, so $\dim\, \Sigma = 11$. Hence the general fiber of the projection $\Sigma \to \mathbf{P}S_1$ is eight-dimensional. We have shown that, for a fixed L_0, the factor $L_0L_3 + L_1L_2$ projectively depends upon eight parameters. Since L_0 projectively depends on 3 parameters, $\dim T^{(2)} = 11$.

Now let $n \geq 4$. Then Q is a cone over a smooth quadric $\widetilde{Q} \subseteq \mathbf{P}^3$ with vertex $\Lambda \cong \mathbf{P}^{n-4}$. (The geometry of Q determines Λ uniquely but not $\widetilde{Q}$.) It follows that each of the $(n-2)$-dimensional linear spaces $L_0 = L_1 = 0$ and $L_0 = L_2 = 0$ is the linear span of Λ and a line on $\widetilde{Q}$. Moreover these two lines must intersect, since the space $L_0 = L_1 = L_2 = 0$ is strictly larger than Λ. Hence $L_0 = 0$ is the span of Λ and the plane in $\mathbf{P}^3$ containing the two lines. Alternately, given complementary spaces Λ, Λ' of dimensions $n-4$ and 3 respectively and a smooth quadric Q' in Λ' such that L_0 is the linear span of one of its tangent planes and Λ, the equation of Q' can be brought into the form $L_0M_3 + M_1M_2$. (Choose coordinates such that $\Lambda' : x_0 = x_1 = \cdots = x_{n-4} = 0$, $\Lambda : x_{n-3} = \cdots = x_n = 0$ and $L_0 = x_n$. Then Q' is defined by a homogeneous quadratic form $f_2(x_{n-3}, x_{n-2}, \ldots, x_n)$, which must factor when x_n is set to zero. Now the assertion is clear.)

Let Y be the space of rank 4 quadrics in $\mathbf{P}^n$ and set up an incidence correspondence $\Sigma \subseteq \mathbf{P}S_1 \times Y$,

$$\Sigma = \{(L, Q) : L \text{ is the linear span of a tangent plane to } \widetilde{Q} \text{ and the vertex of } Q\}.$$

(Although $\widetilde{Q}$ is not uniquely determined by Q, the condition on L is independent of this choice.) We have $\dim Y = 4n-3$ (see [**4**, Example 22.31]) and the general fiber of $\Sigma \to Y$ is two-dimensional. Hence $\dim \Sigma = 4n-1$ and the general fiber of $\Sigma \to \mathbf{P}S_1$ is $(3n-1)$-dimensional. It follows that projectively the factor $L_0L_3 + L_1L_2$ depends on $3n-1$ moduli and L_0 on n. Hence $\dim T^{(2)} = 4n-1$.

2. This will be a special case of the next proposition. □

Proposition 2.3. *Let $d = 2$ and $n \geq 2$. Then*

$$\dim T^{(r)} = 2^r n - (2^{2r-1} - 3.2^{r-1} + 1) \quad \textit{for } n > 2^r - 1,$$

and N otherwise.

For $r = 2$, this reduces to $4n-3$ for $n \geq 2$.

Proof. From the last expression in (4) it is immediate that the cone over $T^{(r)}$ is exactly the set of quadratic forms in $n+1$ variables of rank at most 2^r. Their codimension in S_2 is $[(n+2-2^r)(n+1-2^r)]/2$, [**4**, Example 22.31], hence the assertion. □

The case $n = 1$, i.e., that of a rational normal curve, can be analyzed completely.

Theorem 2.4. *For $n = 1$ and $r < d$, $\dim T^{(r)} = r+1$ and $\deg T^{(r)} = (d-r)(r+1)$.*

Proof. We may write a typical element of $T^{(r)}$ as $L_0^{d-r} G_r$, where G_r is a degree r homogeneous form in two variables. Hence projectively G_r depends on at most r moduli and L_0 on one, so $\dim T^{(r)} \leq r+1$.

We claim that $T^{(r-1)} \subsetneq T^{(r)}$. If not, then $T^{(r-1)}$ would be a proper linear space containing $T^{(0)}$, contradicting the nondegeneracy of the rational normal curve. Now it follows by descending induction on r that $\dim T^{(r)} = r+1$.

For the second part, we identify $\mathbf{P}^d$ with $\mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$. Said differently, we will regard the points of $\mathbf{P}^d$ as members of the canonical g_d^d on $\mathbf{P}^1$, and the rational normal curve with divisors of the form $dP, P \in \mathbf{P}^1$.

Now intersect $T^{(r)}$ with a general $(d-r-1)$-dimensional linear space Λ in $\mathbf{P}^d$. The points on Λ correspond to members of a general subsystem $g_d^{d-r-1} \subseteq g_d^d$. By the argument above, the points of $\Lambda \cap T^{(r)}$ correspond to divisors of the form $(d-r)P + P_{d-r+1} + \cdots + P_d$ in this g_d^{d-r-1}. By De Jonquières's formula (see [**1**, p. 359]) the number of such divisors is the coefficient of $t_1 t_2^r$ in $(1 + (d-r)t_1 + t_2)^{r+1}$, which is $(d-r)(r+1)$. Hence the claim. $\square$

Thus for $n = 1$ the variety $T^{(r)}$ is better described as the set of univariate degree d polynomials having a root of multiplicity at least $d-r$. The hypersurface $T^{(d-2)}$ has degree $2(d-1)$, as it should, since it is defined by the discriminant.

We turn to the cases $r = 3, 4$. The method used is resolutely more algebraic. In each case the problem reduces to calculating the minimal resolution of a certain ideal in a polynomial ring; this is done using Macaulay-2. It would be of some interest to know if the machine computations can be circumvented by more conceptual arguments.

3. Case $r = 3$.

We record the following technical lemma for use in Theorems 3.2 and 4.1.

Lemma 3.1 (Change of rings). *Let $A = k[x_0, \ldots, x_n]$ and $I < A$ a homogeneous ideal such that* proj-dim$_A(A/I) = m < n+1$. *Let*

$$E^\bullet : 0 \longrightarrow E^{-m} \longrightarrow \cdots \longrightarrow E^0(\simeq A) \longrightarrow A/I \longrightarrow 0$$

be its graded minimal resolution. For an integer $p \geq m-1$, let $B = k[y_0, \ldots, y_p]$. Choose general *linear forms $l_0, \ldots, l_n \in B_1$ and define a ring map*

$$f : A \longrightarrow B, \quad x_i \longrightarrow l_i.$$

Then $E^\bullet \otimes_A B$ is a minimal resolution of $B/f(I)$.

Proof.

Case $p < n$. Since the l_i are general, f is surjective and $J = \ker f$ is generated by linear forms $z_1, \dots, z_{n-p} \in A_1$. Define a free module $M = \oplus_{i=1}^{n-p} A e_i$, and set $z = \Sigma z_i e_i$. We have a Koszul resolution

$$\begin{aligned} K_z : 0 \longrightarrow A \longrightarrow \cdots \longrightarrow \wedge^j M \xrightarrow{\wedge z} \wedge^{j+1} M \\ \longrightarrow \cdots \longrightarrow \wedge^{n-p} M \longrightarrow A/J \longrightarrow 0. \end{aligned}$$

By the Auslander-Buchsbaum, $\operatorname{depth}_A(A/I) = n+1-m \geq n-p$; and by genericity we may assume that the (z_i) are an A/I-regular sequence. Hence $H^j(K_z \otimes A/I) = \operatorname{Tor}_j(A/I, A/J) = 0$ for $-(n-p) \leq j \leq -1$. So $E^\bullet \otimes_A B$ is a resolution of $A/(I+J)$. Since all differentials vanish modulo the irrelevant maximal ideal of B, it is minimal.

Case $p \geq n$. In this case f is injective and B is a polynomial algebra over $f(A)$, so the claim is obvious. □

Theorem 3.2. *Let* $n, d \geq 3$. *Then* $\dim T^{(3)} = 8n - 5$.

Proof. From (4),

$$F_3 = L_0^{d-3}(\underbrace{L_0^2 L_7 + L_0(L_1L_6 + L_2L_5 + L_3L_4) + L_1L_2L_4}_{(\star)}).$$

The calculation splits into two cases.

Case $d \geq 4$. We claim that a general point in $\operatorname{im}(\phi_3)$ determines L_0 uniquely up to a scalar. It suffices to show that the polynomial $(\star)$ is irreducible for general choices of L_i. Since irreducibility is an open condition, exhibiting one choice would do. Take $L_0 = L_7 = x_0$, $L_1 = x_1$, $L_2 = x_2$, $L_4 = x_3$ and $L_3 = L_5 = L_6 = 0$, then $(\star) = x_0^3 + x_1x_2x_3$.

Now fix an L_0 and consider the map $(S_1)^7 \xrightarrow{\phi_{L_0}} S_3$, $(L_i) \longrightarrow (\star)$. By virtue of the claim, $\dim T^{(3)} = \dim \operatorname{im}(\phi_{L_0}) + n - 1$. To calculate the dimension of $\mathrm{im}(\phi_{L_0})$ it suffices to calculate the rank of the map $d\phi_{L_0}$ on tangent spaces at a general point of $(S_1)^7$. Polarizing the expression $(\star)$, one sees that $d\phi_{L_0}$ is given by

$$\begin{aligned}(M_1,\dots,M_7) \longrightarrow\ & M_1(L_0L_6+L_2L_4)+M_2(L_0L_5+L_1L_4)\\ &+M_3(L_0L_4)+M_4(L_0L_3+L_1L_2)\\ &+M_5(L_0L_2)+M_6(L_0L_1)+M_7(L_0^2).\end{aligned}$$

An element of $\ker d\phi_{L_0}$ corresponds to a linear syzygy between the multipliers of the M_i terms. Hence the rank of $d\phi_{L_0}$ is $7(n+1)$ minus the number of such linearly independent syzygies.

Now tentatively assume $n=6$; then we may as well assume $L_0,\dots,L_6$ to be the indeterminates $x_0,\dots x_6$, respectively. (Note that L_7 does not appear in the expression for $d\phi_{L_0}$.)

Consider the ideal in S manufactured out of the multipliers of the M_i terms:

$$I=(x_0x_6+x_2x_4,x_0x_5+x_1x_4,x_0x_4,x_0x_3+x_1x_2,x_0x_2,x_0x_1,x_0^2).$$

In the sequel, this will be called the *deformation ideal*. We have a minimal resolution of S/I:

$$\begin{aligned}0\longrightarrow S(-5)\longrightarrow S(-4)^6\longrightarrow S(-3)^{11}&\\ \longrightarrow S(-2)^7\longrightarrow S\longrightarrow S/I\longrightarrow 0.&\end{aligned}\tag{5}$$

The number of linear syzygies is 11 and $\text{proj-dim}_S(S/I)=4$.

Now let $n\ge 3$, and $L_0,\dots,L_7$ be general linear forms in $S'=k[y_0,\dots,y_n]$. Define $S\xrightarrow{f}S'$ by $x_i\to L_i$. By the lemma on change of rings, the resolution above tensored with S' is a resolution of $S'/f(I)$. This is to say that, as long as the L_i are general, although we may no longer assume them to be variables, the number of linear syzygies between the multipliers of the M_i is still 11.

Hence generically $\dim d\phi_{L_0}$ has rank $7(n+1)-11=7n-4$. Since L_0 projectively depends on n parameters, $\dim CT^{(3)}=7n-4+n=8n-4$. So $\dim T^{(3)}=8n-5$.

Case $d=3$. The argument is parallel to the previous case, except that L_0 no longer has any privileged status.

Start with the map

$$\phi_3 : (S_1)^8 \longrightarrow S_3, \quad (L_0, \dots, L_7) \longrightarrow F_3(L_i).$$

The map on tangent spaces is given by

$(M_0, \dots, M_7) \xrightarrow{d\phi_3} M_0(L_0L_7 + L_1L_6 + L_2L_5 + L_3L_4) + M_1(L_0L_6 + L_2L_4) + M_2(L_0L_5 + L_1L_4) + M_3(L_0L_4) + M_4(L_0L_3 + L_1L_2) + M_5(L_0L_2) + M_6(L_0L_1) + M_7(L_0^2)$.

Assume $n = 7$ and $(L_0, \dots, L_7) = (x_0, \dots, x_7)$. The deformation ideal is

$$I = (x_0x_7 + x_1x_6 + x_3x_4 + x_2x_5, x_0x_6 + x_2x_4, x_0x_5 + x_1x_4, x_0x_4, \\ x_0x_3 + x_1x_2, x_0x_2, x_0x_1, x_0^2) < S,$$

with minimal resolution

$$0 \longrightarrow S(-6)^2 \oplus S(-5) \longrightarrow S(-5)^5 \oplus S(-4)^6 \\ \longrightarrow S(-4)^3 \oplus S(-3)^{12} \longrightarrow S(-2)^8 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

There are 12 linear syzygies and proj-dim $S/I = 4$. By the lemma on change of rings, the number of syzygies is still 12 for all $n \geq 3$. Generically $d\phi_3$ has rank $8(n+1) - 12 = 8n - 4$, so $\dim T^{(3)} = 8n - 5$. The proof of Theorem 3.2 is complete. $\square$

We will redo the case $d = 4, n = 3$ where the geometry is most visible. We have an imbedding $\mathbf{P}^3 \longrightarrow \mathbf{P}^{34}$ given by the complete linear system of quartic hypersurfaces. From the expression for F_3 it is clear that a general element of $\mathrm{im}(\phi_3)$ is of the form $H \cup X$, where the hyperplane H intersects the cubic surface X in three lines.

Claim. *The expression $L_0^2L_7 + L_0(L_1L_6 + L_2L_5 + L_3L_4) + L_1L_2L_4$ represents a general cubic surface.*

Indeed, let X be a smooth cubic surface in $\mathbf{P}^3$. Choose a set of three noncoincident intersecting lines on X; such always exist (see e.g. [**3**]), and we may assume that they lie in the plane $x_0 = 0$. Now write the equation of X as $x_0^2h_1 + x_0h_2 + h_3$; here h_2, h_3 are respectively quadratic and cubic forms in x_1, x_2, x_3. By hypothesis, h_3 factors as say $L_1L_2L_4$. Since the lines are not coincident, $\{L_1, L_2, L_4\}$ is a basis of the linear forms in x_1, x_2, x_3; hence one can choose L_3, L_5, L_6 so that $L_1L_6 + L_2L_5 + L_3L_4 = h_2$. Finally choose $L_0 = x_0, L_7 = h_1$. This proves the claim.

Hence X depends on 19 moduli, and it determines H up to a finite ambiguity (since X contains only 27 lines). Thus $\dim T^{(3)} = 19$.

Proposition 3.3. *Let $n = 2$. Then $\dim T^{(3)} = 9$ for $d = 3$ and 11 for $d \geq 4$.*

Write $F_3 = L_0^{d-3}G_3$. We claim that for a fixed L_0, G_3 represents a general plane cubic.

We may assume $L_0 = x_0$, and let f be a general cubic form in x_0, x_1, x_2. Write $f = x_0^2h_1 + x_0h_2 + h_3$ where h_i is a degree i form in x_1, x_2. Now h_3 factors completely as say, $L_1L_2L_4$. Then we can write $h_2 = (aL_1 + bL_2)(cL_1 + dL_2)$ for constants $a, \dots, d$. Now choose $L_6 = a(cL_1 + dL_2), L_5 = b(cL_1 + dL_2), L_3 = 0, L_7 = h_1$. The claim is proved.

For $d = 3$ we have an imbedding $\mathbf{P}^2 \longrightarrow \mathbf{P}^9$, hence in any case dim $T^{(3)} \leq 9$. Since ϕ_3 is dominant, equality must hold.

If $d \geq 4$, then a general plane cubic being irreducible, L_0 is uniquely determined for a general point of $\mathrm{im}(\phi_3)$. Since L_0 depends on two parameters and G_3 on nine, so $\dim T^{(3)} = 11$. $\square$

Case $r = 4$. The calculations are similar to the case $r = 3$, although rather more tedious.

Theorem 4.1. *If $n, d \geq 3$, then $\dim T^{(4)} = 16n - 17$, except possibly in the cases $d = 3$ and $4 \leq n \leq 9$.*

For $d = n = 3$, we have an imbedding $\mathbf{P}^3 \xrightarrow{v_3} \mathbf{P}^{19}$. Since already dim $T^{(3)} = 19$, dim $T^{(4)} = 19$ as well. The case $n = 2$ will be dealt with in Proposition 4.2. I have been unable to settle the cited exceptions.

Proof. We have a morphism

$$\phi_4 : (S_1)^{16} \longrightarrow S_d, \quad (L_0, \dots, L_{15}) \longrightarrow F_4(L_i),$$

where F_4 is as in (4).

Case $d \geq 5$. Write $F_4 = L_0^{d-4} G_4$. A general point of $\mathrm{im}(\phi_4)$ determines L_0 up to a scalar, for this it is enough to show that G_4 is irreducible for general choices of the L_i. Put $L_0 = L_{15} = x_0$, $L_1 = L_2 = L_4 = x_1$, $L_8 = x_2$ and the remaining $L_i = 0$. Then $G_4 = x_0^4 + x_1^3 x_2$ is irreducible, and hence it will remain so for small deformations of the L_i.

So then fix L_0 and consider the map

$$(S_1)^{15} \xrightarrow{\phi_{L_0}} S_4, \quad (L_1, \dots, L_{15}) \longrightarrow G_4(L_i).$$

Since L_0 projectively depends on n parameters,

$$\dim T^{(4)} = \dim CT^{(4)} - 1 = \dim\ \mathrm{im}(\phi_{L_0}) + n - 1.$$

As before $d\phi_{L_0}$ is found out by polarization. Assume $n = 15$ and $(L_0, \dots, L_{15}) = (x_0, \dots, x_{15})$. The deformation ideal is

(6)

$$\begin{aligned} I = x_0^2 x_{14} &+ x_0 x_6 x_8 + x_0 x_2 x_{12} + x_0 x_4 x_{10} + x_2 x_4 x_8, x_0^2 x_{13} + x_0 x_5 x_8 \\ &+ x_0 x_1 x_{12} + x_0 x_4 x_9 + x_1 x_4 x_8, x_0^2 x_{12} + x_0 x_4 x_8, x_0^2 x_{11} + x_0 x_3 x_8 \\ &+ x_0 x_1 x_{10} + x_0 x_2 x_9 + x_1 x_2 x_8, x_0^2 x_{10} + x_0 x_2 x_8, x_0^2 x_9 + x_0 x_1 x_8, x_0^2 x_8, \\ &+ x_0^2 x_7 + x_0 x_1 x_6 + x_0 x_3 x_4 + x_0 x_2 x_5 + x_1 x_2 x_4, x_0^2 x_6 + x_0 x_2 x_4, \\ &+ x_0^2 x_5 + x_0 x_1 x_4, x_0^2 x_4, x_0^2 x_3 + x_0 x_1 x_2, x_0^2 x_2, x_0^2 x_1, x_0^3), \end{aligned}$$

with minimal resolution

(7)
$$\begin{aligned} 0 \longrightarrow S(-7) \longrightarrow S(-6)^8 \longrightarrow S(-5)^{24} \longrightarrow S(-4)^{31} \\ \longrightarrow S(-3)^{15} \longrightarrow S \longrightarrow S/I \longrightarrow 0. \end{aligned}$$

Hence there are 31 linear syzygies and $\text{proj-dim}\, S/I = 5$. Arguing as in the case $r = 3$, we deduce that $\dim\ T^{(4)} = 16n - 17$ for $n \geq 4$.

Case $d \geq 4, n = 3$. We will resort to a direct geometric argument. The quartic surface defined by G_4 evidently contains the four coplanar lines $L_0 = L_j = 0$, $j = 1, 2, 4, 8$.

Tentatively define a *quartet* to be a configuration of four coplanar lines in $\mathbf{P}^3$, and let Y be the space of quartic surfaces containing a quartet. As seen previously, G_4 is uniquely determined up to scalar for a general point of $T^{(4)}$, hence we have a rational map $T^{(4)} - \xrightarrow{f} Y$.

Claim. *The equation of a general surface in Y can be written as $G_4(L_i) = 0$ for some forms L_i. That is to say, f is dominant.*

Proof. Take a quartet in the surface, we may assume that the four lines lie in the plane $x_0 = 0$. Write the equation of the surface as $x_0^3 g_1' + x_0^2 g_2' + x_0 g_3' + g_4' = 0$, where g_i' are degree i polynomials in x_1, x_2, x_3 for $i = 2, 3, 4$. By hypothesis g_4' factors as say $L_1 L_2 L_4 L_8$. By genericity, the sets $\{L_1L_2, L_1L_4, L_1L_8, L_2L_4, L_2L_8, L_4L_8\}$ and $\{L_1, L_2, L_4, L_8\}$ respectively span the vector spaces of all quadratic and linear forms in x_1, x_2, x_3. By the former we can choose $L_{12}, L_{10}, L_6, L_9, L_5$ and L_3 so that $g_3' = g_3(L_i)$. Then by the latter we choose $L_{14}, L_{13}, L_{11}, L_7$ so that $g_2' = g_2(L_i)$. Finally let $L_0 = x_0, L_{15} = g_1'$. The claim is proved. $\square$

Let Q be the 11-dimensional parameter space for quartets and consider the incidence correspondence $\Sigma \subseteq Q \times \mathbf{P}S_4$, $\Sigma = \{(\square, X) : \square \subseteq X\}$. Then Y is the image of the projection $\Sigma \xrightarrow{p_2} \mathbf{P}S_4$.

Claim. *The general fiber of the projection $\Sigma \xrightarrow{p_1} Q$ has dimension* 20. *Hence dim $\Sigma = 31$.*

Proof. A general quartet is a complete intersection of a plane and a degenerate quartic surface (consisting of four planes). So its ideal has a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}(-4) \longrightarrow I_\square \longrightarrow 0.$$

Hence $h^0(I_\square(4)) = h^0(\mathcal{O}_{\mathbf{P}^3}) + h^0(\mathcal{O}_{\mathbf{P}^3}(3)) = 21$. This proves the claim.

Now it is the case that any smooth quartic hypersurface contains at most 64 lines (see [**6**]) and *a fortiori* contains finitely many quartets. Hence the fiber of p_2 over a general point of Y is finite.

Since f is dominant and generically finite, $\dim T^{(4)} = \dim Y = 31$. The proof for the case $d \geq 5$ is complete.

Case $d = 4$.

Proof. The argument parallels the one for the case $r = 3, d = 3$. For the map $\phi_4 : (S_1)^{16} \xrightarrow{\phi_4} S_d$, we calculate the $d\phi_4$. The deformation ideal is

$$\begin{aligned} I' = I + (&x_0^2 x_{15} + 2x_0(x_1x_{14} + x_2x_{13} + x_3x_{12} + x_4x_{11} + x_5x_{10} + x_6x_9 \\ &+ x_7x_8) + x_1x_6x_8 + x_2x_5x_8 + x_3x_4x_8 + x_1x_2x_{12} + x_1x_4x_{10} \\ &+ x_2x_4x_9), \end{aligned}$$

where I is the ideal in (6).

From its minimal resolution

$$\begin{aligned} (8) \quad 0 \longrightarrow S(-7) \longrightarrow S(-8)^3 \oplus S(-6)^8 \longrightarrow S(-7)^7 \oplus S(-5)^{24} \\ \longrightarrow S(-6)^4 \oplus S(-4)^{32} \longrightarrow S(-3)^{16} \longrightarrow S \longrightarrow S/I' \longrightarrow 0, \end{aligned}$$

there are 32 linear syzygies and $\text{proj-dim}\, S/I = 5$. By the change of rings lemma, generically $d\phi_4$ has rank $16(n+1) - 32 = 16n - 16$ for $n \geq 4$. This implies that $\dim CT^{(4)} = 16n - 16$, so $\dim T^{(4)} = 16n - 17$ for $n \geq 4$.

Case $d = 3$. We will be brief since there is nothing new to the argument. The deformation ideal is

$$\begin{aligned} I = (&x_0x_{15} + x_1x_{14} + x_2x_{13} + x_3x_{12} + x_4x_{11} + x_5x_{10} + x_6x_9 + x_7x_8, \\ &x_0x_{14} + x_6x_8 + x_2x_{12} + x_4x_{10}, x_0x_{13} + x_5x_8 + x_1x_{12} + x_4x_9, \\ &x_0x_{12} + x_4x_8, x_0x_{11} + x_3x_8 + x_1x_{10} + x_2x_9, x_0x_{10} + x_2x_8, \\ &x_0x_9 + x_1x_8, x_0x_8, x_0x_7 + x_1x_6 + x_2x_5 + x_3x_4, x_0x_6 + x_2x_4, \\ &x_0x_5 + x_1x_4, x_0x_4, x_0x_3 + x_1x_2, x_0x_2, x_0x_1, x_0^2). \end{aligned}$$

Its minimal resolution begins as

$$\cdots \longrightarrow S(-6)^8 \oplus S(-4)^{32} \oplus S(-3)^{32} \longrightarrow S(-2)^{16} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

and has length 11. Hence dim $T^{(4)} = 16n - 17$ for $n \geq 10$.

This completes the proof of Theorem 4.1. □

Proposition 4.2. *Let* $n = 2$. *Then* $\dim T^{(4)} = 9$ *for* $d = 3$, 14 *for* $d = 4$*; and* 16 *for* $d \geq 5$.

The argument is very similar to Proposition 3.3.

Proof. The claim for $d = 3$ is clear from Proposition 3.3, so assume $d \geq 4$.

Write $F_4 = L_0^{d-4} G_4$. We claim that with L_0 fixed, G_4 represents a general plane quartic f. So assume $L_0 = x_0$ and let $f = x_0^3 h_1 + x_0^2 h_2 + x_0 h_3 + h_4$, where h_i is a form in x_1, x_2. Now h_4 factors completely, as say, $L_1 L_2 L_4 L_8$. By genericity, we can assume that $\{L_1, L_2\}$ are linearly independent, so we can find L_{13}, L_{14} such that $L_1 L_{14} + L_2 L_{13} = h_2$. Assuming the set $\{L_1 L_2, L_1 L_4, L_2 L_4\}$ to be independent, find L_9, L_{10}, L_{12} such that $L_1 L_2 L_{12} + L_1 L_4 L_{10} + L_2 L_4 L_9 = h_3$. Finally let $L_{15} = h_1, L_{3,5,6,7,11} = 0$. This proves the claim.

For $d = 4$ we have an imbedding $\mathbf{P}^2 \longrightarrow \mathbf{P}^{14}$. Since ϕ_4 is dominant, $T^{(4)} = \mathbf{P}^{14}$.

If $d \geq 5$, then a general plane quartic being irreducible, L_0 is unique up to scalars for a general point in ϕ_4. So dim $T^{(4)} = 16$. □

In conclusion, we have the following theorem.

Theorem 4.3 *Let* $v_d : \mathbf{P}^n \longrightarrow \mathbf{P}^N$ *be the Veronese imbedding and let* $\tau(n, d, r)$ *denote the dimension of* $T^{(r)}$. *Then*

(i) $\tau(n, d, 1) = 2n$.

(ii) $\tau(n, d \geq 3, 2) = 4n - 1$.

(iii) $\tau(n \geq 2, 2, 2) = 4n - 3$.

(iv) $\tau(n \geq 2, 2, r) = 2^r.n - (2^{2r-1} - 3.2^{r-1} + 1)$ *for* $n > 2^r - 1$ *and* N *otherwise.*

(v) $\tau(n \geq 2, 2, 2) = 4n - 3$.

(vi) $\tau(1, d, r < d) = r + 1$.

(vii) $\tau(n \geq 3, d \geq 3, 3) = 8n - 5$.

(viii) $\tau(2, 3, 3) = 9$ *and* $\tau(2, d \geq 4, 3) = 11$.

(ix) $\tau(n \geq 3, d \geq 3, 4) = 16n - 17$, *except possibly when* $d = 3$ *and* $4 \leq n \leq 9$.

The interest lies in cases (ii), (vii) and (ix). It appears from the calculations that for d large, $T^{(r)}$ has tangential deficiency $1, 3, 7$, respectively, for $r = 1, 2, 3$. One would like to see the following conjecture settled one way or the other:

Conjecture. *For* $n \geq 3$ *and* $d \gg 0$*, the variety* $T^{(r)}$ *has tangential deficiency* $2^r - 1$.

The deformation ideals considered here have no immediate geometric significance. It is curious however that their resolutions (5) and (7) are linear. It would be of interest to know if these are instances of a general pattern. It is probable that the question is tied up with the conjecture above.

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