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# TRANSCENDENTAL LIMIT CYCLES VIA THE STRUCTURE OF ARBITRARY DEGREE INVARIANT ALGEBRAIC CURVES OF POLYNOMIAL PLANAR VECTOR FIELDS

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ABSTRACT. In this paper we consider planar polynomial vector fields and we show certain structure that their invariant algebraic curves should have. This approach allows to obtain results on the nonexistence of those algebraic curves for arbitrary degree. As an application of this algorithmic method we easily prove that the van der Pol oscillator cannot have any algebraic solution and in particular neither is his limit cycle algebraic. In addition we show that a limit cycle studied by Dolov is not algebraic.

**1. Introduction.** Let us consider a *planar polynomial differential system* of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} = P(x,y) = \sum_{k=0}^{m} P_k(x,y)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y} = Q(x,y) = \sum_{k=0}^{m} Q_k(x,y)\,, \end{aligned} \tag{1}$$

in which $P$, $Q \in \mathbf{R}[x,y]$ are relative prime polynomials in the variables $x$ and $y$ and $P_k$ and $Q_k$ are homogeneous polynomials of degree $k$. Throughout this paper we will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1).

One interesting question to ask is whether some solution of system (1) is algebraic, i.e., can be described implicitly by $f(x,y) = 0$ where $f$ is a polynomial. In general, the answer is not easy but it is very interesting because it is known that the existence of *algebraic solutions* can be used to prove topological properties of system (1) as we will explain.

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In short, algebraic solutions and integrability have a narrow relationship for planar polynomial systems as is clearly shown in the Darboux theory. Darboux showed in his famous paper [**11**] how first integrals of polynomial systems possessing sufficient algebraic solutions are constructed. In particular, he proved that if a polynomial system of degree $m$ has at least $[m(m+1)/2]+1$ invariant algebraic curves, then it has a first integral. Darboux's first idea consists in searching a first integral for system (1) as a function of the form $\prod_{i=1}^{q} f_i^{\lambda_i}(x,y)$, for suitable $\lambda_i \in \mathbb{C}$ not all zero and $f_i(x,y)=0$ being invariant algebraic curves of system (1). The above first integral is called *Darboux first integral*.

These last years, several interesting results linking invariant algebraic curves and *Liapunov constants* have been published. For instance, Cozma and Şubă in [**10**] have proved that a *weak focus* of a polynomial system (1) of degree $m \geq 3$ having the first Liapunov constant zero and $m(m+1)/2-2$ invariant algebraic curves has a Darboux first integral or a Darboux integrating factor. Related with this result, Chavarriga, Giacomini and Giné [**6**] show that if a polynomial system (1) of degree $m$ with an arbitrary linear part has a *center* and admits $m(m+1)/2-[(m+1)/2]$ algebraic solutions, then this system has a Darboux integrating factor. Here [.] denotes as usual the integer part.

On the other hand, the study of *limit cycles* has also been an important topic in the theory of polynomial system (1) since Poincaré first treated it. A limit cycle is a periodic solution which has an annulus-like neighborhood in the *phase plane xy* free of other periodic solutions.

There is a relationship between the theory of limit cycles and algebraic solutions for planar polynomial differential equations as was suggested by Hilbert in the statement of his famous 16th problem, see [**18**], into two parts: (a) about the topology of real algebraic curves, (b) on the maximum number of limit cycles for system (1). This problem has proved to be one of the most difficult of Hilbert's entire collection; indeed, it remains unsolved even for *quadratic systems*, i.e., system (1) with $m=2$.

It is interesting to note that the knowledge of algebraic solutions allows us to understand better the behavior of limit cycles. See for example the classical proof of the fact that a quadratic system with two invariant straight lines does not have limit cycles due to Bautin in [**1**] by using the well known Bendixson-Dulac Criterion to ensure the nonexistence of limit cycles for planar differential systems. In addition a quadratic system with an invariant ellipse, hyperbola or pair of straight lines can have no limit cycles except, perhaps, for the ellipse itself [**9**].

A form of weakening the hypotheses of Hilbert's 16th problem is studying the *algebraic limit cycles*; i.e., algebraic curves $f(x, y) = 0$ which are particular solutions of (1) containing a real closed oval which is a limit cycle of (1). For quadratic systems, the existence of algebraic limit cycles have been studied by Qin Yuan-Xun [**22**] when the invariant curve is of second degree and by Yablonskii [**24**], Filiptsov [**17**], Shen Boqian [**23**] and Chavarriga [**4**] when the invariant curve is of degree four.

On the other hand, Evdokimenko [**14, 15**] and [**16**] has demonstrated the nonexistence of a cubic invariant algebraic curve as a limit cycle for quadratic systems.

The existence of invariant algebraic curves of differential systems is considered by several authors. Druzhkova [**13**] formulate in terms of the coefficients of the quadratic system necessary and sufficient conditions for the existence and uniqueness of an algebraic curve of second degree. Moreover, in [**7**] the authors obtain a complete affine classification of all quadratic systems having a third degree irreducible algebraic solution.

Obviously, if the vector field has a rational first integral then all their solutions are algebraic. However, only a few mathematicians have worked with nonalgebraicity. In this sense it is interesting to note the proof due to Odani [**20**] about the nonalgebraicity of the famous van der Pol limit cycle and the generalization into a family of polynomial Liénard systems. After this work, Zołądek in [**27**] almost completely solves the problem of algebraic invariant curves and algebraic limit cycles for polynomial Liénard systems of arbitrary degree. In general, to show the nonalgebraicity of all solutions of some system (1) is a very hard problem. For instance Jouanolou in [**19**] devotes a large section to showing that one particular system has no algebraic solutions. Other explicit examples of polynomial systems (1) without algebraic solutions are presented by Zołądek in [**26**].

In this context a natural question, due originally to Poincaré [**21**], is the following: *For a fixed degree $m \geq 2$, prove the existence of an upper bound $N(m)$ for the degrees of the irreducible algebraic solutions of all polynomial systems* (1) *of degree $m$ which do not have a rational first* *integral*. Related partial results exist to this question. For instance: Cerveau and Lins [**3**] have proved that if an algebraic solution only possesses *singularities of nodal type*, i.e., ordinary double points, then $N(m) \leq m + 2$; Carnicer [**2**] showing that if the system does not possesses dicritical singularities then $N(m) \leq m + 2$. Recall that a critical point is called *dicritical* if there are infinitely many invariant curves of the system passing through it. Finally, Chavarriga and Llibre [**5**] have proved that if the algebraic solution is not singular, i.e., it does not have multiple points, then in the generic case we have $N(m) \leq m$. Moreover they prove that in the maximal case for which $\deg f = m+1$ then the nonlinear polynomial system has a rational first integral of the form $H(x, y) = f(x, y)/L^{\deg f}(x, y)$ where $L(x, y) = 0$ is an invariant straight line.

But recently Christhopher and Llibre [**8**] have demonstrated that in general $N(m)$ does not exist. In short they have given a family of quadratic systems with irreducible invariant algebraic curves of arbitrarily high degree without rational first integral. This result really shows the difficulty of the problem.

The paper is organized as follows: In the second section we give the main algorithmic result of the article referring to the intrinsic structure of the arbitrary degree algebraic solutions of generic planar vector fields, see Theorem 3. In the third and fourth sections we apply the ideas of the above method to analyze in this context some examples such as the van der Pol limit cycle and a limit cycle studied by Dolov. In short we prove the nonalgebraicity of its solutions.

**2. The main result.** Let us suppose that system (1) has a trajectory (not a singular point) whose path in the phase plane is described implicitly by an algebraic curve $f(x, y) = 0$. It is clear that the derivative of $f$ with respect to time, along the orbits of system (1), should be annulled on the algebraic curve $f(x, y) = 0$. On the other hand, since this derivative is expressed as a polynomial in the variables $x$ and $y$, we are led directly to the following definition.

**Definition 1.** An *invariant algebraic curve* of system (1) is a set of points in $\mathbb{C}^2$ satisfying an equation $f(x, y) = 0$ where $f$ is a polynomial in $x$ and $y$ and such that

$$P\,\frac{\partial f}{\partial x}+Q\,\frac{\partial f}{\partial y}=Kf\,, \tag{2}$$

for some polynomial $K(x,y)$ of degree less than or equal to $m-1$, called cofactor.

**Definition 2.** Polynomial system (1) is degenerate at infinity if it verifies $xQ_m(x,y)-yP_m(x,y)\equiv 0$.

The name *degenerate infinity* is due to the fact that in the Poincaré compactification of (1) all the equator of $S^2$, i.e., the infinity, is filled up of critical points. In other words, the line at infinity ceases to be invariant for the foliation of the projective plane into phase curves.

Now we state the main theorem of this work.

**Theorem 3.** *Assume that polynomial system* (1) *without degenerate infinity possesses an invariant algebraic curve* $f(x,y)=0$ *of degree* $n$ *with associated cofactor* $K(x,y)$ *such that* $f(x,y)=\sum_{k=0}^{n} f_k(x,y)$ *and* $K(x,y)=\sum_{k=0}^{m-1} K_k(x,y)$ *are its developments in homogeneous components. Then the polynomial sequence* $\{\tilde f_i(u)\}$ *with* $i=n,n-1,\ldots,0$*, defined by* $\tilde f_i(u):=f_i(1,u)$*, is recursively obtained from*

$$\tilde f_i(u)=\frac{\int \Lambda_{m-1+i}(u)/\Gamma(u)\exp\left[\int \Gamma_i(u)/\Gamma(u)\,du\right]\,du+C_i}{\exp\left[\int \Gamma_i(u)/\Gamma(u)\,du\right]}\,, \tag{3}$$

*where* $C_i$ *are arbitrary real constants with* $C_n\neq 0$ *and*

$$\Gamma(u):=Q_m(1,u)-uP_m(1,u),\quad \Gamma_i(u):=iP_m(1,u)-K_{m-1}(1,u)\,, \tag{4}$$

*and*

$$\Lambda_{m-1+n}(u)\equiv 0\,, \tag{5}$$

(6)

$$\Lambda_{m-1+i}(u):=\sum_{k=0}^{n-1-i}{}'\Big(\left[uP_{m+i-n+k}(1,u)-Q_{m+i-n+k}(1,u)\right]\frac{d\tilde f_{n-k}(u)}{du}$$
$$+\left[K_{m-1+i-n+k}(1,u)-(n-k)P_{m+i-n+k}(1,u)\right]\tilde f_{n-k}(u)\Big)\,,$$

*where the dash in the previous sum should be understood in the following way: if the index of some term does not make sense then we take null that term.*

*Proof*. Since system (1) admits the invariant algebraic curve $f(x,y) = 0$ with cofactor $K(x,y)$, from Definition 1 the following equation is verified

$$\begin{aligned}\dot{f} &= \left(\sum_{k=0}^{m} P_k(x,y)\right)\frac{\partial}{\partial x}\left(\sum_{k=0}^{n} f_k(x,y)\right)+\left(\sum_{k=0}^{m} Q_k(x,y)\right)\frac{\partial}{\partial y}\left(\sum_{k=0}^{n} f_k(x,y)\right)\\ &= \left(\sum_{k=0}^{m-1} K_k(x,y)\right)\left(\sum_{k=0}^{n} f_k(x,y)\right),\end{aligned}$$

where $P_k$, $Q_k$, $f_k$ and $K_k$ are homogeneous polynomials of degree $k$.

Equaling the homogeneous polynomials of same degree in both members of the previous equation we have a sequence of linear partial differential equations for the homogeneous components $f_i(x,y)$ of the invariant algebraic curve $f = 0$. More concretely this procedure gives, for $i = n, n-1, \dots, 0$, the next sequence of linear partial differential equations

$$(7)\qquad P_m(x,y)\,\frac{\partial f_i}{\partial x} + Q_m(x,y)\,\frac{\partial f_i}{\partial y} - K_{m-1}(x,y) f_i = \Lambda_{m-1+i}(x,y)\,,$$

where the independent terms $\Lambda_{m-1+i}(x,y)$ are homogeneous polynomials of degree $m-1+i$ given by $\Lambda_{m-1+n} \equiv 0$ and

$$\begin{aligned}\Lambda_{m-1+i} = \sum_{k=0}^{n-1-i}{}' \Bigg( & K_{m-1+i-n+k} f_{n-k} - P_{m+i-n+k}\,\frac{\partial f_{n-k}}{\partial x}\\ & - Q_{m+i-n+k}\,\frac{\partial f_{n-k}}{\partial y}\Bigg)\end{aligned}$$

for $i = n-1, n-2, \dots, 0$. Here the dash in the previous sum should be understood as in the statement of the theorem.

Now, we make the change $(x,y) \to (w,u)$ where $w = x$ and $u = y/x$ into equations (7). Taking into account the homogeneity of the involved polynomials in such equations we have $P_m(w,uw) = w^m P_m(1,u)$,

$Q_m(w,uw) = w^m Q_m(1,u)$, $K_{m-1}(w,uw) = w^{m-1}K_{m-1}(1,u)$ and $\Lambda_{m-1+i}(w,uw) = w^{m-1+i}\Lambda_{m-1+i}(1,u)$. We define for sake of simplicity the function $\Lambda_{m-1+i}(u) := \Lambda_{m-1+i}(1,u)$. On the other hand, by the chain rule we have that $\partial/\partial x = \partial/\partial w - u/w\partial/\partial u$ and $\partial/\partial y = 1/w\partial/\partial u$. Hence in the new variables $(w,u)$, partial differential equations (7) become

$$wP_m(1,u)\,\frac{\partial f_i^*}{\partial w} + [Q_m(1,u) - uP_m(1,u)]\frac{\partial f_i^*}{\partial u} - K_{m-1}(1,u)f_i^* \\ = w^i\Lambda_{m-1+i}(u)\,,$$

where $f_i^*(w,u) := f_i(w,uw) = w^i f_i(1,u) = w^i\tilde{f}_i(u)$. In consequence $\partial f_i^*/\partial w = iw^{i-1}\tilde{f}_i(u)$, $\partial f_i^*/\partial u = w^i d\tilde{f}_i(u)/du$ and the above partial differential equations reduce to the following first order linear ordinary differential equations

$$\Gamma(u)\,\frac{d\tilde{f}_i(u)}{du} + \Gamma_i(u)\tilde{f}_i(u) = \Lambda_{m-1+i}(u)\,, \tag{8}$$

where the coefficients $\Gamma(u)$ and $\Gamma_i(u)$ for $i = n, n-1, \ldots, 0$ are given by (4) and the independent terms $\Lambda_{m-1+i}(u)$ by (5) and (6).

Provided that $\Gamma(u) \not\equiv 0$, i.e., if system (1) does not have degenerate infinity, the general solution of equation (8) adopts the form (3) where $C_i$ is an arbitrary real constant obtained due to the made quadrature.

Let us notice that in the case $i = n$, since $\Lambda_{m-1+n}(u) \equiv 0$ we have $\tilde{f}_n(u) = C_n \exp\left(-\int \Gamma_n(u)/\Gamma(u)\,du\right)$. Therefore $C_n \neq 0$ because $\tilde{f}_n(u) \not\equiv 0$ and the theorem is proved. □

In the proof of Theorem 3 the blow-up $(x,y) \to (x,u)$ is used where $u = y/x$. Once we have determined the sequence $\{\tilde{f}_i(u)\}_{i=1}^n$ then $f(x,y) = \sum_{k=0}^n x^k\tilde{f}_k(y/x)$.

*Remark* 1. An interesting case arises when system (1) verifies $\Gamma(u) \equiv 0$ or equivalently when it is degenerate infinity. Except for the Liénard polynomial vector fields of [**27**], as far as we know in the literature, nonlinear polynomial differential systems without algebraic solutions are only known for degenerate infinity systems, see [**19**] and [**26**]. This is just the opposite situation in which Theorem 3 can be applied in order to show polynomial vector fields without any algebraic solution.

*Remark* 2. Another special case in which we cannot obtain any nonalgebraic condition for system (1) is when $\Gamma(u)$ is a constant and $\Gamma_i(u) \equiv 0$ for all $i$. This is the case for instance of the open problem proposed by Zołądek in [**27**] related to the existence of algebraic limit cycles in Liénard polynomial systems $\dot{x} = y$, $\dot{y} = -f(x)y - g(x)$ with $\deg f = 1$ and $\deg g = 3$.

*Remark* 3. Let us notice that we have not made any hypothesis about the irreducibility of the polynomial $f$. Due to the fact that if a polynomial system (1) has an invariant algebraic curve $f = 0$, then $f^l = 0$ is also an invariant algebraic curve for all natural numbers $l$, we remark that the degrees of all invariant algebraic curves of a given polynomial system (1) are not bounded. But, in particular, if some expression $\tilde{f}_k(u)$ of (3) is not a polynomial, then we cannot only conclude that the system does not have any algebraic solution of degree greater than or equal to $k$ but rather we can say that such a system does not have any algebraic solution of degree $j$ with $j$ a divisor of $k$.

*Remark* 4. A simple corollary of Theorem 3 is the following one. Let $L(x, y)$ be a real or complex linear divisor of $f_n(x, y)$ with multiplicity $m_1$. Then $L$ is a divisor of $\Gamma(x, y) := xQ_m(x, y) - yP_m(x, y)$ with multiplicity $m_2$. Moreover $L$ is a divisor of $xK_{m-1} - nP_m$ and $yK_{m-1} - nQ_m$ with multiplicity $m_2 - 1$.

**3. The van der Pol's oscillator.** An important mathematical model introduced by Lord Rayleigh in 1883 and afterwards investigated by van der Pol more extensively when he studied the voltage in a triode circuit is given by the cubic system

$$\begin{aligned} \dot{x} &= P(x, y) = -y\,, \\ \dot{y} &= Q(x, y) = x - \varepsilon(x^2 - 1)y\,. \end{aligned} \tag{9}$$

It will be supposed that the parameter $\varepsilon \neq 0$ because otherwise equation (9) reduces to the harmonic oscillator being all their solutions are algebraic, that is given by concentric circles in the phase plane. In addition, it is well known that van der Pol equation (9) has a limit cycle for $\varepsilon \neq 0$.

This section is devoted to make, by using the ideas of Theorem 3, a shorter proof of the following Odani's result published in [**20**].

**Theorem 4** (Odani). *The van der Pol equation with $\varepsilon \neq 0$ does not have any algebraic solution. In particular, its limit cycle is not algebraic.*

*Proof.* Assume that van der Pol equation (9) has an invariant algebraic curve $f(x,y) = \sum_{i=0}^{n} f_i(x,y) = 0$ of degree $n$ with associated cofactor $K(x,y) = \sum_{i=0}^{2} K_i(x,y)$. Note firstly that system (9) is invariant with respect to the central symmetry $\sigma : (x,y) \to (-x,-y)$. Thus, we can assume that $f = 0$ is $\sigma$-invariant because otherwise we take $f \cdot \sigma^* f = 0$. It follows that the cofactor $K$ is also $\sigma$-invariant, i.e., $K = K_0 + K_2$.

Because $P_3 \partial f_n/\partial x + Q_3 \partial f_n/\partial y = K_2 f_n$ where $P_3 \equiv 0$ and $Q_3 = -\varepsilon x^2 y$, it is clear that $K_2 = -k\varepsilon x^2$ and $f_n = C_n x^{n-k} y^k$ for some $k \in \mathbf{N}$. We will assume $C_n = 1$. Introduce now the operator $\mathcal{L} = Q_3(x,y)\partial/\partial y - K_2(x,y) = -\varepsilon x^2 y \partial/\partial y + k\varepsilon x^2$. Then we get the recursive equations

$$\mathcal{L} f_j = \left( K_0 + y\,\frac{\partial}{\partial x} - (x + \varepsilon y)\,\frac{\partial}{\partial y} \right) f_{j+2}\,. \tag{10}$$

Since both members of (10) are homogeneous polynomials of degree $j+2$ and $\mathcal{L}$ acts on the monomials as $\mathcal{L} x^i y^j = (k-j)\varepsilon x^{i+2} y^j$ it follows that one necessary condition for the solvability of (10) is the vanishing of the coefficient before $x^{j-k+2} y^k$ in the expansion of the righthand side of (10). In particular, for $j = n-2$ we find

$$\mathcal{L} f_{n-2} = (K_0 - k\varepsilon) x^{n-k} y^k + (n-k) x^{n-k-1} y^{k+1} - k x^{n-k+1} y^{k-1}\,.$$

Thus $K_0 = k\varepsilon$ and, similar quadratures as in Theorem 3 gives $f_{n-2} = C_{n-2} x^{n-2-k} y^k + (k - n/\varepsilon) x^{n-k-3} y^{k+1} - (k/\varepsilon) x^{n-k-1} y^{k-1}$ for a constant $C_{n-2}$. At this point we see also that $n \geq 4$. Next, by (10) with $j = n-4$ we obtain

$$\mathcal{L} f_{n-4} = \frac{n}{\varepsilon}\, x^{n-2-k} y^k + \cdots\,,$$

where $\cdots$ means other monomials. Hence we have a contradiction for the solvability of $f_{n-4}$ because $n \neq 0$. $\square$

**4. Dolov's limit cycle is not algebraic.** In [**12**], Dolov considers the cubic system

$$\dot{x} = -y + x^2(-1 + \varepsilon + y) + y^2, \quad \dot{y} = x(1 - x^2)\,, \tag{11}$$

where $\varepsilon \in \mathbf{R}$. Notice that the phase portrait of Dolov's system is symmetric with respect to the $y$-axis. Moreover, in the semi-plane $x \geq 0$ it has for $\varepsilon = 0$ the following critical points: a center at $(0,0)$ due to their symmetry, two saddles at $(1,-1)$ and $(0,1)$ and a first order stable weak focus at $(1,1)$. Taking into account the direction of the vector field associated to (11) on the line segment that joins the points $(1,-1)$ and $(0,1)$ and by using the Dulac function

$$B(x,y) = \left(2y + (1-\sqrt{2})x^2 + 1 + \sqrt{2}\right)^{2/(\sqrt{2}-1)} \exp\left(\frac{y-1}{1-\sqrt{2}}\right)\,,$$

one can assert that there are no limit cycles for $x > 0$.

If in Dolov's system we take $0 < \varepsilon \ll 1$ then the focus $(1, \sqrt{1-\varepsilon})$ changes its stability and hence, by a Hopf's bifurcation, system (11) has a unique limit cycle around $(1, \sqrt{1-\varepsilon})$. We will call such a limit cycle *Dolov's limit cycle* and we have the following result.

**Theorem 5.** *Dolov's limit cycle* (11) *is not algebraic.*

*Proof.* Let us assume that the limit cycle of (11) is contained in a real oval of an invariant algebraic curve $f(x,y) = \sum_{i=0}^{n} f_i(x,y) = 0$ of degree $n$ with associated cofactor $K(x,y) = \sum_{i=0}^{2} K_i(x,y)$.

Let $\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$ with $P(x,y) = -y + x^2(-1+\varepsilon+y) + y^2$ and $Q(x,y) = x(1-x^2)$ be the vector field associated to (11). Since $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$ we observe that Dolov's system (11) is time-reversible, i.e., it is invariant with respect to the change $(x,y,t) \to (-x,y,-t)$. This implies that the axis $x = 0$ is a symmetry axis of their phase portrait. In consequence we can assume that the invariant curve $f(x,y) = 0$ is symmetric with respect to the variable $x$, that is, $f(-x,y) = f(x,y)$. Hence, from (2), the cofactor $K$ must be anti-symmetric with respect to $x$. This leads to $K(x,y) = ax + bxy$. With the notation of the second section, $\Gamma(x,y) := xQ_3(x,y) - yP_3(x,y) = -x^2(x^2+y^2)$, $xK_2(x,y) - nP_3(x,y) = (b-n)x^2y$

and $yK_2(x,y) - nQ_3(x,y) = x(nx^2 + by^2)$. So by Remark 4 and taking into account the symmetry of $f$ we have $f_n(x,y) = x^{2k}(x^2+y^2)^{\bar{n}}$ modulo a multiplicative constant and $K(x,y) = ax + 2kxy$. Of course $n = 2(k+\bar{n})$.

Before proving the existence results based on the ideas of Theorem 3, we pass to the classification of invariant algebraic curves of (11) according to their asymptotic behavior. After a suitable blowing-up of singular points at infinity, we obtain a resolved vector field with some new singular points and then we apply the theory of normal forms to study phase curves near each of these points. More concretely, Dolov's field (11) has three singularities at infinity, namely, $(0:1:0)$ and $(1:\pm i:0)$ in the complex projective plane $\mathbb{CP}^2$. We will study the point $(0:1:0)$. Using the chart $v = x/z$, $z = 1/y$, system (11) becomes

$$\begin{aligned}\dot{z} = zv(v^2 - z^2),\ \dot{v} &= v^2(1+v^2) + z(1+(\varepsilon-2)v^2) - z^2(1+v^2)\\ &= z + v^2 + \cdots .\end{aligned}$$

So, the point $(z,v) = (0,0)$ is a nilpotent singularity which needs further resolution. To do this we perform the change $z = wv^2$ which gives

$$\dot{v} = v(w+1) + \cdots , \quad \dot{w} = 2w(w+1) + v^3(1+\cdots),$$

and we observe a saddle-node singularity (of codimension 2) at the new singular point $(v,w) = (0,-1)$. As any other saddle-node this one has at most two analytic separatrices which are $v = 0$ (strong) and $w = -1 + O(v)$ (center, which may be only formal). In short the separatrix we are looking for is $w = z/v^2 = y/x^2 \approx -1$ and intersects the line at infinity with multiplicity 2. Therefore, it is clear from this analysis that, for system (11), at most one finite analytic separatrix can pass through the singular point $(0:1:0)$ and, if so, then it is of the form $x^2 + y + \cdots = 0$ as $x$, $y$ tend to infinity. This computation shows that, the highest degree homogeneous part $f_n$ of an invariant algebraic curve of (11) is $f_n(x,y) = x^{2k}(x^2+y^2)^{\bar{n}}$ with $k \in \{0,1\}$.

Using polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, we write the Dolov's field as follows

$$\begin{aligned}\mathcal{X} &= (1-x^2)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + [(\varepsilon-1)x^2 + y^2]\,\frac{\partial}{\partial x} \\ &= (1-x^2)\,\frac{\partial}{\partial\varphi} + (r^2+\delta x^2)\,\frac{\partial}{\partial x}\,,\end{aligned}$$

where $\delta := \varepsilon - 2$. Defining the differential operator $\mathcal{L} = -x^2\partial/\partial\varphi - 2kxy$, the next homogeneous component $f_{n-1}(x,y)$ of $f$ verifies the following equation

$$\mathcal{L} f_{n-1} = axf_n - (r^2+\delta x^2)\,\frac{\partial f_n}{\partial x} = r^{2(\bar{n}+k)+1}\Omega(\varphi)\,,$$

where $\Omega(\varphi) := -2k\cos^{2k-1}\varphi + (a-2\bar{n}-2k\delta)\cos^{2k+1}\varphi - 2\bar{n}\delta\cos^{2k-3}\varphi$. Since $f_{n-1}(x,y) = r^{n-1}\tilde{f}_{n-1}(\varphi) = r^{2(k+\bar{n})-1}\tilde{f}_{n-1}(\varphi)$ with $\tilde{f}_{n-1}$ a trigonometric polynomial, the above equation leads to the next linear ordinary differential equation

$$\frac{d\tilde{f}_{n-1}}{d\varphi} + 2k\,\frac{\sin\varphi}{\cos\varphi}\,\tilde{f}_{n-1} = -\frac{\Omega(\varphi)}{\cos^2\varphi}\,,$$

which solution is

$$\tilde{f}_{n-1}(\varphi) = \cos^{2k}\varphi\int\left[\frac{2k}{\cos^3\varphi} - \frac{a-2\bar{n}-2k\delta}{\cos\varphi} + 2\bar{n}\delta\cos\varphi\right]d\varphi\,.$$

Due to the fact that $d\,[\sin\varphi/\cos^2\varphi]/d\varphi = 2/\cos^3\varphi - 1/\cos\varphi$ and $\int 1/\cos\varphi\ d\varphi$ are not trigonometric, the necessary condition to get $\tilde{f}_{n-1}(\varphi)$ a trigonometrical polynomial is $a - 2\bar{n} - 2k\delta = k$, that is,

$$a = 2\bar{n} + (2\delta+1)k\,. \tag{12}$$

In short we have

$$f_{n-1}(x,y) = r^{2(\bar{n}-1)}x^{2(k-1)}y(kr^2 + 2\delta\bar{n}x^2)\,.$$

A new step gives that $f_{n-2}(x,y)$ satisfies the next equation

$$\begin{aligned}\mathcal{L} f_{n-2} &= axf_{n-1} - \frac{\partial f_n}{\partial\varphi} - (r^2+\delta x^2)\,\frac{\partial f_{n-1}}{\partial x} \\ &= r^{2(k+\bar{n})}\cos^{2k-3}\varphi\sin\varphi\Psi(\varphi)\,,\end{aligned}$$

where $\Psi(\varphi) := A + B\cos^2\varphi + C\cos^4\varphi + D\cos^6\varphi$ and $A, B, C, D$ are real constants. In particular, and due to (12), we have

$$C = 4\bar{n}\delta\,. \tag{13}$$

Taking $f_{n-2}(x,y) = r^{2(k+\bar{n})-2}\tilde{f}_{n-2}(\varphi)$ with $\tilde{f}_{n-2}$ a trigonometric polynomial, we get that $\tilde{f}_{n-2}(\varphi)$ verifies the linear ordinary differential equation of first order

$$\frac{d\tilde{f}_{n-2}}{d\varphi} + 2k\,\frac{\sin\varphi}{\cos\varphi}\,\tilde{f}_{n-1} = -\cos^{2k-5}\varphi\sin\varphi\Psi(\varphi)\,.$$

The solution is given by

$$\begin{aligned}\tilde{f}_{n-2}(\varphi) &= -\cos^{2k}\varphi\int\cos^{-5}\varphi\sin\varphi\Psi(\varphi)\,d\varphi\\ &= -\cos^{2k}\varphi\int\left[A\frac{\sin\varphi}{\cos^5\varphi}+B\frac{\sin\varphi}{\cos^3\varphi}+C\frac{\sin\varphi}{\cos\varphi}+D\sin\varphi\cos\varphi\right]d\varphi,\end{aligned}$$

and is a trigonometric polynomial if and only if $C = 0$, i.e., either $\bar{n} = 0$ or $\delta = 0$. The latter case $\delta = 0$ is away from the Hopf bifurcation, $2 < \delta \ll 3$, where the Dolov limit cycle is well established. Nevertheless one easily checks that, for $\delta = 0$, the only real singular points of (11) lie in the symmetry axis $x = 0$ and so there are no limit cycles.

In the case $\bar{n} = 0$ the degree $n$ of the invariant algebraic curve $f = 0$ must be $n = 2$ and in addition it is a parabola of the form $f(x,y) = x^2 + y + \text{cte} = 0$ which of course cannot contain any limit cycle. Hence the theorem is proved. $\square$

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