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SPACES OF λ-MULTIPLIER CONVERGENT SERIES

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ABSTRACT. In this paper, we introduce the quasi 0-gliding hump property of sequence spaces and study a series of elementary properties of spaces of λ-multiplier convergent series.

1. Introduction. Let (X,T) be a Hausdorff locally convex space, X^* the topological dual space of (X,T) and λ a scalar-valued sequence space. A series $\sum_j x_j$ in X is said to be λ-multiplier T-convergent if, for each $(t_j)\in\lambda$, there exists an $x\in X$ such that the series $\sum_{j=1}^\infty t_jx_j$ is T-convergent to x.

Let c_{00} be the scalar valued sequence space which are 0 eventually, the β-dual space of λ to be defined by: $\lambda^\beta=\{(u_j):\sum_j u_jt_j$ is convergence for each $(t_j)\in\lambda\}$. It is obvious that if $c_{00}\subseteq\lambda$, then $[\lambda,\lambda^\beta]$ is a dual pair with respect to the bilinear pairing $[\bar t,\bar u]=\sum_j u_jt_j$, where $\bar t=(t_j)\in\lambda,\ \bar u=(u_j)\in\lambda^\beta$. Let $\tau(\lambda,\lambda^\beta)$ denote the Mackey topology of λ with respect to the dual pair $[\lambda,\lambda^\beta]$, i.e., the topology of uniform convergent on all absolutely convex $\sigma(\lambda^\beta,\lambda)$-compact subsets of λ^β, and $k(\lambda,\lambda^\beta)$ the topology of uniform convergent on all $\sigma(\lambda^\beta,\lambda)$-compact subsets of λ^β. It is clear that $k(\lambda,\lambda^\beta)$ is stronger than $\tau(\lambda,\lambda^\beta)$.

Lemma 1 [**14**]. *Let $c_{00}\subseteq\lambda$ and τ_1 be a vector topology on λ^β such that τ_1 is stronger than the coordinate convergence topology. Then the following states are equivalent*:

(1) *$B\subseteq\lambda^\beta$ is τ_1-compact*;

(2) *$B\subseteq\lambda^\beta$ is τ_1-sequentially compact.*

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Lemma 2 [**17**]. *If (X, T_1) is a sequentially complete locally convex space and $\{x_i\} \subseteq X$ is a T_1 convergent sequence, then the absolutely convex closure of $\{x_i\}$ is a T_1-compact set and is also a T_1-sequentially compact set.*

It follows from Lemmas 1 and 2 easily that:

Lemma 3. *If $\sigma(\lambda^\beta, \lambda)$ is a sequentially complete space, then $k(\lambda, \lambda^\beta) = \tau(\lambda, \lambda^\beta)$.*

A nonzero sequence $\{\bar{t}^{(n)}\}$ in c_{00} is said to be a block sequence if there exists a strictly increasing sequence $\{k_n\}$ of integers with $k_0 = 0$ such that

$$\bar{t}^{(n)} = (0, 0, \dots, 0, t^{(n)}_{k_{n-1}+1}, \dots, t^{(n)}_{k_n}, 0, \dots).$$

The sequence space λ is said to have the signed-weak gliding hump property if, given any $\bar{t} = (t_i) \in \lambda$ and any block sequence $\{\bar{t}^{(n)}\}$ with $\bar{t} = \sum_{n=1}^\infty \bar{t}^{(n)}$ (pointwise sum), then each strictly increasing positive integer sequence $\{m_k\}$ has a further subsequence $\{n_k\}$ and a signed sequence $\{\theta_k\}$ with $\theta_k = 1$ or $\theta_k = -1$, $k \in \mathbf{N}$, such that $\bar{t} = \sum_{k=1}^\infty \theta_k \bar{t}^{(n_k)} \in \lambda$ (pointwise sum) [**3**].

The sequence space λ is said to have the strong gliding hump property if $\{\bar{t}^{(n)}\}$ is a bounded block sequence. Then, for each strictly increasing positive integers sequence, $\{m_k\}$ has a further subsequence $\{n_k\}$ such that $\bar{t} = \sum_{k=1}^\infty \bar{t}^{(n_k)} \in \lambda$ (pointwise sum) [**8**].

Let (λ, τ_0) be a topological vector space, (λ, τ_0) is said to be a K-space, if for each $j_0 \in \mathbf{N}$, the coordinate mapping I_{j_0} of λ to scalar field C, $I_{j_0}((t_j)) = t_{j_0}$ is continuous.

Let $c_{00} \subseteq \lambda$ and $\bar{t} = (t_i) \in \lambda$, denote $\bar{t}^{[n]} = (t_1, t_2, t_3, \dots, t_n, 0, \dots)$. If, for each $\bar{t} \in \lambda$, $\{\bar{t}^{[n]}\}_n$ converges to $\bar{t}$ with respect to the topology τ_0, then (λ, τ_0) is said to be an AK-space.

Let B be a bounded subset of (λ, τ_0), if $\{\bar{t}^{[n]} : \bar{t} \in B,\ n \in \mathbf{N}\}$ is also a bounded subset of (λ, τ_0). Then (λ, τ_0) is said to have the section uniform bounded property.

It is clear that if (λ, τ_0) is a K-space and has the section uniform bounded property, then for each bounded subset B of (λ, τ_0) and $j_0 \in \mathbf{N}$, $\sup\{|t_{j_0}| : (t_j) \in B\} < \infty$.

Now, we introduce the following quasi 0-gliding hump property:

The sequence space (λ, τ_0) is said to have the quasi 0-gliding hump property if, for each bounded block sequence $\{\bar{t}^{(n)}\}$ of (λ, τ_0) and each scalar sequence $\{s_n\}$ which converges to 0, then for each strictly increasing positive integers sequence $\{m_k\}$ has a further subsequence $\{n_k\}$ such that $\sum_{k=1}^{\infty} s_{n_k} \bar{t}^{(n_k)} \in \lambda$ (pointwise sum).

We would like to show that many classical sequence spaces have the quasi 0-gliding hump property:

Example 1. If $c_0 \subseteq S \subseteq l^\infty$, then $(S, ||.||_\infty)$ has the quasi 0-gliding hump property.

Example 2. For each $0 < p < \infty$, $(l^p, ||.||_p)$ has the quasi 0-gliding hump property.

In fact, for each bounded block sequence $\{\bar{t}^{(n)}\}$ of $(l^p, ||.||_p)$ and each scalar sequence $\{s_n\}$ which converges to 0, there exist $M > 0$ and a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $||t^{(n)}||_p \leq M$, $n \in \mathbf{N}$ and $\sum_k |s_{n_k}|^p < \infty$. Thus, $\sum_k s_{n_k} t^{(n_k)} \in l^p$. So $(l^p, ||.||_p)$ has the quasi 0-gliding hump property.

In this paper, the space $X(\lambda) = \{(x_j) :$ for every $(t_j) \in \lambda$, the series $\sum_j t_j x_j$ is T-convergence$\}$ is said to be the λ-multiplier convergent series space.

As we know, the study of the multiplier convergent series is an interesting topic in functional analysis [**2, 5, 7, 10, 13**–**16**]. When (X, T) is a Banach space and $\lambda = l^\infty$, Bu and Wu in [**4**] introduced and studied the bounded multiplier convergent series space $X(l^\infty)$; when (X, T) is a Banach space and $c_0 \subseteq S \subseteq l^\infty$, Aizpuru and Perez-Fernandez in [**1**] introduced and studied the S-multiplier convergent series space $X(S)$.

Now, if (X, T) is a locally convex space and λ has the quasi 0-gliding hump property, we study the λ-multiplier convergent series space $X(\lambda)$.

We obtain a series of elementary properties of the space $X(\lambda)$.

Let $\mathcal{B}$ be all bounded subsets of (λ, τ_0), and $\mathcal{P}$ be all continuous semi-norms of (X, T), for each $B \in \mathcal{B}$, $P \in \mathcal{P}$ and $\bar{x} \in X(\lambda)$, define

$$P_B(\bar{x}) = \sup\left\{P\left(\sum_{j=1}^{\infty} t_j x_j\right) : (t_j) \in B\right\}. \tag{1}$$

2. The uniform bounded principle on $X(\lambda)$.

Theorem 1. *If (λ, τ_0) is a K-space and has the section uniform bounded property and the quasi 0-gliding hump property, then for each $B \in \mathcal{B}$ and $P \in \mathcal{P}$, P_B is a semi-norm of $X(\lambda)$.*

Proof. We only need to prove that, for each $\bar{x} \in X(\lambda)$, $P_B(\bar{x}) < \infty$. If not, we can find an $\bar{x} \in X(\lambda)$ such that $P_B(\bar{x}) = \infty$. Thus, for each $M > 0$, there exists $(t_j) \in B$ such that $P(\sum_j t_j x_j) > M$. Let $M = 1 + 1$, we can pick $\bar{t}^{(1)} \in B$ such that $P(\sum_j t_j^{(1)} x_j) > 1 + 1$. Since the series $\sum_j t_j^{(1)} x_j$ is convergent, there exists a $j_1 \in \mathbf{N}$ such that $P(\sum_{j=j_1+1}^{\infty} t_j^{(1)} x_j) < 1$, so $P(\sum_{j=1}^{j_1} t_j^1 x_j) > 1$. Let $M = \sup\{P(\sum_{j=1}^{j_1} t_j x_j) : (t_j) \in B\} + 2^2 + 1$. Since (λ, τ_0) is a K-space and (λ, τ_0) has the section uniform bounded property, $M < \infty$. Furthermore, we can find a $(t_j^{(2)}) \in B$ such that $P(\sum_j t_j^{(2)} x_j) > M$, so $P(\sum_{j=j_1+1}^{\infty} t_j^{(2)} x_j) > 2^2 + 1$. Similarly, since the series $\sum_j t_j^{(2)} x_j$ is convergent, there exists a $j_2 \in \mathbf{N}$ such that $P(\sum_{j=j_1+1}^{j_2} t_j^{(2)} x_j) > 2^2$. Inductively, we can obtain a bounded block sequence $\{\bar{t}_0^{(n)}\}$ such that

$$P\left(\sum_{j}^{\infty} t_{0j}^{(n)} x_j\right) > n^2, \tag{2}$$

where $\bar{t}_0^{(1)} = (t_{0j}^{(1)}) = (t_1^{(1)}, t_2^{(1)}, \dots, t_{j_1}^{(1)}, 0, \dots)$, $t_0^{(2)} = (t_{0j}^{(2)}) = (0, \dots, 0, t_{j_1+1}^{(2)}, t_{j_1+2}^{(2)}, \dots, t_{j_2}^{(2)}, 0, \dots), \dots$. Let $s^{(n)} = (\bar{t}_0^{(n)})/n$, it follows from the quasi 0-gliding hump property of (λ, τ_0) that there exists a subsequence $\{s^{(n_k)}\}$ of $\{s^{(n)}\}$ such that $\sum_k s^{(n_k)} \in \lambda$ (pointwise convergent).

Note that (x_j) is λ-multiplier convergent, so we have

$$\lim_k P\bigg(\sum_{j=j_{k-1}}^{j_k} s_j^{(n_k)} x_j\bigg) = 0.$$

This contradicts (2) and so the theorem holds. $\square$

Similarly, we can prove the following:

Theorem 2. *If (λ, τ_0) is a K-space and has the section uniform bounded property and the quasi 0-gliding hump property, then for each bounded subset B of (λ, τ_0) and each $(u_j) \in \lambda^\beta$, $\sup\{|\sum_j u_j t_j| : (t_j) \in B\} < \infty$.*

Theorem 1 showed that if (λ, τ_0) is a K-space and has the section uniform bounded property and the quasi-0-gliding hump property, then $X(\lambda)$ equipped by the all semi-norms $\{P_B : B \in \mathcal{B},\ P \in \mathcal{P}\}$, is a locally convex Hausdorff space. We denote the locally convex topology of $X(\lambda)$ by $T_\mathcal{B}$.

Let $M(\lambda, X)$ denote the bounded linear operators mapping (λ, τ_0) to (X, T). Theorem 1 showed that for each $\bar{x} \in X(\lambda)$, $\bar{x} \in M(\lambda, X)$. Now we establish a uniform boundedness principle on $(X(\lambda), T_\mathcal{B})$. That is:

Theorem 3. *If $c_{00} \subseteq \lambda$, (λ, τ_0) is a K-space and has the section uniform bounded property and the quasi 0-gliding hump property, then $(X(\lambda), T_\mathcal{B})$ has the uniform boundedness property, i.e., if $\{\bar{x}^{(\alpha)} : \alpha \in \Lambda\} \subseteq X(\lambda)$ is pointwise bounded on λ, then $\{\bar{x}^{(\alpha)} : \alpha \in \Lambda\} \subseteq X(\lambda)$ is uniformly bounded on each bounded subset of (λ, τ_0), i.e., $\{\bar{x}^{(\alpha)} : \alpha \in \Lambda\}$ is $T_\mathcal{B}$-bounded.*

Proof. Without loss generality, we may assume $\{\bar{x}^{(\alpha)} : \alpha \in \Lambda\} \subseteq X(\lambda)$ is a sequence $\{\bar{x}^{(n)}\}$ of $X(\lambda)$.

If the conclusion is not true, there exists a $P \in \mathcal{P}$ and a $B \in \mathcal{B}$ such that

$$\sup\{P_B(\bar{x}^{(n)}) : n \in \mathbf{N}\} = \infty. \tag{3}$$

Thus, for each $M>0$, there exists an $n \in \mathbf{N}$ such that $P_B(\bar{x}^{(n)}) > M$. Let $M = 1+1$. We can pick a $\bar{x}^{(n_1)}$ such that $P_B(\bar{x}^{(n_1)}) > 1+1$. By the definition of P_B that there exists a $\bar{t}^{(1)} \in B$ and $P(\sum_j t_j^{(1)} x_j^{(n_1)}) > 1+1$. Since the series $\sum_j t_j^{(1)} x_j^{(n_1)}$ is convergent, there exists a $j_1 \in \mathbf{N}$ such that $P(\sum_{j=j_1+1}^{\infty} t_j^{(1)} x_j^{(n_1)}) < 1$, so $P(\sum_{j=1}^{j_1} t_j^{(1)} x_j^{(n_1)}) > 1$. Let $M = \sup\{P(\sum_{j=1}^{j_1} t_j x_j^{(n)}) : (t_j) \in B,\ n \in \mathbf{N}\} + \sum_{n=1}^{n_1} P_B(\bar{x}^{(n)}) + 2^2 + 1$. Note that since $c_{00} \subseteq \lambda$ and $\{\bar{x}^{(n)}\}$ is pointwise bounded on λ, for each $j \in \mathbf{N}$, $\{x_j^{(n)}\}_n$ is a bounded subset of (X, T). Thus, since (λ, τ_0) is a K-space and has the section uniform bounded property, Theorem 1 implies that $M < \infty$. Furthermore, we can find a $\bar{x}^{(n_2)}$ such that $P_B(\bar{x}^{(n_2)}) > M$. So there exists a $\bar{t}^{(2)} \in B$ such that $P(\sum_j t_j^{(2)} x_j^{(n_2)}) > M$. It follows from the definition of M that $n_2 > n_1$ and $P(\sum_{j=j_1}^{\infty} t_j^{(2)} x_j^{(n_2)}) > 2^2 + 1$. Since the series $\sum_j t_j^{(2)} x_j^{(n_2)}$ is convergent, there exists a $j_2 \in \mathbf{N}$ and $j_2 > j_1$ such that $P(\sum_{j=j_2+1}^{\infty} t_j^{(2)} x_j^{(n_2)}) < 1$, so $P(\sum_{j=j_1+1}^{j_2} t_j^{(2)} x_j^{(n_2)}) > 2^2$. Inductively, we can obtain a bounded block sequence $\{\bar{t}_0^{(k)}\}$ of (λ, τ_0) and a subsequence $\{\bar{x}^{(n_k)}\}$ of $\{\bar{x}^{(n)}\}$ such that

$$P\bigg(\sum_j t_{0j}^{(k)} x_j^{(n_k)}\bigg) > k^2, \quad k \in \mathbf{N}.$$

Let $\bar{s}^{(k)} = (\bar{t}_0^{(k)})/k$. Then we have:

$$P\bigg(\sum_j s_j^{(k)} x_j^{(n_k)}\bigg) > k, \quad k \in \mathbf{N}. \tag{4}$$

By the Hahn-Banach theorem we can obtain a sequence of continuous linear functionals $\{f_k\}$ of (X, T) such that $||f_k||_P = \sup\{|f_k(x)| : x \in X, P(x) \le 1\} \le 1$ and

$$f_k\bigg(\sum_j s_j^{(k)} x_j^{(n_k)}\bigg) = P\bigg(\sum_j s_j^{(k)} x_j^{(n_k)}\bigg) > k, \quad k \in \mathbf{N}. \tag{5}$$

That $\{f_k\}$ is an equicontinuous sequence is obvious. Now, we consider the infinite matrix $[(f_i)/i(\sum_j s_j^{(k)} x_j^{(n_i)})]_{ik}$. For each $k \in \mathbf{N}$, since

$\{\bar{x}^{(n)} : n \in \mathbf{N}\}$ is pointwise bounded and $\{f_k\}$ is an equicontinuous sequence,

$$\lim_i \frac{f_i}{i}\left(\sum_j s_j^{(k)} x_j^{(n_i)}\right) = 0$$

is obvious. If $\{k_p\}$ is an increasing sequence from $\mathbf{N}$, it follows from the quasi 0-gliding hump property of (λ, τ_0) that there exists a subsequence $\{k_{p_m}\}$ of $\{k_p\}$ such that $\sum_m \bar{s}^{(k_{p_m})} \in \lambda$. Noting that $\{\bar{x}^{(n)} : n \in \mathbf{N}\}$ is pointwise bounded and $\{f_k\}$ is an equicontinuous sequence, we have

$$\lim_i \frac{f_i}{i}\left(\sum_m \sum_j s_j^{(k_{p_m})} x_j^{(n_i)}\right) = 0.$$

From the basic matrix theorem of Antosik and Mikusinski [**9**], it follows that

$$\lim_k \frac{f_k}{k}\left(\sum_j s_j^{(k)} x_j^{(n_k)}\right) = 0.$$

This contradicts (5), and the theorem is proved. $\square$

Now we present an example to show the necessity of the gliding hump assumptions in Theorem 1 and Theorem 3.

Example 3. Let $\lambda = (c_{00}, ||.||_\infty)$ and C be the complex numbers field. Then λ is a K-space and has the section uniform bounded property, but λ does not have the quasi 0-gliding hump property. The λ-multiplier convergent series space $C(\lambda)$ is the space of all complex numbers sequences ω. Let $\bar{x} = (j)_{j=1}^\infty$ and e_j denote the sequence whose jth coordinate is 1 and other coordinates are 0. Then $\bar{x} \in C(\lambda)$ and $B = \{e_j : j \in \mathbf{N}\}$ is a bounded subset of $(c_{00}, ||.||_\infty)$. But $P_B(\bar{x}) = \infty$. This shows that Theorem 1 and Theorem 3 do not hold.

3. The completeness and Banach-Steinhaus property of $X(\lambda)$. At first, we study the sequentially completeness of $X(\lambda)$. We have:

Theorem 4. *If $c_{00} \subseteq \lambda$, (λ, τ_0) is a K-space and has the section uniform bounded property and the quasi* 0*-gliding hump property, and (X, T) is a sequentially complete Hausdorff space, then $(X(\lambda), T_{\mathcal{B}})$ is also a sequentially complete space.*

Proof. Let $\{\bar{x}^{(n)}\}$ be a $T_{\mathcal{B}}$-Cauchy sequence. It follows from the sequential completeness of (X, T) that there exists a $\bar{x}^{(0)} = (x_j^{(0)})$ satisfying $x_j^{(0)} = \lim_n x_j^{(n)}$ for each $j \in \mathbf{N}$. Now, we only need to prove that $\bar{x}^{(0)} = (x_j^{(0)}) \in X(\lambda)$. For arbitrary $\varepsilon > 0$ and $\bar{t} = (t_j) \in \lambda$, note that (λ, τ_0) has the section uniform bounded property, so $B = \{\bar{t}^{[l]} - \bar{t}^{[k]} : k, l \in \mathbf{N}\} \in \mathcal{B}$. Since $\{\bar{x}^{(n)}\}$ is a $T_{\mathcal{B}}$-Cauchy sequence, there exists $n_0 \in \mathbf{N}$ such that when $m, n \geq n_0$, for any $k, l \in \mathbf{N}$,

$$P\bigg(\sum_{j=k}^{l} t_j (x_j^{(m)} - x_j^{(n)})\bigg) < \frac{\varepsilon}{3}.$$

Since $\bar{x}^{(n_0)} \in X(\lambda)$, there exists $p_0 \in \mathbf{N}$ such that when $p, q \in \mathbf{N}$ and $p, q \geq p_0$,

$$P\bigg(\sum_{p}^{q} t_j x_j^{(n_0)}\bigg) < \frac{\varepsilon}{3}.$$

On the other hand, since $x_j^{(0)} = \lim_n x_j^{(n)}$ for each $j \in \mathbf{N}$, there exists $m_0 \in \mathbf{N}$ such that $m_0 > n_0$ and

$$P\bigg(\sum_{p}^{q} t_j (x_j^{(m_0)} - x_j^{(0)})\bigg) < \frac{\varepsilon}{3}.$$

So, when $p, q \geq p_0$, we have:

(6)

$$P\bigg(\sum_{p}^{q} t_j x_j^{(0)}\bigg) \leq P\bigg(\sum_{p}^{q} t_j (x_j^{(m_0)} - x_j^{(n_0)})\bigg) + P\bigg(\sum_{p}^{q} t_j (x_j^{(m_0)} - x_j^{(0)})\bigg) + P\bigg(\sum_{p}^{q} t_j x_j^{(n_0)}\bigg) \leq \varepsilon.$$

This shows that $\bar{x}^{(0)} = (x_j^{(0)}) \in X(\lambda)$. The theorem is proved. $\square$

Theorem 5. *Let $c_{00} \subseteq \lambda$, (λ, τ_0) be a K-space and have the section uniform bounded property and the quasi 0-gliding hump property, (X,T) a sequentially complete Hausdorff space. If $(\lambda, k(\lambda, \lambda^\beta))$ is an AK-space, then $(X(\lambda), \sigma(X(\lambda), \lambda))$ is sequentially complete, i.e., if $\{\bar{x}^{(n)}\} \subseteq X(\lambda)$ and, for each $\bar{t} = (t_j) \in \lambda$, $\{\sum_j t_j x_j^{(n)}\}_n$ is a Cauchy sequence of (X,T), then $\bar{x}^{(0)} = (x_j^{(0)}) \in X(\lambda)$, and $\{\bar{x}^{(n)}\}$ pointwise converges to $\bar{x}^{(0)} = (x_j^{(0)})$ on λ. Here $\bar{x}^{(0)} = (x_j^{(0)})$ is such that $x_j^{(0)} = \lim_n x_j^{(n)}$ for each $j \in \mathbf{N}$.*

Proof. It follows from (6) that we only need to prove that, for each $(t_j) \in \lambda$, $P \in \mathcal{P}$ and $\varepsilon > 0$, there exist k_0 and n_0 when $k, l \geq k_0$ and $m, n \geq n_0$,

$$P\bigg(\sum_{j=k}^{l} t_j (x_j^{(m)} - x_j^{(n)})\bigg) < \varepsilon.$$

If not, there exist strictly increasing positive integer sequences $\{k_q\}$, $\{l_q\}$, $\{m_q\}$, $\{n_q\}$, and $\varepsilon_0 > 0$, $P \in \mathcal{P}$ such that

$$P\bigg(\sum_{j=k_q}^{l_q} t_j (x_j^{(m_q)} - x_j^{(n_q)})\bigg) \geq \varepsilon_0.$$

By the Hahn-Banach theorem that we can obtain a sequence of continuous linear functionals $\{f_q\}$ of (X,T) such that

$$||f_q||_P = \sup\{|f_q(x)| : x \in X, P(x) \leq 1\} \leq 1,$$

and

$$(7) \quad f_q\bigg(\sum_{j=k_q}^{l_q} t_j (x_j^{(m_q)} - x_j^{(n_q)})\bigg) = P\bigg(\sum_{j=k_q}^{l_q} t_j (x_j^{(m_q)} - x_j^{(n_q)})\bigg) \geq \varepsilon_0.$$

For each $q \in \mathbf{N}$, let $\bar{z}^{(q)} = (z_j^{(q)}) = (x_j^{(m_q)} - x_j^{(n_q)})$. Then, by the condition of Theorem 5, for each $(t_j) \in \lambda$, $\lim_q \sum_j t_j z_j^{(q)} = 0$. Note that, for each $q \in \mathbf{N}$ and $(t_j) \in \lambda$, since the series $\sum_j t_j z_j^{(q)}$ is convergent in (X,T), the series $\sum_j t_j f_q(z_j^{(q)})$ is also convergent, so $(f_q(z_j^{(q)})) \in \lambda^\beta$. It follows from $||f_q||_P = \sup\{|f_q(x)| : x \in X, P(x) \leq 1\} \leq 1$ and $\lim_q \sum_j t_j z_j^{(q)} = 0$ that

$$\lim_q \left(\sum_j t_j f_q(z_j^{(q)}) \right) = 0.$$

So, $\{f_q(z_j^{(q)})\}_q \subseteq \lambda^\beta$ is a $\sigma(\lambda^\beta, \lambda)$-sequentially compact set. It follows from Lemma 1 that $\{f_q(z_j^{(q)})\}_q \subseteq \lambda^\beta$ is also a $\sigma(\lambda^\beta, \lambda)$-compact set. Since $(\lambda, k(\lambda, \lambda^\beta))$ is an AK-space,

$$\lim_q f_q\left(\sum_{j=k_q}^{l_q} t_j (x_j^{(m_q)} - x_j^{(n_q)}) \right) = \lim_q \sum_{j=k_q}^{l_q} t_j f_q(z_j^{(q)}) = 0.$$

This contradicts (7) and so the theorem is proved. □

We know that when λ has the signed-weak gliding hump property, $\sigma(\lambda^\beta, \lambda)$ is a sequentially complete space and $\tau(\lambda, \lambda^\beta)$ is an AK-space [**9**]. Thus, by Lemma 3 and Theorem 5 we have:

Corollary 1. *Let $c_{00} \subseteq \lambda$, (λ, τ_0) be a K-space and have the section uniform bounded property, the quasi-0-gliding hump property and the signed-weak gliding hump property, (X, T) be a sequentially complete Hausdorff space. Then $(X(\lambda), \sigma(X(\lambda), \lambda))$ is sequentially complete.*

Next, we study the Banach-Steinhaus property of $X(\lambda)$.

We will say that the sequence space (λ, τ_0) has the quasi Banach-Steinhaus property, if $\{u^{(n)}\} \subseteq \lambda^\beta$ is pointwise convergent to $u^{(0)} \in \lambda^\beta$ on λ, then for each $B \in \mathcal{B}$, $\{u^{(n)}\}$ converges to $u^{(0)}$ uniformly on B.

Let $(X, ||.||)$ be a normed space. We will say that X is a Grothendieck space if each weak* convergent sequence in X^* is weakly convergent [**1**].

Let $\mathcal{M}$ be a subspace of X^{**} such that $X \subseteq \mathcal{M} \subseteq X^{**}$. We will say that X is $\mathcal{M}$-Grothendieck if each weak* convergent sequence in X^* is $\sigma(X^*, \mathcal{M})$ convergent [**1**].

Example 4. If $c_0 \subseteq S \subseteq l^\infty$ and $(S, ||.||_\infty)$ is an l^∞-Grothendick space, then $(S, ||.||_\infty)$ has the quasi Banach-Steinhaus property.

In fact, it follows from [**1**] that $l^\infty \subseteq S^{**}$, so the condition that $(S, ||.||_\infty)$ is a l^∞-Grothendick space is meaningful. Note that $S^\beta = l^1$. Since $(S, ||.||_\infty)$ is an l^∞-Grothendick space, using the Schur lemma [**11**] it is easy to prove that $(S, ||.||_\infty)$ has the quasi Banach-Steinhaus property.

Theorem 6. *Let $c_{00} \subseteq \lambda$, (λ, τ_0) be a K-space and have the section uniform bounded property and the quasi 0-gliding hump property. If (λ, τ_0) has the quasi Banach-Steinhaus property and (X, T) is a sequentially complete Hausdorff space, then $(X(\lambda), T_{\mathcal{B}})$ has the Banach-Steinhaus property, i.e., if $\{\bar{x}^{(n)}\} \subseteq X(\lambda)$ and, for each $\bar{t} = (t_j) \in \lambda$, $\{\sum_j t_j x_j^{(n)}\}_n$ is a convergence sequence, then there exists an $\bar{x}^{(0)} = (x_j^{(0)}) \in X(\lambda)$ such that $\{\bar{x}^{(n)}\}$ is $T_{\mathcal{B}}$ converges to $\bar{x}^{(0)} = (x_j^{(0)})$.*

Proof. It follows from Theorem 4 that we only need to prove that $\{\bar{x}^{(n)}\} \subseteq X(\lambda)$ is a $T_{\mathcal{B}}$-Cauchy sequence. If not, there exist a $P \in \mathcal{P}$, a $B \in \mathcal{B}$, an $\varepsilon > 0$, and a strictly increasing sequence $\{n_k\} \subseteq \mathbf{N}$ such that

$$P_B(\bar{x}^{(n_k)} - \bar{x}^{(n_{k+1})}) \geq \varepsilon, \quad k \in \mathbf{N}.$$

So, there exists a sequence $(t_j^{(k)}) \in B$ such that

$$P\bigg(\sum_j t_j (x_j^{(n_k)} - x_j^{(n_{k+1})})\bigg) \geq \varepsilon, \quad k \in \mathbf{N}. \tag{8}$$

For each $k \in \mathbf{N}$, let $\bar{z}^{(k)} = (z_j^{(k)}) = (x_j^{(n_k)} - x_j^{(n_{k+1})})$. It is clear that $\bar{z}^{(k)} \in X(\lambda)$ and, for each $(t_j) \in \lambda$, $\lim_k \sum_j t_j z_j^{(k)} = 0$. By the Hahn-Banach theorem again we can obtain a sequence of continuous linear functionals $\{f_k\}$ of (X, T) such that $||f_k||_P = \sup\{|f_k(x)| : x \in X,\ P(x) \leq 1\} \leq 1$, and

$$f_k\bigg(\sum_j t_j^{(k)} z_j^{(k)}\bigg) \geq \varepsilon, \quad k \in \mathbf{N}. \tag{9}$$

Similarly, as in Theorem 5 for each $k \in \mathbf{N}$, $(f_k(z_j^{(k)})) \in \lambda^\beta$, and, for each $(t_j) \in \lambda$, it follows from $||f_k||_P = \sup\{|f_k(x)| : x \in X, P(x) \leq 1\} \leq 1$

and $\lim_k \sum_j t_j z_j^{(k)} = 0$ that

$$\lim_k \left(\sum_j t_j f_k(z_j^{(k)}) \right) = 0.$$

So, $\{f_k(z_j^{(k)})\}_k \subseteq \lambda^\beta$ is pointwise convergent to 0. Thus, by the quasi Banach-Steinhaus property of (λ, τ_0),

$$\lim_k \left(\sum_j t_j^{(k)} f_k(z_j^{(k)}) \right) = \lim_k f_k \left(\sum_j t_j^{(k)} z_j^{(k)} \right) = 0.$$

This contradicts (9) and so the theorem is true. $\square$

It follows from Examples 1 and 4 and Theorem 6 that:

Corollary 2 [**1**]. *If* $c_0 \subseteq S \subseteq l^\infty$ *and* $(S, ||.||_\infty)$ *is an* l^∞*-Grothendick space,* $(X, ||.||)$ *is a Banach space and, if* $\{\bar{x}^{(n)}\} \subseteq X(\lambda)$ *and, for each* $\bar{t} = (t_j) \in S$, $\{\sum_j t_j x_j^{(n)}\}_n$ *is a convergence sequence, then* $\{\bar{x}^{(n)}\}$ *is norm convergent to* $\bar{x}^{(0)} = (x_j^{(0)}) \in X(S)$*, where* $\bar{x}^{(0)} = (x_j^{(0)})$ *is such that* $x_j^{(0)} = \lim_n x_j^{(n)}$ *for each* $j \in \mathbf{N}$.

4. The uniform convergent property of $X(\lambda)$.

Finally, we study when $B \in \mathcal{B}$ and $(t_j) \in \lambda$, under what conditions the series $\sum_j t_j x_j$ converges uniformly with respect to $(t_j) \in B$.

The sequence space (λ, τ_0) is said to have the uniform convergent property if, for each $\sigma(\lambda^\beta, \lambda)$-sequentially compact subset F of λ^β and each $B \in \mathcal{B}$, the series $\sum_j u_j t_j$ converges uniformly with respect to $(u_j) \in F$ and $(t_j) \in B$.

Ronglu Li and Minhyung Cho in [**6**] proved the following important conclusion:

Lemma 4 [**6**, Theorem 1]. *If the sequence space* (λ, τ_0) *has the section uniform bounded property and the strong gliding hump property, then* (λ, τ_0) *has the uniform convergent property.*

Example 5. If $c_0 \subseteq S \subseteq l^\infty$ and $(S, ||.||_\infty)$ is an l^∞-Grothendick space, then $(S, ||.||_\infty)$ also has the uniform convergent property.

Theorem 7. *Let $c_{00} \subseteq \lambda$, (λ, τ_0) be a K-space and have the section uniform bounded property and the quasi 0-gliding hump property. If (λ, τ_0) has the uniform convergent property, then for each $\bar{x} = (x_j) \in X(\lambda)$ and $B \in \mathcal{B}$, the series $\sum_j t_j x_j$ converges uniformly with respect to $(t_j) \in B$.*

Proof. If not, there exist an $\varepsilon_0 > 0$, a $P \in \mathcal{P}$, a sequence $\{\bar{t}^{(k)}\} \subseteq B$ and two strictly increasing subsequences $\{j_k\}$ and $\{l_k\}$ of $\mathbf{N}$ satisfies that

$$P\bigg(\sum_{j=j_k}^{l_k} t_j^{(k)} x_j\bigg) \geq \varepsilon_0, \quad k \in \mathbf{N}.$$

By the Hahn-Banach theorem again we can obtain a sequence of equicontinuous continuous linear functional $\{f_k\}$ of (X, T) such that

$$f_k\bigg(\sum_{j=j_k}^{l_k} t_j^{(k)} x_j\bigg) \geq \varepsilon_0, \quad k \in \mathbf{N}. \tag{10}$$

Let A_1 be the $\sigma(X^*, X)$ closure of $\{f_k\}$. Then, by the famous Alaogue-Bourbaki theorem, A_1 is a $\sigma(X^*, X)$-compact subset of X^* [**12**]. Since $\bar{x} \in X(\lambda)$, for each $(t_j) \in \lambda$, the series $\sum_j t_j x_j$ is convergent. So for each $f \in X^*$, we have

$$f\bigg(\sum_j t_j x_j\bigg) = \sum_j t_j f(x_j).$$

Consider the linear operator $\bar{X} : X^* \to \lambda^\beta$ for $\bar{X}(f) = (f(x_j))_j$. It follows from $\bar{X}(f)(\bar{t}) = \sum_j t_j f(x_j)$ that the linear operator: $\bar{X} : X^* \to \lambda^\beta$ is $\sigma(X^*, X) - \sigma(\lambda^\beta, \lambda)$ continuous. So the image $\bar{X}(A_1)$ of A_1 is a compact subset of $(\lambda^\beta, \sigma(\lambda^\beta, \lambda))$. By Lemma 1, $\bar{X}(A_1)$ is also a sequentially compact subset of $(\lambda^\beta, \sigma(\lambda^\beta, \lambda))$. It follows from the uniform convergent property of (λ, τ_0) that the series $\sum_j t_j f_k(x_j)$ convergent uniformly with respect to $(t_j) \in B$ and $k \in \mathbf{N}$. This contradicts (10) and the theorem is proved. $\square$

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REFERENCES

1. A. Aizpuru and J. Perez-Fernandez, *Spaces of S-bounded multiplier convergent series*, Acta Math. Hungar. **87** (2000), 135–146.

2. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

3. J. Boos and T. Leiger, *The signed weak gliding hump property*, Acta Comn. Univ. Tart. **970** (1994), 13–22.

4. Qingying Bu and Congxin Wu, *Unconditionally convergent series of operators on Banach spaces*, J. Math. Anal. Appl. **207** (1997), 291–299.

5. Ronglu Li and Qingying Bu, *Locally convex spaces containing no Copy of c_0*, J. Math. Anal. Appl. **207** (1993), 205–211.

6. Ronglu Li and Minhyung Cho, *On the gliding hump property*, Taiwanese J. Math. **3** (1999), 115–122.

7. Ronglu Li, Chengri Cui and Minhyung Cho, *An invariant with respect to all admissible polar topologies*, Chinese Ann. Math. **19** (1998), 289–294.

8. C. Swartz, *The gliding hump property in vector sequence spaces*, Monatsh. Math. **116** (1993), 147–158.

9. ———, *Infinite matrices and the gliding hump*, World Scientific Publ., Singapore, 1996.

10. ———, *The Schur lemma for bounded multiplier convergent series*, Math. Ann. **263** (1983), 283–288.

11. ———, *A general Hahn-Schur theorem*, Southeast Asian Bull. Math. **20** (1996), 57–58.

12. A. Wilansky, *Modern methods in topological vector spaces*, McGraw-Hill, New York, 1978.

13. J. Wu and R. Li, *An Orlicz-Pettis theorem with applications to A-spaces*, Studia Sci. Math. Hungar. **35** (1999), 353–358.

14. Junde Wu and Shijie Lu, *A full invariant theorem and some applications*, J. Math. Anal. Appl. **270** (2002), 397–404.

15. ———, *An automatic adjoint theorem and its applications*, Proc. Amer. Math. Soc. **130** (2002), 1735–1741.

16. Junde Wu, Wenbo Qu and Chengri Cui, *On the invariant of λ-multiplier convergent series* **31** (2002), 279–283.

17. Junde Wu and Yajuan Wu, *The null sequences in mapping system*, J. Math. **18** (1998), 264–266.

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