ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 5, 2005

EXPANSIONS OF PRIME IDEALS

P. MALCOLMSON AND F. OKOH

ABSTRACT. When R is an integral domain and S a finitely generated extension of R we investigate the finiteness of the set of prime elements of R that become units in S and the finiteness of the set of prime ideals of R that expand to a given ideal of S. To this end we introduce the notions of GD(1) domains and GD(2) domains. An integral domain is a GD(1), respectively, GD(2), domain if every non-zero element in R is contained in only finitely many principal prime ideals, respectively, prime ideals. We determine when these properties are inherited by subrings, quotient rings, polynomial rings, and power series rings; in this respect GD(1) domains behave like unique factorization domains. A corollary is that if A is an affine (commutative) algebra over a field k, then any field between A and k is algebraic over k. This generalizes the Nullstellensatz. We show that extensions of unique factorization domains studied by Samuel, D.D. Anderson, D.F. Anderson, Zafrullah and many others are proper subclasses of the class of GD(1) domains.

1. Introduction. Let R be an integral domain, and let S be a finitely generated ring extension of R. Thus $S = R[X_1, \dots, X_n]/I$, where I is a constant-free ideal in the polynomial ring $R[X_1, \dots, X_n]$, i.e., $I \cap R = \{0\}$. Our first question is whether the set of (non-associated) prime elements in R that become units in S is finite. For instance, if $R = \mathbf{Z}$, the ring of integers, and $I = \langle p_1 p_2 \dots p_k X - 1 \rangle$, where $p_1, \dots, p_k$ are distinct primes, then only these primes become units in $\mathbf{Z}[X]/I$, while $\mathbf{Z}[X_1, \dots, X_n, \dots]/\langle nX_n - 1 : n = 1, 2, \dots \rangle$ is isomorphic to the field of rational numbers, $\mathbf{Q}$.

The canonical embedding of R into S results in a map

$$\phi : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$$

defined by $\phi(P) = P \cap R$ for all proper prime ideals P of S.

By [**12**, Lemma 6D(2)], $\phi(\operatorname{Spec} S)$ is dense in $\operatorname{Spec} R$. When R is a Dedekind domain (so that every nonzero ideal is contained in only a

Received by the editors on June 14, 2000, and in revised form on January 29, 2002.

finite number of prime ideals of R), we deduce from the denseness and Chevalley's theorem on constructible sets, [**12**, Theorem 6], that only finitely many prime ideals in R become the unit ideal in S.

In order to obtain similar conclusions for integral domains without any chain conditions, we introduce the notions of *robustness*, for $n = 1$, and $\aleph_0$-*robustness*, for n arbitrary, to specify that only a finite number of prime elements in R become units in S. We also introduce the concept of a GD(1) domain, in which every non-zero element lies in only finitely many principal prime ideals. In a Krull domain as defined in [**11**] every nonzero element is contained in only finitely many principal prime ideals. Thus Krull domains are GD(1) domains. The concept of a GD(1) domain also ties this paper to the many papers dealing with extensions of unique factorization domains; see [**1, 2**] for references. We show that GD(1) domains include all the extensions in [**1**] as proper classes.

The second question that we deal with generalizes the first. The setup is the same but this time we ask whether the equation

$$PR[X_1, \dots, X_n] + I = J$$

where $J \supseteq I$ is an ideal of $R[X_1, \dots, X_n]$ has only finitely many solutions, P. To deal with the question we introduce the notions of *super-robustness* (limiting the solutions) and GD(2) domains in which every non-zero element is contained in only finitely many prime ideals. In a Dedekind domain every nonzero element is contained in only a finite number of prime ideals; see [**9**, Theorem 3.16]. Hence, a Dedekind domain is GD(2). We give examples of GD(2) domains that are not Dedekind.

If P is a prime ideal in a GD(2) domain, then clearly R/P is also GD(2). We show that the analogous statement for GD(1) domains is false. While a polynomial ring over a GD(1) domain is shown to be GD(1), we prove that $R[X]$ is a GD(2) domain if and only if R is a field. Neither class is closed under subrings and power series.

We note that the hypotheses in all the results in this paper do not include any chain conditions on the integral domain R.

Notation.

- R stands for an integral domain.
- $R[X]$ is the polynomial ring in the variable X.
- I is a constant-free ideal in $R[X]$, i.e, $I \cap R = \{0\}$.

1. GD(1) domains and robustness. The most important lemma for this paper is a known result.

Lemma 1.1. *Let J be an ideal of $R[X]$ with $J \cap R = \{0\}$. Then there is a polynomial $f(X)$ in J and an element a in R such that for every $g(X)$ in J there is a positive integer m and a polynomial $q(X)$ in $R[X]$ with $a^m g(X) = f(X)q(X)$.*

Proof. If $J = \{0\}$ the statement is obvious, so assume $J \neq \{0\}$. Since $J \cap R = \{0\}$, we may choose a non-zero polynomial $f(X)$ in J with $\deg f(X) = n \geq 1$ as small as possible. The leading coefficient of $f(X)$ is the required a of the lemma. Let $g(X) \in J$. The division algorithm in $R[a^{-1}][X]$ gives

$$g(X) = f(X)q_1(X) + r_1(X) \tag{1}$$

where $r_1(X) = 0$ or $\deg r_1(X) < \deg f(X)$. Multiply (1) by a large enough positive power of a to obtain $a^m g(X) = f(X)q(X) + r(X)$ in $R[X]$. Since $r(X) \in J$ the choice of $f(X)$ forces $r(X)$ to be 0. $\square$

Definitions 1.2 (a). An integral domain R is a GD(1) *domain* if every non-zero element in R is contained in only finitely many principal prime ideals of R.

(b) For a constant-free ideal I of $R[X]$, we say that R is I-*robust* if only finitely many prime elements of R become units in $R[X]/I$. We say R is *robust* if it is I-robust for every constant-free ideal I.

These definitions make sense for a wider class of rings. However, for integral domains the concepts coincide, as we now show.

Proposition 1.3. *An integral domain R is robust if and only if it is a* GD(1) *domain.*

Proof. Suppose R is GD(1), and let I be a constant-free ideal in $R[X]$. Let $f(x) \in R[X]$ and $a \in R$ be as in Lemma 1.1. Now suppose $p \in R$ is a prime element which is a unit in $R[X]/I$. Then $pg(X) - 1 \in I$ for some polynomial $g(X) \in R[X]$. From Lemma 1.1, there exist $q \in R[X]$ and integer $m > 0$ such that

$$a^m(pg(X) - 1) = f(X)q(X). \tag{2}$$

Considering (2) mod$p$ gives

$$-a^m \equiv f(X)q(X) \bmod P \tag{3}$$

where P is the prime ideal in R generated by p. We may consider (3) as an equation in $(R/P)[X]$. If $q(X) = 0$ in $(R/P)[X]$, then $a^m \in P$. Hence, $a \in P$ and p divides a. On the other hand, if $q(X) \neq 0$ in $(R/P)[X]$, then $f(X)$ is a constant in $(R/P)[X]$. Since $\deg f(X) \geq 1$ this implies that $a \in P$. Since R is GD(1), this implies that only finitely many primes become units in $R[X]/I$, i.e., R is robust.

Suppose that R is not GD(1). Then there is a non-zero element a in R in infinitely many principal prime ideals P of R. Let $\langle aX - 1\rangle$ be the ideal of $R[X]$ generated by $aX - 1$. If $a \in P = \langle p\rangle$, then p divides a. Hence, p becomes a unit in $R[X]/\langle aX - 1\rangle$. Thus R is not robust. $\square$

Proposition 1.3 allows us to use robust and GD(1) interchangeably. The latter is more amenable for proofs while the former is more euphonious for the statement of results. In that respect the next proposition is typical.

Proposition 1.4. *An integral domain R is robust if and only if $R[X]$ is robust.*

Proof. If $R[X]$ is GD(1), so is R because a principal prime ideal in R generates a principal prime ideal in $R[X]$.

Suppose that R is GD(1). There are two classes of principal prime ideals in $R[X]$: those generated by prime elements of R and those generated by nonconstant prime elements of $R[X]$. If $0 \neq a \in R$, then a is contained in only finitely many principal prime ideals of the first kind because R is GD(1). Let $f(X) = a_0 + a_1X + \cdots + a_nX^n,\ n \geq 1$, be in $\langle p \rangle$, where p is a prime element in R. Then $a_n \in \langle p \rangle$. Since R is GD(1), $f(X)$ is in only finitely many principal prime ideals of the first kind.

Suppose $p_1, p_2, \ldots, p_k$ are nonconstant, nonassociate prime elements of $R[X]$ such that $f(X) \in \langle p_i \rangle$, for each $i = 1, 2, \ldots, k$. Then, since $\langle p_1 p_2 \ldots p_k \rangle = \langle p_1 \rangle \cap \langle p_2 \rangle \cap \cdots \cap \langle p_k \rangle$, it follows that there exists $q \in R[X]$ such that $f(X) = p_1 p_2 \cdots p_k q_k$. Since k cannot exceed the degree of $f(X)$, $R[X]$ is GD(1). $\square$

We now give examples of GD(1) domains and the relationship of GD(1) domains to other classes of integral domains.

In [**2**] an integral domain R is called a *bounded factorization domain* (BFD) if every element in R is a product of a finite number of irreducible elements in R and, for each non-zero element x of R, there is a positive integer $N(x)$ such that, whenever $x = x_1 \cdots x_n$ is a product of irreducible elements of R, then $n \leq N(x)$. A Noetherian domain is a BFD, see [**1**, Proposition 2.2].

Proposition 1.5. *A bounded factorization domain is robust.*

Proof. Let x be a non-zero element of a BFD, R. Then x cannot be divisible by more than $N(x)$ distinct prime elements. For, if x is divisible by r distinct primes p_i (for $i = 1, 2, \ldots, r$), with $r \geq N(x)+1$, then $x = p_1 p_2 \cdots p_r a$ for some a in R. Since a is a product of irreducibles, and the p_i are irreducible, we have contradicted $N(x) < r$. Hence, R is GD(1). $\square$

The converse of Proposition 1.5 is not true. To see this, let R be a valuation domain whose maximal ideal is not principal (e.g., a union of power series rings over a field in variables $t^{1/2^n}$ for all positive integers n). Proposition 1.6 of [**7**] shows that there are only a finite number of principal prime ideals, so that R is robust. On the other hand, such an R has no irreducible elements and hence is not BFD. (For if $x \in R$ is irreducible, then let m be a nonzero element in the maximal ideal. Since ideals are linearly ordered, $mR \subseteq xR$ by irreducibility. Hence x generates the maximal ideal, a contradiction.)

GD(1) *domains need not be* idf *nor* FFD. An integral domain R is a *finite factorization domain* (FFD) if every non-zero element in R has only a finite number of non-associate divisors, while R is *irreducible divisor finite* (idf) if every non-zero element has only a finite number of non-associate irreducible divisors. Therefore idf domains and FFD's are GD(1). A Krull domain is FFD, by Example 3 or Theorem 2, both in [**2**]. We will see in Proposition 2.11 that a GD(1) domain need not be idf nor FFD.

Examples 1.6. *Non-noetherian robust rings.* (1) Let R be a UFD; then $R[X_1, \dots, X_n, \dots]$ is a UFD, hence robust.

(2) As noted above, a Krull domain is robust. The integral closure of a noetherian domain need not be noetherian but it is Krull; see [**14**, Theorem 33.10].

(3) There are non-noetherian domains with only finitely many prime ideals; see [**7**, Exercise 8, p. 13 and Proposition 1.7]. Such domains are trivially GD(1).

(4) A valuation domain whose maximal ideal is not principal has no irreducible elements, as in the discussion following Proposition 1.5. Hence it is not noetherian and is vacuously GD(1).

All arrows lead to GD(1). A GD(1) domain can be alternatively described as *prime divisor finite* (pdf). The comments since Proposition 1.5 put pdf at the extreme right of the chart on page 2 of [**1**] and no arrow goes out of GD(1).

Since the class of robust domains is so inclusive, it is necessary to give examples of non-robust domains.

We recall that a *Jacobson* ring is a commutative ring with identity in which every prime ideal is an intersection of maximal ideals. (Here we follow Bourbaki, rather than many authors, for example [**11**] and [**13**], who use the term *Hilbert.*) The class of Jacobson rings is closed under quotients and polynomial rings, see [**11**]. In particular, polynomial rings in several variables over principal ideal domains are Jacobson. However, an ascending union of Jacobson rings need not be Jacobson. A domain with non-zero Jacobson radical cannot be a Jacobson ring. Thus $\mathbf{Z}_2$, the localization of $\mathbf{Z}$ at the prime 2, is not Jacobson. But it is an ascending union of A_k where A_k is the ring generated over $\mathbf{Z}$ by

$$\left\{\frac{1}{3}, \dots, \frac{1}{2k+1}\right\}.$$

Each A_k is a quotient of the Jacobson ring $\mathbf{Z}[X_1, \dots, X_k]$.

The following result is a consequence of [**11**, Theorem 147], but the proof below is brief.

Lemma 1.7. *Let A be a Jacobson ring. Then a prime ideal P of A that is not maximal is contained in infinitely many maximal ideals of A.*

Proof. Let $P = \cap_{j \in J} M_j$, where each M_j is a maximal ideal of R containing P. Since P is not maximal, J has at least two members. If J were finite, then the Chinese remainder theorem gives

$$R/P \cong \coprod_{j \in J} (R/M_j). \tag{4}$$

Since the righthand side of (4) is not an integral domain, (4) gives a contradiction. Hence J is an infinite set. $\square$

Examples 1.8. *Examples of non-robust rings.* 1. The following result is proved in [**13**]: If D is a PID with infinitely many prime ideals with K as its quotient field, then $R = D + XK[X]$ is a Jacobson domain in which every maximal ideal is principal. Since $XK[X]$ is a prime ideal that is not maximal then by Lemma 1.8, X is contained in infinitely many maximal ideals. Thus R is not GD(1). Hence R is not robust, by Proposition 1.3.

2. Let A be an integral domain. Let $R = A[X_0, X_1, X_2, \dots, Y_1, Y_2, \dots]$. Let I be the ideal generated by $\{X_i - X_{i+1}Y_{i+1} : i = 0, 1, 2, \dots\}$. Consider the sequence of rings $R_0 = R[X_0]$, $R_1 = R[X_1, Y_1]$, $R_2 = R[X_2, Y_1, Y_2]$, $R_n = R[X_n, Y_1, Y_2, \dots, Y_n]$ with $R_i \subset R_{i+1}$ via $X_i = X_{i+1}Y_{i+1}$. Here we are dealing with an ascending union of polynomial rings (with a non-standard embedding). This shows that R/I is an integral domain and that the coset $X_0 + I$ is contained in the infinitely many prime ideals $\langle Y_i + I \rangle$, $i = 1, 2, \dots$. Hence R/I is not robust.

Corollary 1.9. (a) *If P is a prime ideal of a robust ring R, R/P need not be robust.*

(b) *A subring of a robust domain need not be robust.*

Proof. (a) Choose the coefficient ring A of Example 1.8.2 to be a UFD. Then by Example 1.6.1, the ring R of Example 1.8.2 is robust with a quotient that is a domain and non-robust.

(b) The non-robust example in Example 1.8.1 is a subring of the robust domain $K[X]$. In this case both rings have the same field of quotients, that is, $K[X]$ is an overring of $D + XK[X]$. $\square$

We shall need some lemmas in order to deal with robustness and power series rings.

Lemma 1.10. *Let $S = R[[X]]$. Let $p, b \in S$, and let the constant term of p be $p_0 \neq 0$. Then p divides b in S if and only if for every element $q \in S$ with $b - pq$ having lowest order term $c_k X^k$ with $k \geq 0$, we must have p_0 divides c_k in R.*

Proof. Starting with $q = 0$, the assumption p_0 divides c_k allows us to obtain each term of the quotient by induction on k. Conversely, if $b = pr$ in S, then for each q we get $b - pq = p(r - q)$. The lowest order term $c_k X^k$ of $b - pq$ must therefore have p_0 divides c_k. $\square$

Lemma 1.11. *If the constant term p_0 of a series p in $S = R[[X]]$ is a nonzero prime element in R, then p is a prime element in $R[[X]]$.*

Proof. Let b and c be two elements of S neither of which is divisible by p in S. Then by Lemma 1.10, there must exist elements of S, q for b and r for c, such that $b - pq$ has lowest order term $d_0 X^k$ with d_0 not divisible by p_0, and $c - pr$ has lowest order term $e_0 X^m$ with e_0 not divisible by p_0. Then $(b-pq)(c-pr)$ has lowest order term $d_0 e_0 X^{k+m}$, not divisible by p_0. But $(b - pq)(c - pr) = bc - p(qc + br - pqr)$. Thus, by Lemma 1.10, p does not divide bc. $\square$

We now adapt the second example of a non-robust ring to show that R robust does not imply that $R[[X]]$ is robust.

Proposition 1.12. *Let R be the ring generated over $\mathbf{Z}$ by variables $X_i, i = 0, 1, \dots,$ and Y, with relations $X_i = YX_{i+1},\ i = 0, 1, \dots$. Let $S = R[[t]]$ be the power series ring over R. Then R is robust while S is not robust.*

Proof. By Lemma 1.11, $Y + nt$ is a prime element of S for every integer n. Each of these is a divisor of X_0, since $X_0 = (Y + nt)(X_1 - nX_2t + n^2X_3t^2 - n^3X_4t^3 + \cdots)$. If $m \neq n$, then $Y + mt$ and $Y + nt$ are not associates in S, since $Y + nt = (Y + mt)(1 + kt + \cdots)$ implies $n = m + kY$, which is not solvable in R. Thus S is not robust.

Let a be a nonzero element of R that is not a unit. Factor it uniquely into primes in the smallest subring $\mathbf{Z}[X_i, Y] \subset R$ in which it lies. We will show that these are the only primes dividing a in R. If a prime element p in $\mathbf{Z}[X_i, Y]$ factors in $\mathbf{Z}[X_k, Y]$, $k > i$, then using $X_{k-1} = YX_k$ a sufficiently high power Y^m of Y results in a factorization of Y^mp in $\mathbf{Z}[X_i, Y]$. Uniqueness of factorization in the latter ring implies that p remains irreducible in $\mathbf{Z}[X_k, Y]$. $\square$

We now extend the above concepts to polynomials in more than one variable, $R[\mathbf{X}] = R[X_1 \dots, X_n]$. Let I be a constant-free ideal in $R[\mathbf{X}]$. We say that R is *I-n-robust* if only finitely many prime elements of R become units in $R[\mathbf{X}]/I$. If n is fixed and R is I-n-robust for every constant-free ideal I in $R[\mathbf{X}]$, we say that R is *n-robust*. If R is n-robust for every positive integer n, we say that R is $\aleph_0$-robust.

The next lemma is well-known (see, e.g., [**14**, 10.5]): we use it to prove that robust already implies $\aleph_0$-robust.

Lemma 1.13. *Let $R_1 \subseteq R_2$ be an integral extension of rings, and suppose that $x \in R_1$ has an inverse $y \in R_2$; then $y \in R_1$.*

Proof. Let y satisfy the monic polynomial $y^n + r_1y^{n-1} + \cdots + r_{n-1}y + r_n = 0$, where $r_i \in R_1$. Multiplying this equation by x^{n-1} yields $y + (r_1 + r_2x + \cdots + r_nx^{n-1}) = 0$. Thus, $y \in R_1$. $\square$

Theorem 1.14. *An integral domain R is a* GD(1) *domain if and only if it is $\aleph_0$-robust.*

Proof. Suppose that R is GD(1); we have to show that if I is a constant-free ideal in $S = R[X_1, \dots , X_n]$, then only finitely many prime elements of R become units in S/I. The proof is by induction on n, and Proposition 1.3 starts the induction.

Suppose that $I \cap R[X_i] \neq \{0\}$ for each $i \in \{1, 2, \dots , n\}$. Let f_i be the polynomial of least positive degree in $I \cap R[X_i]$ with leading coefficient a_i. Applying the division algorithm to each f_i leads to the conclusion that S/I is finitely generated as a module over $T = R[a^{-1}]$, where $a = a_1a_2 \cdots a_n$. Since $I \cap R = \{0\}$, R embeds in S/I and we have that $(S/I)[a^{-1}]$ is an integral extension of T. If $p \in R$ is a prime and becomes a unit in S/I, then $1/p \in T$, by Lemma 1.13. Then $1/p = r/a^m$ for some $r \in R$ and $m \geq 0$. Multiplying this equation by a suitable power of a leads to the conclusion that p divides a in R. Since R is GD(1) we conclude that only finitely many non-associated prime elements become units in S/I.

Suppose that $I \cap R[X_i] = \{0\}$ for some i, say $i = 1$ by relabeling. Then $S = (R[X_1])[X_2, \dots , X_n]$. By Proposition 1.4, $R[X_1]$ is robust. Hence by the induction hypothesis, only finitely many prime elements in $R[X_1]$ become units in S/I. Since prime elements in R remain prime in $R[X_1]$, we deduce that only finitely many prime elements of R become units in S/I, i.e., that R is $\aleph_0$-robust. □

Zariski's version of the Nullstellensatz states that a field which is finitely generated as an algebra over a subfield is algebraic over that subfield [**8**, p. 31]. The following is a generalization.

Corollary 1.15. *Let A be an affine* (*commutative*) *algebra over a field k, and let F be a field over k which is contained in A. Then F is algebraic over k.*

Proof. If not, then F contains $k[x]$ for some x transcendental over k. The algebra A is finitely generated over $k[x]$, and $k[x]$ is $\aleph_0$-robust.

Hence, only finitely many primes of $k[x]$ become units in A. But $k[x]$ has infinitely many primes, all of which are units in F, a contradiction. $\square$

Remark 1.16. A consequence of the results in this section is that the field of rational numbers $\mathbf{Q}$ cannot be contained in a finitely generated commutative ring extension of $\mathbf{Z}$, since $\mathbf{Z}$ is GD(1). Non-commutative analogues of our results would be useful in the implementation of some ideas of Makar-Limanov (private communication, by analogy with [**10**]) on studying skew fields of characteristic zero by specializing to finite characteristic.

2. GD(2) domains and super-robustness. In addition to Lemma 1.13 we shall need the following result.

Lemma 2.1. *Suppose $C \subseteq D$ is an inclusion of rings with D integral over C. Suppose P is a prime ideal of C and J is an ideal of D with $PD = J$. Then $J \cap C = P$.*

Proof. By the lying over theorem (see [**5**, Proposition 4.15]) there is a prime ideal Q of D with $Q \cap C = P$. Since $PD \subseteq Q$ and $PD = J$, we have that $J \subseteq Q$. From $J \cap C \subseteq Q \cap C = P$ and $J \cap C \supseteq P$ we get $J \cap C = P$. $\square$

Definitions 2.2. (a) An integral domain R is a GD(2) *domain* if every non-zero element in R is contained in only finitely many prime ideals of R.

(b) Let J be an ideal of $R[X]$ that contains a constant-free ideal I. We shall say that R is *I-super-robust with respect to J* if the set of prime ideals, P, of R that satisfy

$$PR[X] + I = J \tag{5}$$

is finite. If R is I-super-robust with respect to every ideal J of $R[X]$ that contains I we say that R is *I-super-robust*, while R is *super-robust* if it is I-super-robust for every constant-free ideal I of $R[X]$.

Proposition 2.3. *An integral domain R is super-robust if and only if it is a GD(2) domain.*

Proof. Suppose that R is a GD(2) domain. We want to show that R is super-robust. Let I be a constant-free ideal in $R[X]$, and let J be an ideal of $R[X]$ containing I. Recall the element a of Lemma 1.1. Since R is GD(2), there are only finitely many prime ideals P that contain a.

Now let P be a prime ideal of R that satisfies (5) with $a \notin P$. Let $c \in J \cap R$. If $c = 0$, then $c \in P$. Otherwise from (5) we get that $c = p(X) + b$ for some $p(X) \in PR[X]$ and some $b \in I$. Using Lemma 1.1 and its notation, we get that $a^m c = a^m p(X) + f(X)q(X)$. Reducing mod P leads to $a^m c \in P$ as in the proof of Proposition 1.3. Since $a \notin P$, we get that $c \in P$. Since $P \subseteq J \cap R$, we have proved that $J \cap R = P$. Thus there is only one P satisfying (5) with $a \notin P$. We conclude that R is super-robust.

Suppose R is not GD(2). Then there is a non-zero element a of R that is contained in infinitely many prime ideals of R. Let I be any prime ideal of $R[X]$ that contains $aX - 1$. (Note that $aX - 1$ is irreducible but not necessarily prime.) Let P be a prime ideal of R that contains a. Then $PR[X] + I = R[X]$ because $1 = aX + (1 - aX)$. Therefore R is not super-robust. $\square$

Proposition 2.4 is in contrast to Proposition 1.4.

Proposition 2.4. *The polynomial ring $R[X]$ is super-robust if and only if R is a field.*

Proof. If R is a field, then $R[X]$ is GD(2), hence super-robust by Proposition 2.3. If R is not a field, then it has a non-zero prime ideal, P. By Theorem 36 of [**11**], there are infinitely many prime ideals of $R[X]$ lying over the prime ideal $PR[X]$. Hence any non-zero element of P is contained in infinitely many prime ideals of $R[X]$. $\square$

It follows from Proposition 2.4 that the property of super-robustness is not closed under subrings nor polynomial extensions. This is in contrast to the properties of BFD, FFD, and UFD, which are closed under polynomial extensions, see [**1**, Propositions 2.6, 5.3].

Corollary 2.5. *A bounded factorization domain, even a UFD, need not be super-robust.*

Example 2.6. Example 1.6.3 is also a non-noetherian example of a super-robust domain.

Proposition 2.7 below gives an even more interesting example of a super-robust ring. The ring $R = \mathbf{R} + X\mathbf{C}[X] \subset \mathbf{C}[X]$ is one of the examples of an integral domain that is not FFD, see [**2**]: think of R as the set of complex polynomials, $f(X)$, with $f(0)$ real. Thus the imaginary number i is not in R. For each real number r the element $(r+i)X$ divides X^2. Since $1/(r+i) \notin R$, X^2 has infinitely many non-associated divisors. Hence R is neither idf nor FFD.

We shall now show that R is a GD(2) domain. First note that the ideal $\langle X\rangle$ is properly contained in $X\mathbf{C}[X]$ because iX is not in $\langle X\rangle$. Also no element of $X\mathbf{C}[X]$ is a prime element because $Xf(X)$ divides $(iXf(X))(iXf(X))$ but $Xf(X)$ does not divide $iXf(X)$. Let $P \subseteq X\mathbf{C}[X]$ be a prime ideal of R. If $X \in P$, then $(Xi)(Xi) \in P$. Hence $Xi \in P$. Thus $P = X\mathbf{C}[X]$. Assume that $X \notin P$. Thus there is an element $0 \neq X^n f(X) \in P$, where $f(X) = \lambda + Xg(X)$, $0 \neq \lambda \in \mathbf{C}$, $n \geq 1$, and $g(X) \neq 0$. We cannot say a priori that $f(X)$ is in R. Write $\lambda^{-1}f(X) = 1 + \lambda^{-1}Xg(X)$. Then $(\lambda^{-1}X)(X^n f(X)) = X^{n+1}(1 + X\lambda^{-1}g(X))$. Since $1 + X\lambda^{-1}g(X)$ is in R and $X \notin P$, $1 + X\lambda^{-1}g(X)$ is in P. But no such element is in $X\mathbf{C}[X]$. Hence $X \in P$ and $Xi \in P$. Thus $P = X\mathbf{C}[X]$. Therefore, $X\mathbf{C}[X]$ is a prime ideal with no prime elements. Every element in R of the form $Xf(X)$, $f(X) \neq 0$, is contained only in the prime ideal, $X\mathbf{C}[X]$.

We shall now determine the other prime ideals of R. Given any non-zero complex number α the ideal $\langle 1 - \alpha^{-1}X\rangle$ is a maximal ideal of R, since the evaluation map $\phi_\alpha : R \to \mathbf{C}$ defined by $\phi_\alpha(f(X)) = f(\alpha)$ is onto $\mathbf{C}$. Certainly $\langle 1 - \alpha^{-1}X\rangle \subseteq \operatorname{Ker}\phi_\alpha$. Suppose $f(X) \in \operatorname{Ker}\phi_\alpha$. Then for appropriate polynomials, $f(X) = (1 - \alpha^{-1}X)(a_m X^m + \cdots + a_1 X + a_0)$ in $\mathbf{C}[X]$. Since $f(X) \in R$, a_0 is real. Thus $f(X) \in \langle 1 - \alpha^{-1}X\rangle$. Hence $\langle 1 - \alpha^{-1}X\rangle$ is a maximal ideal in R. Any prime ideal, P, of R not contained in $X\mathbf{C}[X]$ has an element of the form $r + Xf(X)$, where $r \neq 0$, $f(X) \neq 0$.

An element in R is of the form $x = r + Xf(X)$ where r is some non-zero real number, and $f(X)$ in $\mathbf{C}[X]$ is non-zero. This element x factorizes in $\mathbf{C}[X]$ and in R as $r(1-\alpha_1^{-1}X)\cdots(1-\alpha_k^{-1}X)$. Any prime ideal, P, of R not in $X\mathbf{C}[X]$ contains an element of the form $r+Xf(X)$, where $r \neq 0$, $f(X) \neq 0$. From the above argument we conclude that $P = \langle 1-\alpha^{-1}X\rangle$ for some complex number α. Moreover x is contained in only finitely many prime ideals, viz., $\langle 1-\alpha_j^{-1}X\rangle$, $j = 1, \ldots, k$. Thus R is GD(2).

We summarize the above discussion in Proposition 2.7.

Proposition 2.7. *The ring of complex polynomials R with real constant term is neither an* FFD *domain nor an* idf *domain, but it is a* GD(2) *domain. Hence it is super-robust.*

Remark 2.8. Robustness is implied by various factorization properties, but super robustness is not. Nonrobust rings are hard to find, but examples of domains that are not super-robust are plentiful, see Proposition 2.4.

In comparison with Corollary 1.9, we state the following.

Corollary 2.9. (a) *If P is a prime ideal of a super-robust ring, then R/P is super-robust.*

(b) *A subring of a super-robust domain need not be super-robust.*

Proof. (a) follows from the definition of super-robustness while (b) follows from Proposition 2.4: $\mathbf{Z}[X]$ is not super-robust while $\mathbf{Q}[X]$ is super-robust. $\square$

Remark 2.10. As already remarked, a Dedekind domain is super-robust. The super-robust domain R in Proposition 2.7 is not Dedekind because it is not integrally closed since $i \notin R$.

Proposition 2.11. *A* GD(1) *domain need not be an* FFD *or* idf.

Proof. A GD(2) domain is clearly also GD(1). The domain in Proposition 2.7 is therefore GD(1), but it is neither FFD nor idf. □

If p is a prime element in R, then the element X is contained in the prime ideal of $R[[X]]$ generated by $\{p, X\}$. Thus, if R has infinitely many primes p we have that $R[[X]]$ is not GD(2).

Proposition 2.12. *If R has infinitely many prime elements, then $R[[X]]$ is not super-robust.*

Definition 2.13. Let I be a constant-free ideal in $R[X_1, \dots, X_n]$. Then R is *I-n-super-robust with respect to* an ideal $J \supseteq I$ in $R[X_1, \dots, X_n]$ if there are only finitely many prime ideals, P, of R that satisfy

$$PR[X_1, \dots, X_n] + I = J. \tag{6}$$

If, for a fixed positive integer n and for a fixed constant-free ideal I, (6) has only finitely many solutions P for every ideal $J \supseteq I$ in $R[X_1, \dots, X_n]$, we say that R is *I-n-super-robust*. If, for a fixed n, R is I-n-super-robust for every constant-free ideal I, we say that R is *n-super-robust*. If R is n-super-robust for every positive integer n, we say that R is *$\aleph_0$-super-robust*.

Theorem 2.14. *An integral domain R is a* GD(2) *domain if and only if it is $\aleph_0$-super-robust.*

Proof. If R is $\aleph_0$-super-robust, then it is super-robust. Hence by Proposition 2.3, it is GD(2). Suppose that R is GD(2). We want to show that, for every positive integer n and every constant-free ideal I in $S = R[X_1, \dots, X_n]$ and every ideal $J \subseteq S$ containing I, the equation (6) has only finitely many primes P of R as solutions. Note that I and J are fixed and P is the variable.

Let $k \geq 0$ be the maximum number of variables with the property (after possibly relabeling) that $I \cap R[X_1, \dots, X_k] = \{0\}$. Let $A = R[X_1, \dots, X_k]$. (If $k = 0$, then $A = R$.) Rewrite (6) as

$$PA[X_{k+1}, \dots, X_n] + I = J. \tag{7}$$

In (7) PA is a prime ideal of A consisting of polynomials with coefficients in P. Let $P_1 \neq P_2$ be prime ideals of R solving (6). Then P_1A and P_2A remain distinct solutions of (7). By the choice of k, $I \cap A[X_{k+i}] \neq 0$ for $i = 1, \dots, n-k$. Let

$f_i(X_{k+i})$ be a polynomial in $I \cap A[X_{k+i}]$ of minimum degree in X_{k+i} with $a_i \in A$ as its leading coefficient.

Localize A at the multiplicative set $\{a^m : m = 1, 2, \dots\}$ where $a = a_1 a_2 \cdots a_{n-k}$ to get $A[a^{-1}]$. Only finitely many P's from (6) intersect $\{a^m : m = 1, 2, \dots\}$ because R is GD(2). Such primes are no longer proper ideals in $A[a^{-1}]$. If these are the only solutions of (6) then we would be done.

If $P \cap \{a^m : m = 1, 2, \dots\} = \varnothing$, then $PA[a^{-1}]$ is a prime ideal in $A[a^{-1}]$. However, it is possible that distinct solutions of (6) may become the same prime ideals of $A[a^{-1}]$.

Fix P a prime ideal of R which is a solution of (6). Suppose

$$PA[a^{-1}] = P_1A[a^{-1}] \tag{8}$$

in $A[a^{-1}]$ with P_1 a solution of (6) different from P. Since R is GD(2), P is contained in only finitely many P_1's that are solutions of (6). Thus in (8) we may restrict to P_1 with $P \not\subset P_1$.

Let $x \in P \backslash P_1$. Then, from (8), $x = g(a^{-1})$ where g is some polynomial in a^{-1} with coefficients in P_1. By multiplying by a large enough power of a we get $xy \in P_1A$ where y is a power of a. Therefore, $a \in P_1A$ since $x \notin P_1$. Hence the coefficients of a are in P_1. Since R is GD(2), the coefficients of a are contained in only finitely many P_1's. Thus (8) holds for only finitely many P_1's. Therefore, only finitely many solutions of (6) are lost in the following equation

$$PA[a^{-1}][X_{k+1}, \dots, X_n] + I[a^{-1}] = J[a^{-1}]. \tag{9}$$

Since $I[a^{-1}] \cap A[a^{-1}] = \{0\}$, we may consider $A[a^{-1}]$ as equal to $(A[a^{-1}] + I[a^{-1}])/I[a^{-1}]$ and therefore a subring of $T = A[a^{-1}] [X_{k+1}, \dots, X_n]/I[a^{-1}]$. Because of the choice of $f_i(X_{k+i})$, the division algorithm gives that T is an integral extension of $A[a^{-1}]$. Equation (9) implies that the hypotheses of Lemma 2.1 are satisfied, with

$C = A[a^{-1}]$, $D = T$, and $J = J[a^{-1}]/I[a^{-1}]$. Therefore Lemma 2.1 implies that

$$(J[a^{-1}]/I[a^{-1}]) \cap A[a^{-1}] = (PA[a^{-1}] + I[a^{-1}])/I[a^{-1}]. \tag{10}$$

Thus after discarding finitely many solutions of (6) along the way only the lefthand side of (10) survives. Therefore, (6) has only finitely many solutions. Thus R is $\aleph_0$-super-robust. $\square$

Remark 2.15. We originally intended to use irreducible elements in place of prime elements. The role of GD(1) domains would be played by *irreducible divisor finite* (idf) domains as defined in [**1**]. We would also need *irreducible divisors of powers finite* (idpf) domains. This means that if a is a non-zero element of R the set of irreducible non-associate divisors of powers of a is finite. While *prime factors of powers finite* domains and GD(1) domains coincide, it is no longer clear that idpf domains and idf domains coincide.

For integral domains $R \subseteq S$, S is called an *overring* of R if the quotient fields coincide. One might ask whether every overring of a (super)robust ring is (super)robust? We note that this is true for super-robustness if S is a localization of R, see [**11**, Theorem 34]. The corresponding statement is false for robust rings by an example of Coykendall (private communication). In general Coykendall has shown in [**3, 4**] that integral closures of GD(1) (respectively GD(2), HFD) domains need not be of the same type. For the behavior of other generalizations of UFD under localization, see [**1**].

We conclude the paper with a problem suggested by the referee. When are rings of the form $D+M$ GD(1) or GD(2)? In Proposition 2.7 we used $D = \mathbf{R}$ and $M = X\mathbf{C}[X]$. For a general definition see [**6**]. More generally, characterize which pullbacks are GD(1) and GD(2) domains.

Note added in proof. The concept of idpf is considered by the authors in a paper to appear in the Houston Journal of Mathematics. With regard to the problem suggested by the referee, Coykendall and Dumitrescu have described the rings of the form $A + XB[X]$ which are idf.

REFERENCES

1. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19.

2. D.D. Anderson and B. Mullins, *Finite factorization domains*, Proc. Amer. Math. Soc. **124** (1996), 389–396.

3. J. Coykendall, GD(1) *and* GD(2) *are not preserved in integral closure*, in *Commutative rings*, Nova Science Publ., Hauppage, New York, 2002.

4. J. Coykendall, *Counterexamples relating to the integral closure of a half-factorial domain*, preprint, 1999.

5. D. Eisenbud, *Commutative algebra*, Grad. Texts in Math., vol. 150, Springer, New York, 1994.

6. M. Fontana, J. Huckaba and I. Papick, *Prüfer domains*, Monographs Textbooks Pure Appl. Math., vol. 203, Marcel Dekker, Inc., New York, 1997.

7. L. Fuchs and L. Salce, *Modules over valuation domains*, Lecture Notes in Pure and Appl. Math., vol. 97, Marcel Dekker, Inc., New York, 1985.

8. W. Fulton, *Algebraic curves*, W.A. Benjamin, Inc., New York, 1969.

9. G. Janusz, *Algebraic number fields*, Academic Press, New York, 1973.

10. A. Joseph, *Some ring-theoretic techniques and open problems in enveloping algebras*, in *Noncommutative rings*, (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 24, Springer, New York, 1992, pp. 27–67.

11. I. Kaplansky, *Commutative rings*, rev. ed., University of Chicago Press, Chicago, 1974.

12. H. Matsumura, *Commutative algebra*, W.A. Benjamin, Inc., New York, 1970.

13. J.L. Mott and M. Zafrullah, *Unruly Hilbert domains*, Canad. Math. Bull. **33** (1990), 106–109.

14. M. Nagata, *Local rings*, Interscience Publl., New York, 1962.

Department of Mathematics, Wayne State University, Detroit, MI 48202
E-mail address: `petem@math.wayne.edu`

Department of Mathematics, Wayne State University, Detroit, MI 48202
E-mail address: `okoh@math.wayne.edu`