ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 6, 2006

ON THE BEHAVIOR OF THE SOLUTIONS FOR CERTAIN FIRST ORDER LINEAR AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS

CH.G. PHILOS AND I.K. PURNARAS

ABSTRACT. Some results are given concerning the behavior of the solutions for scalar first order linear autonomous delay as well as neutral delay differential equations. These results are obtained by the use of two distinct real roots of the corresponding characteristic equation.

1. Introduction. This paper deals with the behavior of the solutions of scalar first order linear autonomous delay differential equations as well as neutral delay differential equations. Our results are obtained via two distinct real roots of the corresponding characteristic equations and are motivated by a result due to Driver [**3**, see Theorem 2]. The case of delay differential equations is treated in Section 2, while Section 3 is devoted to the case of neutral delay differential equations. Our results for delay differential equations can be derived as a special case from the results for the more general case of neutral delay differential equations, under some additional restrictions. This is the reason for which the case of delay differential equations is considered separately.

Some closely related asymptotic results for delay differential equations or neutral delay differential equations have been given by Driver [**3**], Driver, Sasser and Slater [**6**], Graef and Qian [**8**], Kordonis, Niyianni and Philos [**12**], Philos [**13**], and Philos and Purnaras [**14, 15**], see also Arino and Pituk [**1**], Driver [**4**] and Györi [**9**] for certain related results. We must also refer here to the very recent interesting article by Frasson and Verduyn Lunel [**7**] concerning the large time behavior of linear functional differential equations.

It is an interesting problem to extend the results of this paper for the more general case of periodic delay differential equations, such as in [**13**], as well as of periodic neutral delay differential equations, cf. [**14**].

2000 AMS *Mathematics Subject Classification.* Primary 34K25, 34K40.
Key words and phrases. Delay differential equation, neutral delay differential equation, behavior of solutions, characteristic equation.
Received by the editors on January 13, 2004.

It will be the subject of a future work to present an analogous treatment for scalar first order linear autonomous delay or neutral delay differential equations with distributed type delays.

For the general theory of delay differential equations as well as of neutral delay differential equations, the reader is referred to the books by Diekmann et al. [**2**], Driver [**5**], Hale [**10**] and Hale and Verduyn Lunel [**11**].

2. Delay differential equations. Consider the delay differential equation

$$x'(t) = ax(t) + \sum_{j\in J} b_j x(t-\tau_j), \tag{E}$$

where J *is an initial segment of natural numbers*, a *and* $b_j \neq 0$ *for* $j \in J$ *are real constants, and* τ_j *for* $j \in J$ *positive real numbers such that* $\tau_{j_1} \neq \tau_{j_2}$ *for* $j_1, j_2 \in J$ *with* $j_1 \neq j_2$*.*

Define

$$\tau = \max_{j\in J} \tau_j.$$

(τ is a positive real number.)

By a *solution* of the delay differential equation (E), we mean a continuous real-valued function x defined on the interval $[-\tau, \infty)$, which is continuously differentiable on $[0, \infty)$ and satisfies (E) for all $t \geq 0$.

Let $C([-\tau, 0], \mathbf{R})$ be the space of all continuous real-valued functions on the interval $[-\tau, 0]$. It is well known, see, for example, Diekmann et al. [**2**], Driver [**5**], Hale [**10**] or Hale and Verduyn Lunel [**11**], that, for any given *initial function* $\phi \in C([-\tau, 0], \mathbf{R})$, there exists a unique solution x of the differential equation (E) which satisfies the *initial condition*

$$x(t) = \phi(t) \quad \text{for } t \in [-\tau, 0]; \tag{C}$$

this solution x will be called the solution of the *initial problem* (E)–(C) or, more briefly, the solution of (E)–(C).

The *characteristic equation* of (E) is

$$\lambda = a + \sum_{j\in J} b_j e^{-\lambda\tau_j}. \tag{$*$}$$

Theorem 2.0 below is a special case of some more general results obtained by Philos [**13**] for periodic delay differential equations, by Kordonis, Niyianni and Philos [**12**] for autonomous neutral delay differential equations, and by Philos and Purnaras [**14**] for periodic neutral delay differential equations. This theorem constitutes a fundamental asymptotic criterion for the solutions of the delay differential equation (E).

Theorem 2.0. *Let* λ_0 *be a real root of the characteristic equation* $(*)$ *with the property*

$$\sum_{j\in J} |b_j|\, \tau_j e^{-\lambda_0 \tau_j} < 1.$$

(*Note that this property guarantees that* $1 + \sum_{j\in J} b_j \tau_j e^{-\lambda_0\tau_j} > 0$.)

Then, for any $\phi \in C([-\tau, 0], \mathbf{R})$, *the solution* x *of* (E)–(C) *satisfies*

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t)\right] = \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j\in J} b_j \tau_j e^{-\lambda_0 \tau_j}},$$

where

$$L_{\lambda_0}(\phi) = \phi(0) + \sum_{j\in J} b_j e^{-\lambda_0\tau_j} \int_{-\tau_j}^{0} e^{-\lambda_0 s}\phi(s)\, ds.$$

Our main purpose in this section is to establish the following theorem.

Theorem 2.1. *Suppose that*

$$b_j < 0 \quad \text{for } j \in J,$$

and let λ_0 *and* $\lambda_1, \lambda_0 \neq \lambda_1$, *be two real roots of the characteristic equation* $(*)$.

Then, for any $\phi \in C([-\tau, 0], \mathbf{R})$, *the solution* x *of* (E)–(C) *satisfies*

$$M_1(\lambda_0, \lambda_1; \phi) \le e^{-\lambda_1 t}\left[x(t) - \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j\in J} b_j\tau_j e^{-\lambda_0\tau_j}}\, e^{\lambda_0 t}\right] \le M_2(\lambda_0, \lambda_1; \phi) \quad \text{for all } t \ge 0,$$

where $L_{\lambda_0}(\phi)$ *is defined as in Theorem* 2.0 *and*:

$$M_1(\lambda_0, \lambda_1; \phi) = \min_{t \in [-\tau, 0]} \left\{ e^{-\lambda_1 t} \left[\phi(t) - \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}} \, e^{\lambda_0 t} \right] \right\}$$

and

$$M_2(\lambda_0, \lambda_1; \phi) = \max_{t \in [-\tau, 0]} \left\{ e^{-\lambda_1 t} \left[\phi(t) - \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}} \, e^{\lambda_0 t} \right] \right\}.$$

Note. By Lemma 2.1 below, we always have

$$1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j} \neq 0.$$

We immediately observe that the double inequality in the conclusion of Theorem 2.1 can equivalently be written as follows

$$M_1(\lambda_0, \lambda_1; \phi) e^{(\lambda_1 - \lambda_0) t} \leq e^{-\lambda_0 t} x(t) - \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}}$$
$$\leq M_2(\lambda_0, \lambda_1; \phi) e^{(\lambda_1 - \lambda_0) t} \quad \text{for all } t \geq 0,$$

and consequently

$$\lim_{t \to \infty} \left[e^{-\lambda_0 t} x(t) \right] = \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}},$$

provided that $\lambda_1 < \lambda_0$.

Moreover, we see that an equivalent form of the double inequality in the conclusion of Theorem 2.1 is the following one

$$M_1(\lambda_0, \lambda_1; \phi) e^{\lambda_1 t} + \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}} \, e^{\lambda_0 t} \leq x(t)$$
$$\leq M_2(\lambda_0, \lambda_1; \phi) e^{\lambda_1 t} + \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}} \, e^{\lambda_0 t} \quad \text{for all } t \geq 0.$$

Before we proceed to prove Theorem 2.1, we will give a lemma about the real roots of the characteristic equation $(*)$.

Lemma 2.1. *Suppose that*

$$b_j < 0 \quad \textit{for } j \in J.$$

(I) *Let* λ_0 *be a real root of the characteristic equation* $(*)$. *Then*

$$1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j} > 0$$

if $(*)$ *has another real root less than* λ_0, *and*

$$1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j} < 0$$

if $(*)$ *has another real root greater than* λ_0.

(II) *In the interval* $[a, \infty)$, *the characteristic equation* $(*)$ *has no roots.*

(III) *Assume that*

$$\text{(H)} \qquad \tau \sum_{j \in J} (-b_j) e^{-(a-(1/\tau))\tau_j} < 1.$$

Then

(i) $\lambda = a - (1/\tau)$ *is not a root of the characteristic equation* $(*)$.

(ii) *In the interval* $(a - (1/\tau), a)$, $(*)$ *has a unique root.*

(iii) *In the interval* $(-\infty, a - (1/\tau))$, $(*)$ *has a unique root.*

Proof. We first observe that, if μ is a real root of the characteristic equation $(*)$, then

$$\mu - a = \sum_{j \in J} b_j e^{-\mu \tau_j} < 0$$

and so $\mu < a$. This shows Part (II).

In order to prove Parts (I) and (III), we set

$$F(\lambda) = \lambda - a - \sum_{j\in J} b_j e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R}.$$

We have

$$F'(\lambda) = 1 + \sum_{j\in J} b_j \tau_j e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R}.$$

Furthermore, we obtain

$$F''(\lambda) = \sum_{j\in J} (-b_j)\tau_j^2 e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R}$$

and consequently

$$F''(\lambda) > 0 \quad \text{for all } \lambda \in \mathbf{R}. \tag{2.1}$$

Now, we will show Part (I). To this end, let us consider a real root λ_0 of the characteristic equation $(*)$. We see that

$$1 + \sum_{j\in J} b_j \tau_j e^{-\lambda_0\tau_j} = F'(\lambda_0).$$

Assume that $(*)$ has another real root λ_1 with $\lambda_1 < \lambda_0$ (respectively, $\lambda_1 > \lambda_0$). Since $F(\lambda_0) = F(\lambda_1) = 0$, from Rolle's theorem it follows that there exists a point ξ with $\lambda_1 < \xi < \lambda_0$, respectively $\lambda_0 < \xi < \lambda_1$, such that $F'(\xi) = 0$. On the other hand, (2.1) implies that F' is strictly increasing on $\mathbf{R}$ and hence, as $F'(\xi) = 0$, it follows that F' is positive on (ξ, ∞), respectively F' is negative on $(-\infty, \xi)$. Thus, we always have $F'(\lambda_0) > 0$, respectively $F'(\lambda_0) < 0$.

Next, we shall prove Part (III). For this purpose, let us assume that (H) holds. Assumption (H) means that

$$F\left(a - \frac{1}{\tau}\right) < 0. \tag{2.2}$$

This, in particular, implies that $\lambda = a - (1/\tau)$ is not a root of the characteristic equation $(*)$. We immediately observe that

$$F(a) > 0. \tag{2.3}$$

Furthermore, it is not difficult to verify that

$$F(-\infty) = \infty. \tag{2.4}$$

From (2.1), (2.2) and (2.3) it follows that, in the interval $(a-(1/\tau), a)$, $(*)$ has a unique root. Moreover, (2.1), (2.2) and (2.4) guarantee that, in the interval $(-\infty, a-(1/\tau))$, $(*)$ has also a unique root.

The proof of the lemma is complete.

Proof of Theorem 2.1. Let ϕ be an arbitrary initial function in $C([-\tau, 0], \mathbf{R})$ and consider the solution x of (E)–(C). Set

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for } t \geq -\tau.$$

Furthermore, let us define

$$z(t) = y(t) - \frac{L_{\lambda_0}(\phi)}{1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}} \quad \text{for } t \geq -\tau.$$

Following the procedure applied by Philos [**13**] for periodic delay differential equations, by Kordonis, Niyianni and Philos [**12**] for autonomous neutral delay differential equations as well as by Philos and Purnaras [**14**] for the more general case of periodic neutral delay differential equations, we can verify that the fact that x satisfies (E) for $t \geq 0$ is equivalent to the fact that z satisfies

$$z(t) = -\sum_{j \in J} b_j e^{-\lambda_0 \tau_j} \int_{t-\tau_j}^{t} z(s)\, ds \quad \text{for } t \geq 0. \tag{2.5}$$

Next, consider the function w defined by

$$w(t) = e^{(\lambda_0 - \lambda_1) t} z(t) \quad \text{for } t \geq -\tau.$$

Then it is easy to see that (2.5) can equivalently be written in the form

$$w(t) = -\sum_{j \in J} b_j e^{-\lambda_0 \tau_j} \int_{t-\tau_j}^{t} e^{(\lambda_0 - \lambda_1)(t-s)} w(s)\, ds \quad \text{for } t \geq 0. \tag{2.6}$$

From the definitions of y, z and w it follows immediately that

$$w(t) = e^{-\lambda_1 t}\left[x(t) - \frac{L_{\lambda_0}(\phi)}{1+\sum_{j\in J} b_j\tau_j e^{-\lambda_0\tau_j}}\, e^{\lambda_0 t}\right] \quad \text{for } t \geq -\tau.$$

Hence, by taking into account the initial condition (C) as well as the way of definition of $M_1(\lambda_0, \lambda_1; \phi)$ and $M_2(\lambda_0, \lambda_1; \phi)$, we can conclude that all we have to prove is that w satisfies

$$\min_{s\in[-\tau,0]} w(s) \leq w(t) \leq \max_{s\in[-\tau,0]} w(s) \quad \text{for all } t \geq 0.$$

We restrict ourselves to show that

$$w(t) \geq \min_{s\in[-\tau,0]} w(s) \quad \text{for all } t \geq 0. \tag{2.7}$$

By an analogous procedure, one can establish that

$$w(t) \leq \max_{s\in[-\tau,0]} w(s) \quad \text{for every } t \geq 0.$$

It remains to prove (2.7). To this end, let us consider an arbitrary real number M with $M < \min_{s\in[-\tau,0]} w(s)$. Then

$$w(t) > M \quad \text{for } t \in [-\tau, 0]. \tag{2.8}$$

We claim that

$$w(t) > M \quad \text{for all } t \geq 0. \tag{2.9}$$

Otherwise, in view of (2.8), there exists a point $t_0 > 0$ so that

$$w(t) > M \quad \text{for } t \in [-\tau, t_0), \quad \text{and} \quad w(t_0) = M.$$

Then, by using the assumption that $b_j < 0$ for $j \in J$, from (2.6) we obtain

$$\begin{aligned} M = w(t_0) &= -\sum_{j\in J} b_j e^{-\lambda_0\tau_j} \int_{t_0-\tau_j}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)} w(s)\,ds \\ &> -M \sum_{j\in J} b_j e^{-\lambda_0\tau_j} \int_{t_0-\tau_j}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)}\,ds \\ &= \frac{M}{\lambda_0-\lambda_1} \sum_{j\in J} b_j e^{-\lambda_0\tau_j} \left[1 - e^{(\lambda_0-\lambda_1)\tau_j}\right] \\ &= \frac{M}{\lambda_0-\lambda_1} \sum_{j\in J} b_j \left(e^{-\lambda_0\tau_j} - e^{-\lambda_1\tau_j}\right) \\ &= \frac{M}{\lambda_0-\lambda_1} \left(\sum_{j\in J} b_j e^{-\lambda_0\tau_j} - \sum_{j\in J} b_j e^{-\lambda_1\tau_j} \right) \\ &= \frac{M}{\lambda_0-\lambda_1} \left[(\lambda_0 - a) - (\lambda_1 - a)\right] = M. \end{aligned}$$

We have thus arrived at a contradiction. This contradiction establishes our claim, i.e., (2.9) holds true. Finally, since (2.9) is satisfied for all real numbers M with $M < \min_{s\in[-\tau,0]} w(s)$, it follows that (2.7) is always fulfilled. So, the proof of our theorem is complete. $\square$

3. Neutral delay differential equations. Let us consider the neutral delay differential equation

$$(\widehat{\mathrm{E}}) \qquad \left[x(t) + \sum_{i\in I} c_i x(t-\sigma_i)\right]' = ax(t) + \sum_{j\in J} b_j x(t-\tau_j),$$

where I and J are initial segments of natural numbers, c_i for $i \in I$, a and $b_j \neq 0$ for $j \in J$ are real constants, and σ_i for $i \in I$ and τ_j for $j \in J$ are positive real numbers such that $\sigma_{i_1} \neq \sigma_{i_2}$ for i_1, $i_2 \in I$ with $i_1 \neq i_2$ and $\tau_{j_1} \neq \tau_{j_2}$ for j_1, $j_2 \in J$ with $j_1 \neq j_2$.

Define

$$\sigma = \max_{i\in I} \sigma_i, \quad \tau = \max_{j\in J} \tau_j, \quad \text{and} \quad r = \max\{\sigma, \tau\}.$$

(Clearly, σ, τ and r are positive real numbers.)

As usual, a continuous real-valued function x defined on the interval $[-r, \infty)$ will be called a *solution* of the neutral delay differential equation $(\widehat{\text{E}})$ if the function $x(t) + \sum_{i \in I} c_i x(t - \sigma_i)$ is continuously differentiable for $t \geq 0$ and x satisfies $(\widehat{\text{E}})$ for all $t \geq 0$.

In the sequel, by $C([-r, 0], \mathbf{R})$ we will denote the set of all continuous real-valued functions on the interval $[-r, 0]$. It is well known, see, for example, Diekmann et al. [**2**], Hale [**10**] or Hale and Verduyn Lunel [**11**], that, for any *initial function* ϕ in $C([-r, 0], \mathbf{R})$, the differential equation $(\widehat{\text{E}})$ has a unique solution x which satisfies the *initial condition*

$$(\widehat{\text{C}}) \qquad x(t) = \phi(t) \quad \text{for } t \in [-r, 0];$$

we shall call this function x the solution of the *initial problem* $(\widehat{\text{E}})$–$(\widehat{\text{C}})$ or, more briefly, the solution of $(\widehat{\text{E}})$–$(\widehat{\text{C}})$.

With the neutral delay differential equation $(\widehat{\text{E}})$ we associate its *characteristic equation*

$$(\widehat{*}) \qquad \lambda\left(1 + \sum_{i \in I} c_i e^{-\lambda \sigma_i}\right) = a + \sum_{j \in J} b_j e^{-\lambda \tau_j}.$$

We will now present a known asymptotic result for the solutions of $(\widehat{\text{E}})$, i.e., Theorem 3.0 below. This theorem has been established by Kordonis, Niyianni and Philos [**12**]. Note that Theorem 3.0 can also be obtained as a special case from a more general asymptotic criterion (for periodic neutral delay differential equations) due to Philos and Purnaras [**14**].

Theorem 3.0. *Let* λ_0 *be a real root of the characteristic equation* $(\widehat{*})$ *with the property*

$$\sum_{i \in I} |c_i| \left(1 + |\lambda_0| \sigma_i\right) e^{-\lambda_0 \sigma_i} + \sum_{j \in J} |b_j| \tau_j e^{-\lambda_0 \tau_j} < 1$$

and set

$$\gamma_{\lambda_0} = \sum_{i \in I} c_i \left(1 - \lambda_0 \sigma_i\right) e^{-\lambda_0 \sigma_i} + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j}.$$

(*Note that the property of* λ_0 *guarantees that* $1 + \gamma_{\lambda_0} > 0$.)

Then, for any $\phi \in C([-r,0],\mathbf{R})$*, the solution* x *of* $(\widehat{\mathrm{E}})$–$(\widehat{\mathrm{C}})$ *satisfies*

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}},$$

where

$$\widehat{L}_{\lambda_0}(\phi) = \phi(0) + \sum_{i\in I} c_i\left[\phi(-\sigma_i) - \lambda_0 e^{-\lambda_0\sigma_i}\int_{-\sigma_i}^{0} e^{-\lambda_0 s}\phi(s)\,ds\right] + \sum_{j\in J} b_j e^{-\lambda_0\tau_j}\int_{-\tau_j}^{0} e^{-\lambda_0 s}\phi(s)\,ds.$$

The main result in this section is the following theorem.

Theorem 3.1. *Suppose that*

$$c_i \le 0 \quad \text{for } i\in I, \quad \text{and} \quad b_j < 0 \quad \text{for } j \in J.$$

Let λ_0 *be a nonpositive real root of the characteristic equation* $(\widehat{*})$ *with*

$$1+\gamma_{\lambda_0} \neq 0,$$

where γ_{λ_0} *is defined as in Theorem* 3.0*. Let also* λ_1 *be a real root of* $(\widehat{*})$ *with* $\lambda_1 \neq \lambda_0$.

Then, for any $\phi \in C([-r,0],\mathbf{R})$*, the solution* x *of* $(\widehat{\mathrm{E}})$–$(\widehat{\mathrm{C}})$ *satisfies*

$$\widehat{M}_1(\lambda_0,\lambda_1;\phi) \le e^{-\lambda_1 t}\left[x(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\,e^{\lambda_0 t}\right] \le \widehat{M}_2(\lambda_0,\lambda_1;\phi) \quad \text{for all } t \ge 0,$$

where $\widehat{L}_{\lambda_0}(\phi)$ *is defined as in Theorem* 3.0 *and*:

$$\widehat{M}_1(\lambda_0,\lambda_1;\phi) = \min_{t\in[-r,0]}\left\{e^{-\lambda_1 t}\left[\phi(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\,e^{\lambda_0 t}\right]\right\}$$

and

$$\widehat{M}_2(\lambda_0, \lambda_1; \phi) = \max_{t \in [-r,0]} \left\{ e^{-\lambda_1 t} \left[\phi(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \, e^{\lambda_0 t} \right] \right\}.$$

Note. By Lemma 3.1 below, we always have $1+\gamma_{\lambda_0} \neq 0$ if λ_1 is also nonpositive.

We see that the double inequality in the conclusion of the above theorem is equivalent to

$$\widehat{M}_1(\lambda_0, \lambda_1; \phi) e^{(\lambda_1 - \lambda_0)t} \le e^{-\lambda_0 t} x(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \le \widehat{M}_2(\lambda_0, \lambda_1; \phi) e^{(\lambda_1 - \lambda_0)t}$$
$$\text{for all } t \ge 0$$

and so

$$\lim_{t \to \infty} \left[e^{-\lambda_0 t} x(t) \right] = \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}},$$

provided that $\lambda_1 < \lambda_0$. Moreover, we immediately observe that this double inequality can equivalently be written in the form

$$\widehat{M}_1(\lambda_0, \lambda_1; \phi) e^{\lambda_1 t} + \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \, e^{\lambda_0 t}$$
$$\le x(t) \le \widehat{M}_2(\lambda_0, \lambda_1; \phi) e^{\lambda_1 t} + \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \, e^{\lambda_0 t} \quad \text{for all } t \ge 0.$$

Proof of Theorem 3.1. Let $\phi \in C([-r,0], \mathbf{R})$ and x be the solution of $(\widehat{\text{E}})$–$(\widehat{\text{C}})$. Furthermore, let y and z be defined by

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for } t \ge -r, \quad \text{and} \quad z(t) = y(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}$$
$$\text{for } t \ge -r.$$

As it has been shown by Kordonis, Niyianni and Philos [**12**], see, also, Philos and Purnaras [**14**] for the more general case of periodic delay differential equations, the fact that x satisfies $(\widehat{\mathrm{E}})$ for $t \geq 0$ is equivalent to

$$\begin{aligned} & z(t) + \sum_{i \in I} c_i e^{-\lambda_0 \sigma_i} z(t - \sigma_i) \\ & \quad = \lambda_0 \sum_{i \in I} c_i e^{-\lambda_0 \sigma_i} \int_{t-\sigma_i}^{t} z(s)\, ds - \sum_{j \in J} b_j e^{-\lambda_0 \tau_j} \int_{t-\tau_j}^{t} z(s)\, ds \\ & \qquad \text{for } t \geq 0. \end{aligned} \tag{3.1}$$

Next, let us define

$$w(t) = e^{(\lambda_0 - \lambda_1)t} z(t) \quad \text{for } t \geq -r.$$

By the use of the function w, (3.1) becomes

$$\begin{aligned} & w(t) + \sum_{i \in I} c_i e^{-\lambda_1 \sigma_i} w(t - \sigma_i) \\ & \quad = \lambda_0 \sum_{i \in I} c_i e^{-\lambda_0 \sigma_i} \int_{t-\sigma_i}^{t} e^{(\lambda_0 - \lambda_1)(t-s)} w(s)\, ds \\ & \qquad - \sum_{j \in J} b_j e^{-\lambda_0 \tau_j} \int_{t-\tau_j}^{t} e^{(\lambda_0 - \lambda_1)(t-s)} w(s)\, ds \quad \text{for } t \geq 0. \end{aligned} \tag{3.2}$$

By way of the definition of y, z and w, we have

$$w(t) = e^{-\lambda_1 t} \left[x(t) - \frac{\widehat{L}_{\lambda_0}(\phi)}{1 + \gamma_{\lambda_0}}\, e^{\lambda_0 t} \right] \quad \text{for } t \geq -r.$$

Thus, from the initial condition $(\widehat{\mathrm{C}})$ and the definitions of the constants $\widehat{M_1}(\lambda_0, \lambda_1; \phi)$ and $\widehat{M_2}(\lambda_0, \lambda_1; \phi)$, it follows that the double inequality in the conclusion of our theorem can equivalently be written as follows

$$\min_{s \in [-r,0]} w(s) \leq w(t) \leq \max_{s \in [-r,0]} w(s) \quad \text{for all } t \geq 0.$$

The proof of the theorem will be accomplished by proving this double inequality.

We will confine our attention to establish that

$$w(t) \geq \min_{s\in[-r,0]} w(s) \quad \text{for every } t \geq 0. \tag{3.3}$$

In a similar way, it can be shown that

$$w(t) \leq \max_{s\in[-r,0]} w(s) \quad \text{for every } t \geq 0.$$

To prove (3.3), we consider an arbitrary real number M such that $M < \min_{s\in[-r,0]} w(s)$. Clearly,

$$w(t) > M \quad \text{for } t \in [-r, 0]. \tag{3.4}$$

We will show that

$$w(t) > M \quad \text{for all } t \geq 0. \tag{3.5}$$

To this end, let us assume that (3.5) fails to hold. Then, because of (3.4), there exists a point $t_0 > 0$ so that

$$w(t) > M \quad \text{for } t \in [-r, t_0), \quad \text{and} \quad w(t_0) = M.$$

Thus, by using the hypothesis that $c_i \leq 0$ for $i \in I$ and $b_j < 0$ for $j \in J$ and taking into account the fact that $\lambda_0 \leq 0$, from (3.2) we derive

$$\begin{aligned}
M &= w(t_0) \\
&= -\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} w(t_0-\sigma_i) + \lambda_0 \sum_{i\in I} c_i e^{-\lambda_0\sigma_i} \\
&\qquad\qquad \times \int_{t_0-\sigma_i}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)} w(s)\, ds \\
&\quad - \sum_{j\in J} b_j e^{-\lambda_0\tau_j} \int_{t_0-\tau_j}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)} w(s)\, ds \\
&> M \Bigg[-\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} + \lambda_0 \sum_{i\in I} c_i e^{-\lambda_0\sigma_i} \int_{t_0-\sigma_i}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)}\, ds \\
&\qquad\qquad - \sum_{j\in J} b_j e^{-\lambda_0\tau_j} \int_{t_0-\tau_j}^{t_0} e^{(\lambda_0-\lambda_1)(t_0-s)}\, ds \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= M\Bigg\{ -\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} + \lambda_0 \sum_{i\in I} c_i e^{-\lambda_0\sigma_i}\left(-\frac{1}{\lambda_0-\lambda_1}\right)\big[1-e^{(\lambda_0-\lambda_1)\sigma_i}\big] \\
&\qquad - \sum_{j\in J} b_j e^{-\lambda_0\tau_j}\left(-\frac{1}{\lambda_0-\lambda_1}\right)\big[1-e^{(\lambda_0-\lambda_1)\tau_j}\big]\Bigg\} \\
&= \frac{M}{\lambda_0-\lambda_1}\Bigg[-(\lambda_0-\lambda_1)\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} - \lambda_0\sum_{i\in I} c_i\big(e^{-\lambda_0\sigma_i}-e^{-\lambda_1\sigma_i}\big) \\
&\qquad\qquad + \sum_{j\in J} b_j\big(e^{-\lambda_0\tau_j}-e^{-\lambda_1\tau_j}\big)\Bigg] \\
&= \frac{M}{\lambda_0-\lambda_1}\Bigg(\lambda_1\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} - \lambda_0\sum_{i\in I} c_i e^{-\lambda_0\sigma_i} \\
&\qquad\qquad + \sum_{j\in J} b_j e^{-\lambda_0\tau_j} - \sum_{j\in J} b_j e^{-\lambda_1\tau_j}\Bigg) \\
&= \frac{M}{\lambda_0-\lambda_1}\Bigg[\Bigg(-\lambda_0\sum_{i\in I} c_i e^{-\lambda_0\sigma_i} + \sum_{j\in J} b_j e^{-\lambda_0\tau_j}\Bigg) \\
&\qquad\qquad - \Bigg(-\lambda_1\sum_{i\in I} c_i e^{-\lambda_1\sigma_i} + \sum_{j\in J} b_j e^{-\lambda_1\tau_j}\Bigg)\Bigg] \\
&= \frac{M}{\lambda_0-\lambda_1}\,[(\lambda_0-a)-(\lambda_1-a)] = M.
\end{aligned}
$$

This is a contradiction and hence (3.5) is always satisfied. We have thus proved that (3.5) holds true for all real numbers M with $M < \min_{s\in[-r,0]} w(s)$. This guarantees that (3.3) is fulfilled and so the proof of our theorem is complete. □

Now, we will give a lemma which is concerned with the real roots of the characteristic equation $(\widehat{*})$.

Lemma 3.1. *Suppose that*

$$c_i \le 0 \quad \textit{for } i\in I, \quad \textit{and} \quad b_j < 0 \quad \textit{for } j \in J.$$

(I) *Let* λ_0 *be a nonpositive real root of the characteristic equation*

$(\widehat{*})$ *and let* γ_{λ_0} *be defined as in Theorem* 3.0. *Then*

$$1 + \gamma_{\lambda_0} > 0$$

if $(\widehat{*})$ *has another real root less than* λ_0, *and*

$$1 + \gamma_{\lambda_0} < 0$$

if $(\widehat{*})$ *has another nonpositive real root greater than* λ_0.

(II) *If* $a = 0$, *then* $\lambda = 0$ *is not a root of the characteristic equation* $(\widehat{*})$.

(III) *Assume that* $a = 0$ *and that*

$$(\mathrm{H}_1) \qquad \sum_{i \in I} (-c_i) \leq 1.$$

Then the characteristic equation $(\widehat{*})$ *has no positive real roots.*

(IV) *Assume that*

$$(\mathrm{H}_2) \qquad \sum_{j \in J} (-b_j) \geq a$$

and

$$(\mathrm{H}_3) \qquad \sum_{i \in I} (-c_i) + \sum_{j \in J} (-b_j) \tau_j \leq 1.$$

Then the characteristic equation $(\widehat{*})$ *has no positive real roots.*

(V) *Assume that* (H_2) *holds, and that*

$$(\mathrm{H}_4) \qquad ar < 1$$

and

$$(\mathrm{H}_5) \qquad (1 - ar) \sum_{i \in I} (-c_i) e^{-(a-(1/r))\sigma_i} + r \sum_{j \in J} (-b_j) e^{-(a-(1/r))\tau_j} < 1.$$

Then

(i) $\lambda = a - (1/r)$ *is not a root of the characteristic equation* $(\widehat{*})$.

(ii) *In the interval* $(a-(1/r), 0]$, $(\widehat{*})$ *has a unique root.*

(iii) *In the interval* $(-\infty, a-(1/r))$, $(\widehat{*})$ *has a unique root.*

Proof. We first consider the particular case where $a = 0$. In this case, the characteristic equation $(\widehat{*})$ becomes

$$(\widehat{*})_0 \qquad \lambda\left(1+\sum_{i\in I} c_i e^{-\lambda\sigma_i}\right) = \sum_{j\in J} b_j e^{-\lambda\tau_j}.$$

It follows immediately that $\lambda = 0$ is not a real root of $(\widehat{*})_0$, which establishes Part (II). Furthermore, let us assume that (H_1) is satisfied and suppose, for the sake of contradiction, that $(\widehat{*})_0$ has a positive real root μ. We obtain

$$1+\sum_{i\in I} c_i e^{-\mu\sigma_i} \geq 1+\sum_{i\in I} c_i = 1-\sum_{i\in I}(-c_i) \geq 0$$

and consequently

$$\mu\left(1+\sum_{i\in I} c_i e^{-\mu\sigma_i}\right) \geq 0.$$

But, we obviously have

$$\sum_{j\in J} b_j e^{-\mu\tau_j} < 0.$$

We have thus arrived at a contradiction, which proves Part (III).

Now, for the rest of the proof, we define

$$F(\lambda) = \lambda\left(1+\sum_{i\in I} c_i e^{-\lambda\sigma_i}\right) - a - \sum_{j\in J} b_j e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R}.$$

We have

$$F'(\lambda) = 1-\sum_{i\in I}(-c_i)e^{-\lambda\sigma_i} + \lambda\sum_{i\in I}(-c_i)\sigma_i e^{-\lambda\sigma_i} - \sum_{j\in J}(-b_j)\tau_j e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R}.$$

Assume that (H_2) and (H_3) hold. Assumption (H_2) means that

$$F(0) \geq 0. \tag{3.6}$$

Furthermore, by assumption (H_3), we obtain for $\lambda > 0$

$$F'(\lambda) > 1 - \sum_{i\in I}(-c_i) - \sum_{j\in J}(-b_j)\tau_j \geq 0,$$

and consequently F is strictly increasing on the interval $(0, \infty)$. This fact together with (3.6) guarantee that $(\widehat{*})$ has no roots in the interval $(0, \infty)$. We have thus shown Part (IV).

In order to establish Parts (I) and (V), we obtain

$$F''(\lambda) = 2\sum_{i\in I}(-c_i)\sigma_i e^{-\lambda\sigma_i} - \lambda\sum_{i\in I}(-c_i)\sigma_i^2 e^{-\lambda\sigma_i} + \sum_{j\in J}(-b_j)\tau_j^2 e^{-\lambda\tau_j} \quad \text{for } \lambda \in \mathbf{R},$$

and so we have

$$F''(\lambda) > 0 \quad \text{for all } \lambda \in (-\infty, 0]. \tag{3.7}$$

To show Part (I), we consider a nonpositive real root λ_0 of the characteristic equation $(\widehat{*})$. By the definition of γ_{λ_0}, we have

$$\begin{aligned} 1 + \gamma_{\lambda_0} &= 1 + \sum_{i\in I} c_i(1 - \lambda_0\sigma_i)e^{-\lambda_0\sigma_i} + \sum_{j\in J} b_j\tau_j e^{-\lambda_0\tau_j} \\ &= 1 - \sum_{i\in I}(-c_i)e^{-\lambda_0\sigma_i} + \lambda_0\sum_{i\in I}(-c_i)\sigma_i e^{-\lambda_0\sigma_i} - \sum_{j\in J}(-b_j)\tau_j e^{-\lambda_0\tau_j} \\ &= F'(\lambda_0). \end{aligned}$$

Let us assume that there exists another real root λ_1 of $(\widehat{*})$ with $\lambda_1 < \lambda_0$, respectively $0 \geq \lambda_1 > \lambda_0$. Since $F(\lambda_0) = F(\lambda_1) = 0$, we can apply Rolle's theorem to conclude that $F'(\xi) = 0$ for some point ξ such that $\lambda_1 < \xi < \lambda_0$, respectively $\lambda_0 < \xi < \lambda_1$. Furthermore, we observe that, in view of (3.7), F' is strictly increasing on $(-\infty, 0]$. Thus, since $F'(\xi) = 0$, it follows that F' is positive on $(\xi, 0]$, respectively

F' is negative on $(-\infty, \xi)$. So, we must have $F'(\lambda_0) > 0$, respectively $F'(\lambda_0) < 0$.

Finally, we will prove Part (V). Assume that (H_2), (H_4) and (H_5) are satisfied. Assumption (H_2) means that (3.6) holds, while assumptions (H_4) and (H_5) mean respectively

$$a - \frac{1}{r} < 0 \tag{3.8}$$

and

$$F\left(a - \frac{1}{r}\right) < 0. \tag{3.9}$$

The last inequality guarantees, in particular, that $\lambda = a - (1/r)$ is not a root of the characteristic equation $(\widehat{*})$. Furthermore, it is not difficult to verify that

$$F(-\infty) = \infty. \tag{3.10}$$

By taking into account (3.8), from (3.6), (3.7) and (3.9) we can conclude that, in the interval $(a - (1/r), 0]$, $(\widehat{*})$ has a unique root. Moreover, in view of (3.8), from (3.7), (3.9) and (3.10) it follows that, in the interval $(-\infty, a - (1/r))$, $(\widehat{*})$ has a unique root.

The proof of our lemma is now complete.

Before closing this section and ending the paper, let us concentrate our interest on the special case of the (non-neutral) delay differential equation (E) considered in Section 2. Equation (E) can be obtained, as a special case, from $(\widehat{\text{E}})$ by taking $c_i = 0$ for $i \in I$ and considering the initial segment of natural numbers I and the delays σ_i for $i \in I$ to be chosen arbitrarily so that: σ_i for $i \in I$ are positive real numbers such that $\sigma_{i_1} \neq \sigma_{i_2}$ for i_1, $i_2 \in I$ with $i_1 \neq i_2$; and $\sigma \le \tau$. (For example, it can be considered that $I = J$, and $\sigma_i = \tau_i$ for $i \in I$.) As it concerns the (non-neutral) delay differential equation (E), we have the number τ in place of r and the initial condition (C) instead of $(\widehat{\text{C}})$. Also, the characteristic equation $(\widehat{*})$ reduces to $(*)$.

By applying Theorem 3.1 to the (non-neutral) delay differential equation (E), we are led to Theorem 2.1, *under the additional hypothesis* *that the root* λ_0 *of the characteristic equation* $(*)$ *is nonpositive and such that* $1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j} \neq 0$. (Note that we always have $1 + \sum_{j \in J} b_j \tau_j e^{-\lambda_0 \tau_j} \neq 0$ if the other root λ_1 of $(*)$ is also nonpositive.) But, this (additional) hypothesis is not needed for Theorem 2.1 to hold. This is the reason for which we have examined separately the special case of delay differential equations.

REFERENCES

1. O. Arino and M. Pituk, *More on linear differential systems with small delays*, J. Differential Equations **170** (2001), 381–407.

2. O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel and H.-O. Walther, *Delay equations: Functional-, complex-, and nonlinear analysis*, Springer-Verlag, New York, 1995.

3. R.D. Driver, *Some harmless delays*, in *Delay and functional differential equations and their applications*, Academic Press, New York, 1972, pp. 103-119.

4. ———, *Linear differential systems with small delays*, J. Differential Equations **21** (1976), 148–166.

5. ———, *Ordinary and delay differential equations*, Springer-Verlag, New York, 1977.

6. R.D. Driver, D.W. Sasser and M.L. Slater, *The equation* $x'(t) = ax(t) + bx(t-\tau)$ *with "small" delay*, Amer. Math. Monthly **80** (1973), 990–995.

7. M.V.S. Frasson and S.M. Verduyn Lunel, *Large time behaviour of linear functional differential equations*, Integral Equations Operator Theory **47** (2003), 91–121.

8. J.R. Graef and C. Qian, *Asymptotic behavior of forced delay equations with periodic coefficients*, Comm. Appl. Anal. **2** (1998), 551–564.

9. I. Györi, *Invariant cones of positive initial functions for delay differential equations*, Appl. Anal. **35** (1990), 21–41.

10. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, 1977.

11. J.K. Hale and S.M. Verduyn Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

12. I.-G.E. Kordonis, N.T. Niyianni and Ch.G. Philos, *On the behavior of the solutions of scalar first order linear autonomous neutral delay differential equations*, Arch. Math. (Basel) **71** (1998), 454–464.

13. Ch.G. Philos, *Asymptotic behaviour, nonoscillation and stability in periodic first-order linear delay differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1371–1387.

14. Ch.G. Philos and I.K. Purnaras, *Periodic first order linear neutral delay differential equations*, Appl. Math. Comput. **117** (2001), 203–222.

15. ———, *Asymptotic properties, nonoscillation, and stability for scalar first order linear autonomous neutral delay differential equations*, Electron. J. Differential Equations **2004** (2004), 1–17.

Department of Mathematics, University of Ioannina, P.O. Box 1186, 451 10 Ioannina, Greece
E-mail address: cphilos@cc.uoi.gr

Department of Mathematics, University of Ioannina, P.O. Box 1186, 451 10 Ioannina, Greece
E-mail address: ipurnara@cc.uoi.gr