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CARDINAL INVARIANTS OF THE TOPOLOGY OF UNIFORM CONVERGENCE ON COMPACT SETS ON THE SPACE OF MINIMAL USCO MAPS

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ABSTRACT. For a Baire space X the set of all minimal USCO real-valued maps on X coincides with the space $D^*(X)$ of locally bounded densely continuous real-valued forms on X. When X is a locally compact space, the space $D_k^*(X)$ of locally bounded densely continuous real-valued forms on X, under the topology of uniform convergence on compact sets, is a locally convex linear topological space. This paper gives characterizations and bounds for the cardinal function properties on $D_k^*(X)$ of character, pseudocharacter, density, weight, netweight and cellularity. Examples are given to show how these properties can be the same or different. We answer also some questions posed in [**17**].

1. Introduction. For Hausdorff spaces X and Y, a *densely continuous form* from X to Y [**12**] is the closure in $X \times Y$ of $f \restriction C(f)$, where f is a function from X to Y such that the set of points $C(f)$ in X at which f is continuous is dense in X and where $f \restriction C(f)$ is the restriction of f to $C(f)$ (considered as a subset of $X \times Y$).

A densely continuous form can be considered as a set-valued map that has a kind of minimality property found in the theory of minimal USCO maps. These are maps, first appearing in complex analysis, that have been studied in many papers; for example, see [**5, 8**].

The set $D(X, Y)$ of all densely continuous forms from X to Y contains several subsets of interest. It contains the set $C(X, Y)$ of all continuous functions from X to Y. If X is a Baire space and Y is locally compact and second countable, then $D(X, Y)$ contains the functions from X to Y that have closed graphs. If X is a Baire space and Y is a metric

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space, then $D(X,Y)$ contains the minimal USCO mappings from X to Y.

By using results of Christensen [**5**], Saint-Raymond [**22**] and Drewnowski-Labuda [**8**] we can see that if X is a metric Baire space, Y a Banach space and $F : X \to Y$ a minimal weakly USCO mapping (i.e. F is USCO into $(Y, weak)$), then F is a densely continuous form from X into $(Y, weak)$. Starting with Christensen's paper [**5**], a series of "multi-valued Namioka theorems" were discovered [**22, 16, 8**]. These theorems show that, under unexpectedly general assumptions on X and Y, a minimal USCO map $F : X \to Y$ reduces to a (point-valued) function f on a dense subspace of X. There is also a connection between differentiability properties of convex functions and densely continuous forms as expressed via subdifferentials of convex functions, which are a kind of convexification of minimal USCO maps, see [**3**].

An additional important and useful example of a densely continuous form is a so-called Argmin multifunction, see [**13**].

If $Y = \mathbf{R}$, the space of real numbers, then $D(X,Y)$ is denoted by $D(X)$. In this case, when X is a Baire space, $D(X)$ can be identified with the set of all equivalence classes of semi-continuous functions on X under an appropriate equivalence relation [**17**].

Let $C_k(X,Y)$ denote the space $C(X,Y)$ with the topology of uniform convergence on compact sets. This topology can be extended in a natural way to a topology on $D(X,Y)$, whose topological space is denoted by $D_k(X,Y)$ [**12**]. If $Y = \mathbf{R}$, then $D_k(X)$ and $C_k(X)$ denote $D_k(X,Y)$ and $C_k(X,Y)$, respectively. There is a growing literature concerning topologies and convergences on spaces of set-valued maps, such as $D(X,Y)$; for example, see [**2, 7, 12–14, 17, 18**].

The set $D(X)$ has a subspace $D^*(X)$ consisting of those densely continuous forms in $D(X)$ that are locally bounded; that is, bounded on some neighborhood of each point of X [**17**].

The space $D_k^*(X)$ is especially nice when X is locally compact because it is then a locally convex linear topological space [**17**].

This paper continues the study of $D_k^*(X)$ by looking at some cardinal function properties of this space, such as the character, the pseudocharacter, the weight, the netweight and the density. This improves on results in [**17**] and eliminates the need for the continuum hypothesis that was assumed in that paper. Also several examples are given to illustrate the kinds of properties that $D_k^*(X)$ can have.

2. Preliminaries. In what follows let X be a Hausdorff nontrivial topological space, i.e., X is at least countable. The set $D(X)$ of densely continuous real-valued forms ([**12, 17**]) is defined by

$$D(X) = \{\overline{f \upharpoonright C(f)} : C(f) \text{ dense in } X, f : X \to \mathbf{R}\},$$

where $\overline{f \upharpoonright C(f)}$ is the closure of $f \upharpoonright C(f)$ in $X \times \mathbf{R}$.

The densely continuous forms from X to $\mathbf{R}$ are not, in general, functions mapping X into $\mathbf{R}$. They may be considered as multi-functions. For each $x \in X$ and $\Phi \in D(X)$, define

$$\Phi(x) = \{t \in R : (x, t) \in \Phi\}.$$

A multi-function Φ from X to $\mathbf{R}$ is upper semi-continuous at $x \in X$ if for every open set $U \subset \mathbf{R}$ such that $\Phi(x) \subset U$, there is an open neighborhood G of x with $\Phi(z) \subset U$ for every $z \in G$. A multi-function Φ is upper semi-continuous if it is upper semi-continuous at every $x \in X$. Following [**5**] we say that Φ is a USCO map if it is upper semi-continuous with nonempty compact values. If $\Phi \in D(X)$ and $A \subset X$, we say that Φ is bounded on A [**17**] provided that the set $\Phi(A) = \cup\{\Phi(x) : x \in A\}$ is a bounded subset of $\mathbf{R}$. Then Φ is locally bounded provided that each point of X has a neighborhood on which Φ is bounded. Now define $D^*(X)$ to be the set of members of $D(X)$ that are locally bounded.

If $\Phi \in D^*(X)$, then Φ is a minimal USCO map. In fact, if $\Phi \in D^*(X)$, then $\Phi(x)$ is a nonempty compact set for every $x \in X$. By a result of Berge [**4**, p. 112] any multi-function with closed graph which has a compact range is upper semi-continuous. Since upper semi-continuity is a local property, every $\Phi \in D^*(X)$ is upper semi-continuous. Now we can apply Theorem 4.7 in [**8**] to argue that every $\Phi \in D^*(X)$ is a minimal USCO mapping.

Thus, if X is a Baire space, the set $D^*(X)$ coincides with the set of all minimal USCO real-valued mappings.

The topology of $D_k^*(X)$ can be defined using the Hausdorff metric, H, on the space of nonempty compact subsets of $\mathbf{R}$. This metric is defined for nonempty compact subsets A and B of R by

$$H(A,B) = \max\{\max\{d(a,B) : a \in A\},\ \max\{d(b,A) : b \in B\}\},$$

where $d(s,T) = \inf\{|s-t| : t \in T\}$. Then, for each Φ in $D^*(X)$, compact set A in X and real $\varepsilon > 0$, define $W(\Phi, A, \varepsilon)$ to be the set of all Ψ in $D^*(X)$ such that

$$\sup\{H(\Phi(a), \Psi(a)) : a \in A\} < \varepsilon.$$

The family of all $W(\Phi, A, \varepsilon)$ is a base for the topology of $D_k^*(X)$ [**12**]. The metrizability and complete metrizability of $D_k^*(X)$ were studied in [**12, 13, 17**].

The cardinal function properties of $D_k^*(X)$ that are considered here are mainly the *cellularity*, the *density*, the *netweight* and the *weight*. These are defined, respectively, as the supremum of the cardinalities of all pairwise disjoint families of nonempty open subsets, the minimum of the cardinalities of all dense subsets, the minimum of the cardinalities of all networks and the minimum of the cardinalities of all bases. For a general space Z, these cardinal functions are denoted by $c(Z)$, $d(Z)$, $nw(Z)$ and $w(Z)$, respectively. They are in general related by the inequalities

$$c(Z) \le d(Z) \le nw(Z) \le w(Z).$$

When Z is metrizable,

$$c(Z) = d(Z) = nw(Z) = w(Z).$$

The metrizability of $D_k^*(X)$ can be characterized with the following property. The space X is *hemi-compact* provided that in the family $K(X)$ of all nonempty compact subsets of X, there is a countable cofinal subfamily; that is, every member of $K(X)$ is contained in some member of the subfamily. Then the following theorem is shown in [**13**].

Theorem 2.1. *The space $D_k^*(X)$ is metrizable if and only if X is hemi-compact.*

Corollary 2.2. *If X is hemi-compact,*

$$c(D_k^*(X)) = d(D_k^*(X)) = nw(D_k^*(X)) = w(D_k^*(X)).$$

Concerning complete metrizability of $D_k^*(X)$, the following result was proved in [**13**].

Theorem 2.3. *Let X be a locally compact hemi-compact space. Then $D_k^*(X)$ is completely metrizable.*

Theorem 2.4. *If X is a locally compact hemi-compact space, then $D_k^*(X)$ is a completely metrizable locally convex linear topological space.*

It is known [**23**] that if two infinite-dimensional completely metrizable locally convex linear topological spaces have the same density, then they are homeomorphic. The density for such $D_k^*(X)$ is established in the next section.

A partial characterization for the netweight of $D_k^*(X)$ is given in [**17**]. This involves a cardinal function on X called the *peripheral k-network weight* of X. Define a family, $\mathcal{P}$, of subsets of X to be a peripheral k-network for X provided that for every regular open subset U of X and every compact subset A of $\overline{U}$, there exists a $P \in \mathcal{P}$ such that $P \subseteq U$ and every net in U that clusters at some point of A is cofinally in P. Now the peripheral k-network weight of X, $\mathrm{pknw}\,(X)$, is the minimum cardinality of a peripheral k-network for X. In general,

$$\mathrm{pknw}\,(X) \leq |\tau(X)| \leq 2^{w(X)},$$

where $|\tau(X)|$ is the cardinality of the topology of $\tau(X)$ of X and if X is regular, then

$$w(X) \leq \mathrm{pknw}\,(X).$$

A topological space X is Volterra [**10**] if for each pair $f, g : X \to \mathbf{R}$ of functions such that $C(f)$ and $C(g)$ are both dense in X the set $C(f) \cap C(g)$ is dense in X.

Of course every Baire space is Volterra and there are Volterra spaces which are not of second category, hence not Baire [**11**].

Theorem 2.5. *For every Volterra space X,* $\mathrm{nw}\,(D_k^*(X)) \leq \mathrm{pknw}\,(X)$. *If X is locally compact, then* $\mathrm{nw}\,(D_k^*(X)) = \mathrm{pknw}\,(X)$.

Proposition 2.6. *The set $C(X)$ of continuous real-valued functions on X is closed in $D_p^*(X)$, the space $D^*(X)$ equipped with the topology of pointwise convergence.*

Proof. Let $\Phi \in D^*(X)$ be in the closure of $C(X)$ in $D_k^*(X)$. It is easy to verify that $\Phi(x)$ is a singleton set for every $x \in X$. Thus, Φ is a function with closed graph which is locally bounded, i.e., $\Phi \in C(X)$. $\square$

Thus $C(X)$ is closed also in $D_k^*(X)$.

The following example shows that Proposition 2.6 does not hold in $D_p(X)$, the space $D(X)$ equipped with the topology of pointwise convergence.

Example 2.7. Put $X = \{0\} \cup \{1/n : n \in \mathbf{N}\}$ with the natural topology. Define the function $f : X \to \mathbf{R}$ as follows: $f(0) = 0$, $f(1/n) = 0$ if n is even and $f(1/n) = n$ if n is odd. It is easy to verify that $f \in D(X)$ since $C(f) = X \setminus \{0\}$ and $f = \overline{f \restriction C(f)}$. For every $n \in \mathbf{N}$, let f_n be the continuous function from X to $\mathbf{R}$ defined as follows: $f_n(x) = f(x)$ if $x \in [1/n, 1] \cap X$ and $f_n(x) = 0$ otherwise. It is easy to see that $\{f_n : n \in \mathbf{N}\}$ pointwise converges to f, but of course f is not continuous at 0.

3. New theorems. The pseudocharacter and the diagonal degree of $D_k^*(X)$ can be expressed by using the so-called *weak k-covering number* of X. The weak k-covering number of X is defined to be

$$\operatorname{wkc}(X) = \aleph_0 + \min\left\{|\beta| : \beta \subset K(X),\ \overline{\bigcup \beta} = X\right\}.$$

The diagonal degree of X is

$$\Delta(X) = \aleph_0 + \min\left\{|\mathcal{G}| : \mathcal{G} \text{ is a family of open sets in } X \times X, \bigcap \mathcal{G} = \Delta_X\right\},$$

where Δ_X is the diagonal in $X \times X$.

Theorem 3.1. *For every regular space X, $\Psi(D_k^*(X)) = \Delta(D_k^*(X)) = \mathrm{wkc}\,(X)$.*

Proof. To prove that $\mathrm{wkc}\,(X) \leq \Psi(D_k^*(X))$, let f be the zero function on X. Put $\Phi = \overline{f \restriction C(f)} = f$. Let $\{W(\Phi, A_t, \varepsilon_t) : A_t \in K(X),\ \varepsilon_t > 0,\ t \in T\}$ be such that

$$\Phi = \bigcap\{W(\Phi, A_t, \varepsilon_t) : t \in T\} \text{ and } |T| \leq \Psi(D_k^*(X))\}.$$

We claim that $X = \overline{\cup\{A_t : t \in T\}}$. Suppose that there is $x \in X \setminus \overline{\cup\{A_t : t \in T\}}$. Let V be an open set in X such that $x \in V \subset \overline{V} \subset X \setminus \overline{\cup\{A_t : t \in T\}}$. Let g be a function from X to $\mathbf{R}$ such that $g(z) = 1$ for $z \in \overline{V}$ and $g(z) = 0$ otherwise. The set $C(g)$ is dense in X; thus, $\Gamma = \overline{g \restriction C(g)} \in D^*(X)$ and

$$\Gamma \in \bigcap\{W(\Phi, A_t, \varepsilon_t) : t \in T\},$$

a contradiction since $\Gamma(x) = 1$ and $\Phi(x) = 0$. Thus

$$\mathrm{wkc}\,(X) \leq |T| \leq \Psi(D_k^*(X)) \leq \Delta(D_k^*(X)).$$

To prove that $\Delta(D_k^*(X)) \leq \mathrm{wkc}\,(X)$, let $\beta \subset K(X)$ be such that $\mathrm{wkc}\,(X) = |\beta|$ and $\overline{\cup\beta} = X$.

For every $A \in \beta$ and $n \in \mathbf{N}$ put

$$\mathcal{G}_{A,n} = \bigcup\{W(\Phi, A, 1/n) \times W(\Phi, A, 1/n) : \Phi \in D^*(X)\}$$

and we claim that

$$\bigcap\{\mathcal{G}_{A,n} : A \in \beta,\ n \in \mathbf{N}\} = \Delta_{D^*(X)}.$$

Let $\Sigma, \Phi \in D^*(X)$ be such that $\Sigma \neq \Phi$. Thus there is $z \in X$ such that $\Sigma(z) \neq \Phi(z)$. We can suppose that there is $y \in \Sigma(z) \setminus \Phi(z)$; the other case is symmetric. Let $\varepsilon > 0$ be such that $S_{4\varepsilon}[y] \cap \Phi(z) = \varnothing$, where by $S_\eta[a]$ we mean $\{s \in \mathbf{R} : |s - a| < \eta\}$. Since Φ is upper semi-continuous at z, there is an open set $O \subset X$ such that $z \in O$ and $\Phi(v) \cap S_{3\varepsilon}[y] = \varnothing$ for every $v \in O$.

Let $g : X \to \mathbf{R}$ be such that the set $C(g)$ is dense in X and $\Sigma = \overline{g \upharpoonright C(g)}$. Thus there is $s \in C(g)$ such that $s \in O$ and $g(s) \in S_\varepsilon[y]$. The continuity of g at s implies that there is an open set $V \subset O$ with $s \in V$ and $g(l) \subset S_\varepsilon[y]$ for every $l \in V$; then $\Sigma(l) \subset \overline{S_\varepsilon[y]}$ for every $l \in V$. Now, $s \in \overline{\cup\beta}$; thus, there is $A \in \beta$ with $V \cap A \neq \varnothing$. Let $n \in \mathbf{N}$ be such that $1/n < \varepsilon/2$. Suppose that there is $\Gamma \in D^*(X)$ such that $(\Sigma, \Phi) \in W(\Gamma, A, 1/n) \times W(\Gamma, A, 1/n)$. Let $t \in V \cap A$. Then $\Sigma(t) \subset S_{2\varepsilon}[y]$ and $\Phi(t) \cap S_{3\varepsilon}[y] = \varnothing$. Thus $\Gamma(t) \subset S_{\varepsilon/2}[\Sigma(t)] \subset S_{5\varepsilon/2}[y]$ and simultaneously $\Gamma(t) \cap S_{5\varepsilon/2}[y] = \varnothing$, a contradiction. $\square$

Remark. If X is a Tychonoff space, then the pseudocharacter and the diagonal degree of $C_k(X)$ coincide with the pseudocharacter and the diagonal degree of $D_k^*(X)$, since by [**19**] we have $\Psi(C_k(X)) = \Delta(C_k(X)) = wkc(X)$.

Corollary 3.2. *Let X be a regular space. The following are equivalent*:

(1) *Each point of $D_k^*(X)$ is a G_δ-set*;

(2) *Each compact subset of $D_k^*(X)$ is a G_δ-set*;

(3) *$D_k^*(X)$ has a G_δ-diagonal*;

(4) *X is almost σ-compact, i.e., there is a countable family $\beta \subset K(X)$ with $\overline{\cup\beta} = X$.*

The character and π-character of $D_k^*(X)$ can be expressed by using a generalization of the concept of X being hemi-compact. The *k-cofinality* of X is defined to be

$$k\text{cof}\,(X) = \aleph_0 + min\{|\beta| : \beta \text{ is a cofinal subfamily of } K(X)\}.$$

To define the π-character of X, we first need a notion of a local π-base. If $x \in X$, a collection η of nonempty open subsets of X is called a local π-base at x provided that for each open neighborhood U of x, there exists a $V \in \eta$ which is contained in U. Define the π-character of X by

$$\pi_\chi(X) = \aleph_0 + \sup\{\pi_\chi(X, x) : x \in X\},$$

where $\pi_\chi(X, x) = \min\{|\eta| : \eta \text{ is a local } \pi\text{-base at } x\}$.

Theorem 3.3. *For every space X, $\pi\chi(D_k^*(X)) = \chi(D_k^*(X)) = k\,cof(X)$.*

Proof. To prove that $k\mathrm{cof}\,(X) \leq \pi\chi(D_k^*(X))$, let f be the zero function on X. Put $\Phi = \overline{f \restriction X} = f$. Let $\{W(\Phi_t, A_t, \varepsilon_t) : t \in T\}$ be a local π-base of Φ in $D_k^*(X)$ with $|T| \leq \pi\chi(D_k^*(X)$.

We claim that $\{A_t : t \in T\}$ is a cofinal family in $K(X)$. Let $A \in K(X)$. There must exist $t \in T$ with

$$W(\Phi_t, A_t, \varepsilon_t) \subset W(\Phi, A, 1).$$

Then we have $A \subset A_t$. Suppose there is $a \in A \setminus A_t$. Let U be an open set such that $a \in U$ and $\overline{U} \cap A_t = \varnothing$.

Let $f_t : X \to \mathbf{R}$ be such that $\Phi_t = \overline{f_t \restriction C(f_t)}$.

Let $g : X \to \mathbf{R}$ be de defined as follows:

$$g(z) = 1 \quad \text{for } z \in \overline{U} \text{ and } g(z) = f_t(z) \text{ otherwise.}$$

The set $C(g)$ is dense in X. Put $\Gamma = \overline{g \restriction C(g)}$. Then $\Gamma \in D^*(X)$. It is easy to verify that $\Gamma(s) = \Phi_t(s)$ for every $s \notin \overline{U}$; thus also for every $s \in A_t$. Thus $\Gamma \in W(\Phi_t, A_t, \varepsilon_t)$, but $\Gamma \notin W(\Phi, A, 1)$, a contradiction. Thus

$$k\mathrm{cof}\,(X) \leq |T| \leq \pi\chi(D_k^*(X)) \leq \chi(D_k^*(X)). \qquad \square$$

Remark. If X is a Tychonoff space the character and the π-character of $C_k(X)$ coincide with the character and π-character of $D_k^*(X)$, since by [**19**] we have $\pi\chi(C_k(X)) = \chi(C_k(X)) = k\mathrm{cof}\,(X)$.

Corollary 3.4 [**13**]. *The following are equivalent*:

(1) $D_k^*(X)$ *is first countable*;

(2) $\pi\chi(D_k^*(X))$ *is countable*;

(3) X *is hemi-compact.*

A collection β of nonempty open subsets of a space X is called a π-base for X provided that every nonempty open subset of X contains some member of β. Then the π-weight of X is defined to be

$$\pi w(X) = \aleph_0 + \min\{|\beta| : \beta \text{ is a } \pi\text{-base for } X\}.$$

The following theorem shows that the π-weight and weight of $D_k^*(X)$ coincide; the π-weight of $D_k^*(X)$ can be expressed in terms of its density and k-*cofinality* of X and the weight of $D_k^*(X)$ in terms of its cellularity and k-*cofinality* of X.

Theorem 3.5. *For every space X, $\pi w(D_k^*(X)) = w(D_k^*(X)$. In fact, $\pi w(D_k^*(X)) = k\mathrm{cof}(X) \cdot d(D_k^*(X))$ and $w(D_k^*(X)) = k\mathrm{cof}(X) \cdot c(D_k^*(X))$.*

Proof. First we prove that $\pi w(D_k^*(X)) \geq k\mathrm{cof}(X) \cdot d(D_k^*(X))$. The inequality $d(D_k^*(X)) \leq \pi w(D_k^*(X))$ is clear. By Theorem 3.3

$$k\mathrm{cof}(X) = \pi_\chi(D_k^*(X)) \leq \pi w(D_k^*(X));$$

thus, we have

$$k\mathrm{cof}(X) \cdot d(D_k^*(X)) \leq \pi w(D_k^*(X)).$$

Now we prove that $w(D_k^*(X)) \leq k\mathrm{cof}(X) \cdot c(D_k^*(X))$. This part of the proof is similar to the proof of Theorem 5.1 in [**17**]. Let $\mathcal{A}$ be a cofinal subfamily of $K(X)$ with $|\mathcal{A}| = k\mathrm{cof}(X)$. For each $A \in \mathcal{A}$ and $n \in \mathbf{N}$, there exists, by Zorn's lemma, a maximal pairwise disjoint family, $\mathcal{W}_{A,n}$, of basic open subsets of $D_k^*(X)$ of the form $W(\Phi, A, \varepsilon)$ where $\Phi \in D^*(X)$ and ε is a positive real number such that $\varepsilon \leq 1/n$. For each $A \in \mathcal{A}$ and $n \in \mathbf{N}$, define

$$\mathcal{B}_{A,n} = \{W(\Phi, A, q) : q \in \mathbf{Q}_+ \text{ and } q > \varepsilon \text{ for some } W(\Phi, A, \varepsilon) \in \mathcal{W}_{A,n}\},$$

where $\mathbf{Q}_+$ is the set of positive rational numbers. Define

$$\mathcal{B} = \bigcup\{\mathcal{B}_{A,n} : A \in \mathcal{A} \text{ and } n \in \mathbf{N}\}.$$

Then $|\mathcal{B}| \leq k\mathrm{cof}(X) \cdot c(D_k^*(X))$.

It remains to show that $\mathcal{B}$ is a base for $D_k^*(X)$. Fix $\Psi \in D_k^*(X)$ and its open neighborhood $W(\Psi, B, \delta)$, where $B \in K(X)$ and $\delta > 0$. We look for $\Phi \in D^*(X)$, $A \in \mathcal{A}$ and $q \in \mathbf{Q}$ such that

$$(*) \qquad \Psi \in W(\Phi, A, q) \subseteq W(\Psi, B, \delta).$$

We may and do assume that $\delta \in \mathbf{Q}$ and $A = B \in \mathcal{A}$ because $\mathcal{A}$ is cofinal in $K(X)$ (otherwise we take $A \in \mathcal{A}$ with $B \subseteq A$). We have two cases:

1^0. $W(\Psi, A, \delta) \in \mathcal{B}_{A,n}$ for some $n \in \mathbf{N}$. Then we take $\Phi = \Psi$ and $q = \delta$, and the condition $(*)$ is satisfied.

2^0. Assume that, for every $n \in \mathbf{N}$, $W(\Psi, A, \delta) \notin \mathcal{B}_{A,n}$, i.e., for every n and every $0 < \varepsilon < \delta$, $W(\Psi, A, \varepsilon) \notin \mathcal{W}_{A,n}$. In this case, by maximality of $\mathcal{W}_{A,n}$, there exist $\Phi \in D^*(X), \varepsilon > 0$ and $n \in \mathbf{N}$ such that $\varepsilon \leq 1/n \leq \delta/2$, and

$$W(\Psi, A, \varepsilon) \bigcap W(\Phi, A, \varepsilon) \neq \varnothing.$$

Then $\Psi \in W(\Phi, A, 2\varepsilon) \subseteq W(\Psi, A, 2/n) \subseteq W(\Psi, A, \delta)$, and the condition $(*)$ is satisfied. $\square$

Recall that a k-network for X is a family $\mathcal{B}$ of subsets of X such that if $A \in K(X)$ and U is open in X with $A \subset U$, then there exists a finite subfamily $\mathcal{B}' \subset \mathcal{B}$ with $A \subset \cup\mathcal{B}' \subset U$. The *$k$-netweight* of X is defined by

$$\operatorname{knw}(X) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } k\text{-network for } X\}.$$

It is clear that $\operatorname{nw}(X) \leq \operatorname{knw}(X)$ and $\operatorname{knw}(X) \leq w(X)$. Thus in locally compact spaces we have also that $w(X) = \operatorname{knw}(X) = \operatorname{nw}(X)$.

As a result of Theorem 3.5, the weight of $D_k^*(X)$ can be equated to the netweight of $D_k^*(X)$ whenever X is locally compact.

Theorem 3.6. *For every regular space X,* $\operatorname{knw}(X) \leq \operatorname{nw}(D_k^*(X))$. *For locally compact X,* $w(D_k^*(X)) = \operatorname{nw}(D_k^*(X)) = \operatorname{pknw}(X)$.

Proof. Let $\mathcal{N}$ be a network for $D_k^*(X)$. For each $N \in \mathcal{N}$, define

$$N^* = \left\{ x \in X : \Psi(x) \bigcap (0, \infty) \neq \varnothing \text{ for all } \Psi \in N \right\},$$

and let

$$\mathcal{N}^* = \{N^* : N \in \mathcal{N}\}.$$

To show that $\mathcal{N}^*$ is a k-network for X, let U be an open subset of X and let $A \subset U$ be compact. Let V be an open set in X such that $A \subset V \subset \overline{V} \subset U$.

Define the function $f : X \to \mathbf{R}$ as follows: $f(x) = 1$ if $x \in V$ and $f(x) = 0$ if $x \in X \setminus V$. The set $C(f)$ is dense in X. Put $\Phi = \overline{f \restriction C(f)}$. Then $\Phi \in D^*(X)$, so there is $N \in \mathcal{N}$ with

$$\Phi \in N \subset W(\Phi, A, 1).$$

Note that $N^* \subset U$, because if $x \notin U$, then $\Phi(x) = \{0\}$, implying that $x \notin N^*$. Now we prove that $A \subset N^*$. Let $x \in A$. Then $\Phi(x) = \{1\}$. Since for every $\Psi \in N$ we have $\Psi \in W(\Phi, A, 1)$, $\Psi(x) \cap (0, \infty) \neq \varnothing$.

To prove that $w(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X))$ by Theorem 3.5, it suffices to show that $k\mathrm{cof}\,(X) \leq \mathrm{nw}\,(D_k^*(X))$. By above we know that $\mathrm{knw}\,(X) \leq \mathrm{nw}\,(D_k^*(X))$, thus $w(X) \leq nw(D_k^*(X))$ because X is locally compact. Also because X is locally compact, it has a base $\mathcal{B}$ of relatively compact sets such that $|\mathcal{B}| = w(X)$. Then the family of all finite unions of members of $\{\overline{B} : B \in \mathcal{B}\}$ is cofinal in $K(X)$ and has cardinality $w(X)$. Therefore $k\mathrm{cof}\,(X) \leq w(X) \leq nw(D_k^*(X))$. $\square$

Remark. Of course for X Tychonoff the proof of $\mathrm{knw}\,(X) \leq \mathrm{nw}\,(D_k^*(X))$ can be done easily in the following way: by [**19**] $\mathrm{knw}\,(X) = \mathrm{nw}\,(C_k(X))$ and $\mathrm{nw}\,(C_k(X)) \leq \mathrm{nw}\,(D_k^*(X))$ since the compact-open topology on $C(X)$ coincides with the topology induced from $D_k^*(X)$ on $C(X)$.

The inequality in Theorem 3.6 can be strict. If X is a nondiscrete locally compact second countable space then $\mathrm{knw}\,(X) = w(X) = \aleph_0$ and $\mathrm{nw}\,(D_k^*(X) = c$, see Corollary 3.12 below.

From Theorem 2.5, for a Volterra space X $\mathrm{pknw}\,(X)$ gives an upper bound for most of these cardinal functions on $D_k^*(X)$. To get a lower bound, consider the following concept. Let $UK(X)$ be the collection of all pairwise disjoint families $\mathcal{U}$ of nonempty open subsets of X such that $\overline{\cup \mathcal{U}}$ is compact.

Theorem 3.7. *For each $\mathcal{U}$ in $UK(X)$, $2^{|\mathcal{U}|} \leq c(D_k^*(X))$.*

Proof. Let $\mathcal{U} \in UK(X)$, and let $m = |\mathcal{U}|$. Let $2^{\mathcal{U}}$ denote the set of functions from $\mathcal{U}$ to $\{0,1\}$. For each $\phi \in 2^{\mathcal{U}}$, define $f_\phi : X \to \mathbf{R}$ by

$$f_\phi(x) = \begin{cases} \phi(U) & \text{if } x \in U \text{ for some } U \in \mathcal{U}, \\ 0 & \text{otherwise.} \end{cases}$$

For each $\phi \in 2^{\mathcal{U}}$, the set $C(f_\phi)$ is dense in X. So each Φ_ϕ, defined as $\overline{f_\phi \restriction C(f_\phi)}$, is a member of $D^*(X)$. Now for each $\phi \in 2^{\mathcal{U}}$, define $B_\phi = W(\Phi_\phi, \overline{\cup\mathcal{U}}, 1/4)$. Then $\{B_\phi : \phi \in 2^{\mathcal{U}}\}$ is a pairwise disjoint family of nonempty open subsets of $D_k^*(X)$. Therefore $2^{|\mathcal{U}|} \leq c(D_k^*(X))$. $\square$

Corollary 3.8. *If X is compact, then $2^{<c(X)} \leq c(D_k^*(X))$.*

The next corollary gives some special cases of spaces satisfying Theorem 3.7. In this corollary, $\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the continuum.

A topological space X is almost locally compact ([**1, 21**]) provided that every nonempty open subset contains a compact set with a nonempty interior. Of course every locally compact space is almost locally compact. The Michael line is an example of an almost locally compact space, which is not locally compact. Another example is given by the subspace $\{(x,y) \in R^2 : y > 0\} \cup \{(0,0)\}$ of the real plane [**13**].

Corollary 3.9. *For a space X, $\mathfrak{c} \leq c(D_k^*(X))$ if either of the following conditions hold*:

(1) *X is almost locally compact and has a non-isolated point with a countable base, or*

(2) *X has a non-isolated point with a compact neighborhood.*

Corollary 3.10. *If $k\mathrm{cof}(X) \leq 2^{|\mathcal{U}|}$ for some $\mathcal{U}$ in $UK(X)$, then*

$$c(D_k^*(X)) = d(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X)) = w(D_k^*(X)).$$

Corollary 3.11. *If $kcof(X) \leq \mathfrak{c}$ and either of the conditions in Corollary* 3.9 *hold, then*

$$c(D_k^*(X)) = d(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X)) = w(D_k^*(X)).$$

Corollary 3.12. *If X is a nondiscrete locally compact second countable space, then*

$$c(D_k^*(X)) = d(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X)) = w(D_k^*(X)) = \mathfrak{c}.$$

Corollary 3.13. *For every two nondiscrete locally compact second countable spaces X and Y, $D_k^*(X)$ and $D_k^*(Y)$ are homeomorphic.*

Proof. If X and Y are two nondiscrete locally compact second countable spaces, then by Corollary 3.12 $d(D_k^*(X)) = d(D_k^*(Y)) = \mathfrak{c}$. By Theorem 2.4, both $D_k^*(X)$ and $D_k^*(Y)$ are completely metrizable locally convex linear topological spaces. By the result of Torunczyk [**23**] we are done.

Corollaries 3.12 and 3.13 improve Corollary 5.4 and Theorem 5.4 in [**17**] and answer questions posed in [**17**].

Note that, for a locally compact space X,

$$w(X) \le w(D_k^*(X)) \le 2^{w(X)}.$$

Corollary 3.12 shows that if X is a nondiscrete locally compact second countable space, then

$$w(X) < w(D_k^*(X)).$$

Example 4.3 in the next section gives a compact space X for which

$$w(D_k^*(X)) < 2^{w(X)}.$$

4. Examples. The examples in this section illustrate the various possibilities for these cardinal function properties of $D_k^*(X)$.

Example 4.1. If X is a countably infinite discrete space and Y is the one-point-compactification of X, then

$$d(D_k^*(X)) = \aleph_0 < \mathfrak{c} = d(D_k^*(Y)).$$

Proof. The last equality follows from Corollary 3.12. Because X is discrete, $D_k^*(X)$ is homeomorphic to the countably infinite product of copies of $\mathbf{R}$; so its density is $\aleph_0$. $\square$

Example 4.2. If X is the space of countable ordinals, ω_1, with the topology induced on ω_1 by the linear order [**9**, p. 56], then $D_k^*(X)$ is a nonmetrizable locally convex linear topological space such that

$$c(D_k^*(X)) = d(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X)) = w(D_k^*(X)) = \mathfrak{c}.$$

Proof. First observe that $k\mathrm{cof}\,(X) = \aleph_1$ because the subfamily of all intervals $[0, \alpha]$ for $\alpha < \omega_1$ is cofinal in $K(X)$ and has the minimum cardinality of such cofinal subfamilies of $K(X)$. Therefore by Theorem 2.1, $D_k^*(X)$ is not metrizable, and by Corollary 3.11,

$$c(D_k^*(X)) = d(D_k^*(X)) = \mathrm{nw}\,(D_k^*(X)) = w(D_k^*(X)).$$

Since X satisfies condition (1) in Corollary 3.9, it follows that $\mathfrak{c} \leq c(D_k^*(X))$.

By Theorem 3.6, it remains to show that $\mathrm{pknw}\,(X) \leq \mathfrak{c}$. Let $\mathcal{P}$ be the family of all open sets in X that are contained in the interval $[0, \alpha)$ for some $\alpha < \omega_1$. Then $\mathcal{P}$ has cardinality $\mathfrak{c}$. To show that $\mathcal{P}$ is a peripheral k-network of X, let U be a regular open subset of X and let A be a compact subset of $\overline{U}$. Now A is contained in $[0, \alpha)$ for some $\alpha < \omega_1$. Then $P = [0, \alpha) \cap U$ is a member of $\mathcal{P}$ that is contained in U. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in U which clusters at $a \in A$. Then $a \in [0, \alpha)$, so $[0, \alpha)$ is a neighborhood of a. Thus for every $\lambda \in \Lambda$ there is an $\eta \geq \lambda$ with $x_\eta \in [0, \alpha)$. Thus the net $\{x_\lambda : \lambda \in \Lambda\}$ is cofinally in $[0, \alpha)$, i.e., the net $\{x_\lambda : \lambda \in \Lambda\}$ must be cofinally in P. $\square$

Example 4.3. If $D(m)$ is the discrete space of infinite cardinality m and X is the Čech-Stone compactification, $\beta D(m)$, of $D(m)$, then

$$d(D_k^*(X)) = 2^m < 2^{2^m} = |\tau(X)|.$$

Proof. If $\mathcal{U}$ consists of the singleton subsets of $D(m)$, then $\mathcal{U} \in UK(X)$. So by Theorem 3.7, $2^m \leq c(D_k^*(X)) = d(D_k^*(X))$. For the reverse inequality, observe that there is a natural injection from $D^*(\beta D(m))$ into $C(D(m))$ defined by mapping each element of $D^*(\beta D(m))$ to its restriction to $D(m)$, which is a singleton-valued function on $D(m)$. Therefore,

$$d(D_k^*(X)) \leq |D^*(\beta D(m))| \leq |C(D(m))| \leq \mathfrak{c}^m = (2^{\aleph_0})^m \leq 2^m. \qquad \square$$

Example 4.4. As a special case of Example 4.3, if $X = \beta N$, then $d(D_k^*(X)) = \mathfrak{c}$. On the other hand, if $Y = \beta N \setminus N$, then $d(D_k^*(Y)) = 2^{\mathfrak{c}}$ so that Y is a compact subspace of compact space X with the property that $d(D_k^*(X)) < d(D_k^*(Y))$.

Proof. The cellularity and weight of Y are both equal to $\mathfrak{c}$, see [**24**, Theorem 3.22, p. 77]. So by Theorem 3.7 and Theorem 2.5,

$$2^{\mathfrak{c}} \leq c(D_k^*(Y)) = d(D_k^*(Y)) \leq \operatorname{pknw}(Y) \leq 2^{\mathfrak{c}}. \qquad \square$$

Example 4.5. If X is a nondiscrete locally compact second countable space and Y is the disjoint topological sum of $2^{2^{\mathfrak{c}}}$ copies of X, then $D_k^*(Y)$ is a locally convex linear topological space such that

$$\mathfrak{c} = c(D_k^*(Y)) < d(D_k^*(Y)) < w(D_k^*(Y)) = 2^{2^{\mathfrak{c}}}.$$

Proof. Let $m = 2^{2^{\mathfrak{c}}}$. As indicated in [**12**], the space $D_k^*(Y)$ is homeomorphic to the product of m copies of $D_k^*(X)$. Now $\operatorname{pknw}(X) = \mathfrak{c}$, so that $\operatorname{pknw}(Y) = m \cdot \mathfrak{c} = m$. Then by Theorem 3.6, $w(D_k^*(Y)) = m = 2^{2^{\mathfrak{c}}}$. By Corollary 3.12, the cellularity of $D_k^*(X)$ is $\mathfrak{c}$, so that the product of m copies of $D_k^*(X)$ has cellularity no greater than $\mathfrak{c}$, see [**9**]. Thus, $c(D_k^*(Y)) = \mathfrak{c}$. Again by Corollary 3.12, the density of $D_k^*(X)$ is $\mathfrak{c}$, so that the product of m copies of $D_k^*(X)$ has density no greater than $2^{\mathfrak{c}}$, see [**9**, Theorem 2.3.15, p. 81]. Therefore, $d(D_k^*(Y)) \leq 2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}}$.

It remains to show that $\mathfrak{c} < d(D_k^*(Y))$. The proof of Theorem 4.2.1 in [**19**] shows that for general spaces W and Z such that Z contains a nontrivial path, $ww(W) \leq d(C_p(W, Z))$, where p denotes the topology of pointwise convergence and $ww(W)$ is the *weak weight* of W, defined to be the minimum of the cardinalities of the weights of all continuous one-to-one Hausdorff images of W. Since $D_k^*(X)$ is a locally convex linear topological space, it contains a nontrivial path. The product of m copies of $D_k^*(X)$ is homeomorphic to $C_p(D(m), D_k^*(X))$, so that $ww(D(m)) \leq d(D_k^*(Y))$. To show that $\mathfrak{c} < ww(D(m))$, let Z be a Hausdorff space with $w(Z) = ww(D(m))$. Then $c < w(Z)$ because otherwise $m = |Z| \leq 2^{w(Z)} \leq 2^{\mathfrak{c}}$, which is a contradiction. $\square$

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