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FURTHER CONSEQUENCES OF A SEXTUPLE PRODUCT IDENTITY

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ABSTRACT. Presented here are representations of values of the sum-of-divisors function σ at odd arguments. A corollary giving necessary conditions for primality of these arguments is then presented.

1. Introduction. In [**2**, p. 1287] the author presented the following sextuple product identity, which is valid for each triple of complex numbers a, b, x such that $a \neq 0$, $b \neq 0$ and $|x| < 1$.

$$
\begin{aligned}
\prod^{\infty}(1-x^{2n})^2&(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1}) \\
&\cdot(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1}) \\
&= \sum_{-\infty}^{\infty} x^{2m^2}a^{2m}\sum_{-\infty}^{\infty} x^{2n^2}b^{2n} \\
&\quad + x\sum_{-\infty}^{\infty} x^{2m(m+1)}a^{2m+1}\sum_{-\infty}^{\infty} x^{2n(n+1)}b^{2n+1}.
\end{aligned} \tag{1.1}
$$

In fact, it was demonstrated there that this identity is an easy and straightforward consequence of the classical Gauss-Jacobi triple product identity:

$$
\prod_1^{\infty}(1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}t^n, \tag{1.2}
$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. Identity (1.1) was then used to derive two formulas for representing numbers by sums of four triangular numbers and by sums

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of eight triangular numbers. In this paper we propose to derive two further results of arithmetical interest. As these results involve several arithmetical functions, we collect these in the following definition.

Definition 1.1. $\mathbf{P} := \{1, 2, 3, \dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$. Then, for each $n \in \mathbf{N}$,

$$t_2^2(n) := \left| \left\{ (j,k) \in \mathbf{N}^2 \mid n = \frac{j(j+1)}{2} + \frac{k(k+1)}{2} \right\} \right|$$

and

$$r_2(n) := |\{(j,k) \in \mathbf{Z}^2 | n = j^2 + k^2\}|.$$

For each $k \in \mathbf{N}$ and each $n \in \mathbf{P}$, $\sigma_k(n)$: is the sum of the kth powers of all positive divisors of n. For simplicity, $\sigma(n) := \sigma_1(n)$.

For each $i \in \{1, 3\}$ and each $n \in \mathbf{P}$,

$$d_i(n) := \sum_{\substack{d|n \\ d \equiv i \pmod 4}} 1.$$

Then, for each $n \in \mathbf{P}$, $\delta(n) := d_1(n) - d_3(n)$.

We are now prepared to state our main result.

Theorem 1.2. *For each* $m \in \mathbf{N}$,

$$\sigma(4m+1) = \delta(4m+1) + 4 \sum_{k=1}^{m} \delta(4m+1-4k)\delta(2k) \tag{1.3}$$

and

$$\sigma(4m+3) = 4 \sum_{k=0}^{m} \delta(4m+3-4k-2)\delta(2k+1). \tag{1.4}$$

The counting functions $t_2(n)$ and $r_2(n)$, $n \in \mathbf{N}$, arise naturally in the proof of this result. They are eliminated in the final statement, owing to the following formulas.

$$r_2(n) = 4\delta(n), \quad n \in \mathbf{P}, \tag{1.5}$$

$$t_2(n) = \delta(4n+1), \quad n \in \mathbf{N}. \tag{1.6}$$

Of course, $r_2(0) = t_2(0) = 1$. Formula (1.5) is a classical result due to Jacobi. For a proof, see [**3**, pp. 241–242]. For a proof of (1.6), see [**1**, pp. 753–755].

2. Proofs. First of all we recall the following identity due to Gauss:

$$\prod_{1}^{\infty} \frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{0}^{\infty} x^{n(n+1)/2}, \quad |x| < 1.$$

This is actually an easy special case of (1.2). Our proof then turns on being able to express the infinite product $x\prod_{n=1}^{\infty}(1-x^{4n})^6$ in two different ways. First we observe that the square of the righthand side of Gauss's identity generates the sequence $t_2(n)$, $n \in \mathbf{N}$, i.e.,

$$\left\{\sum_{n=0}^{\infty} x^{n(n+1)/2}\right\}^2 := \sum_{n=0}^{\infty} t_2(n)x^n, \quad |x| < 1.$$

Hence we (i) square both sides of Gauss's identity, (ii) let $x \to x^4$ and (iii) multiply the resulting identity by x to get

$$\begin{aligned} x\prod_{n=1}^{\infty} \frac{(1-x^{8n})^2}{(1-x^{8n-4})^2} &= x\bigg(\sum_{n=0}^{\infty} x^{2n(n+1)}\bigg)^2 \\ &= \sum_{n=0}^{\infty} t_2(n)x^{4n+1}. \end{aligned}$$

Then, on the one hand, by the foregoing identity and (1.6), we get

$$\begin{aligned} x\prod_{1}^{\infty}(1-x^{4n})^6 &= \prod_{1}^{\infty}(1-x^{4n})^4(1-x^{8n-4})^4 \cdot x\prod_{1}^{\infty}\frac{(1-x^{8n})^2}{(1-x^{8n-4})^2} \\ &= \prod_{1}^{\infty}(1-x^{4n})^4(1-x^{8n-4})^4 \cdot \sum_{n=0}^{\infty} t_2(n)x^{4n+1} \\ &= \prod_{1}^{\infty}(1-x^{4n})^4(1-x^{8n-4})^4 \cdot \sum_{n=0}^{\infty} \delta(4n+1)x^{4n+1}. \end{aligned} \tag{2.1}$$

On the other hand,

$$x\prod_{1}^{\infty}(1-x^{4n})^6 = \prod_{1}^{\infty}(1-x^{2n})^2(1-x^{4n-2})^2 \sum_{n=0}^{\infty} \sigma(2n+1)x^{2n+1}. \tag{2.2}$$

To see this, we begin with the identity

$$x\prod_1^\infty \frac{(1-x^{4n})^4}{(1-x^{4n-2})^4} = \sum_0^\infty \sigma(2n+1)x^{2n+1}, \quad |x|<1.$$

For a proof, see [**2**, p. 1291]. Then, owing to the classical identity of Euler

$$\prod_1^\infty (1+x^n)(1-x^{2n-1}) = 1, \quad |x|<1,$$

we get

$$\begin{aligned}\prod_1^\infty (1-x^{2n})^2(1-x^{4n-2})^2 \cdot \sum_{n=0}^\infty \sigma(2n+1)x^{2n+1} \\ &= x\prod_1^\infty (1-x^{2n})^2(1-x^{4n-2})^2 \frac{(1-x^{4n})^4}{(1-x^{4n-2})^4} \\ &= x\prod_1^\infty (1-x^{2n})^2(1+x^{2n})^2(1-x^{4n})^4 \\ &= x\prod_1^\infty (1-x^{4n})^6.\end{aligned}$$

This proves (2.2).

In (1.2) let $t=1$ to get

$$\prod_{n=1}^\infty (1-x^{2n})(1+x^{2n-1})^2 = \sum_{-\infty}^\infty x^{n^2}.$$

Note that the square of the righthand side of this identity generates the sequence $r_2(n)$, $n\in\mathbf{N}$, i.e.,

$$\left\{\sum_{-\infty}^\infty x^{n^2}\right\}^2 := \sum_{n=0}^\infty r_2(n)x^n, \quad |x|<1.$$

Hence, in the foregoing identity we (i) let $x\to x^2$ and (ii) square both sides of the resulting identity to get

$$\begin{aligned}\prod_1^\infty (1-x^{4n})^2(1+x^{4n-2})^4 &= \left\{\sum_{-\infty}^\infty x^{2n^2}\right\}^2 \\ &= \sum_{n=0}^\infty r_2(n)x^{2n}.\end{aligned}$$

Now between (2.1) and (2.2) we eliminate $x\prod(1-x^{4n})^6$ and divide the resulting identity by the product $\prod(1-x^{2n})^2(1-x^{4n-2})^2$ to get

$$\begin{aligned}\sum_0^\infty \sigma(2n+1)x^{2n+1} &= \prod_1^\infty (1-x^{4n})^2(1+x^{4n-2})^4 \cdot \sum_{n=0}^\infty \delta(4n+1)x^{4n+1}\\ &= \sum_{n=0}^\infty r_2(n)x^{2n} \cdot \sum_{n=0}^\infty \delta(4n+1)x^{4n+1}\\ &= \left\{1+\sum_{n=1}^\infty r_2(n)x^{2n}\right\}\sum_{n=0}^\infty \delta(4n+1)x^{4n+1}\\ &= \left\{1+4\sum_{n=1}^\infty \delta(n)x^{2n}\right\}\sum_{n=0}^\infty \delta(4n+1)x^{4n+1}.\end{aligned}$$

(In the last two steps we've used $r_2(0)=1$ and (1.5).) Next we expand the product of the two series and separate the terms according to whether the exponents of the powers of x are congruent to $1 \pmod 4$ or congruent to $3 \pmod 4$. Then, equating coefficients of like powers of x, we prove our theorem.

Corollary 2.1. (i) *For each* $n \in \mathbf{P}$, *if* $n \equiv 1 \pmod 4$ *and* n *is prime, then*

$$\frac{n-1}{4} = \sum_{k=1}^{(n-1)/4} \delta(n-4k)\delta(2k).$$

(ii) *For each* $n \in \mathbf{P}$, *if* $n \equiv 3 \pmod 4$ *and* n *is prime, then*

$$\frac{n+1}{4} = \sum_{k=0}^{(n-3)/4} \delta(n-4k-2)\delta(2k+1).$$

Proof. (i) To see this, suppose that $n=4m+1$ for some $m \in \mathbf{P}$, and, further, suppose that n is prime. Then $\sigma(n)=\sigma(4m+1)=4m+2$ and $\delta(4m+1)=2$, whence the desired conclusion, owing to (1.3).

(ii) Suppose that $n=4m+3$, for some $m \in \mathbf{N}$ and n is prime. Then $\sigma(n)=\sigma(4m+3)=4m+4$, whence the desired conclusion owing to (1.4).

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